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Day convolution for enriched bicategories with an application to Mackey 2-functors

Convolution de Day pour les bicatégories enrichies avec une application aux 2-foncteurs de Mackey

Thèse de doctorat en Mathématiques et leurs interactions

Présentée et soutenue publiquement par

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Abstract

This thesis consists of two parts. In the first one we study bicategories enriched over a monoidal bicategory, as defined by Garner–Shulman. The novelty is the introduction of a braiding on the base bicategory, which allows us to form opposites and tensor products of enriched bicategories. This lets us build a novel theory of ends and coends in this context.

The second part concerns the notion of Mackey 2-functors as introduced by Balmer–Dell'Ambrogio for studying the bicategorical structures arising from finite group actions throughout equivariant mathematics. We prove that the bicategory of Mackey 2-motives (namely the target of the universal Mackey 2-functor) admits a canonical monoidal structure. This is combined with the results of the first part in order to define a Day convolution product of Mackey 2-functors. In its turn, the latter product allows us to redefine the notion of a Green 2-functor simply as being a pseudomonoid in the braided monoidal bicategory of Mackey 2-functors. This is a conceptual clarification of the theory, in that now we can see the perfect analogy to the classical theory of Mackey and Green functors in the representation theory of finite groups.

Contents

Introduction

Bicategories were first introduced in [Bén67] as both a vertical categorification of the notion of category, as well as an horizontal categorification of the notion of monoidal category. There, the complication brought by non-strict categorical constraint is justified by the number of examples provided which would be excluded by more rigid requirements (those defining a 2-category, most notably). One of the first examples introduced in [Bén67] is the bicategory of spans of a category with pullbacks $\mathcal{C}$, which consists of the same objects as those of $\mathcal{C}$, while morphisms from an object X to an object Y are pairs of morphisms in $\mathcal{C}$ of the form $X \leftarrow P \rightarrow Y$ called "spans". Pullbacks, which are unique only up to isomorphisms (whence the non-strictness of the definition) are used to form the composition, while morphisms between two such spans $X \leftarrow P \rightarrow Y$ and $X \leftarrow P' \rightarrow Y$ are morphisms between P and P' making the obvious triangles commute. Spans provide rich categorical and algebraic structures: they serve very generally as a place to look for adjunctions (as shown in [LWW10]), and they often inherit some structure from the base category. A sufficiently interesting case of study is the bicategory of spans of the (2,1)-category gpd of finite groupoids, considered together with its faithful morphisms, and more in general of a spannable pair (Definition 3.2.2). This notion was developed in order to formally axiomatize and understand the following phenomenon, involving most notably finite groups and their representation theory. If $\mathcal{C}$ is a category and $G' \rightarrow G$ a functor, the induced functor $\mathcal{C}^G \rightarrow \mathcal{C}^{G'}$ often have a left and a right adjoint. If moreover $\mathcal{C}$ is linear and $G' \rightarrow G$ is of "finite type", there is a strong tendency for the two adjoints to coincide (see [Law86], Section 2). This axiomatization is done in in [BD20] by means of *Mackey pseudofunctors*, which were introduced there first.

Mackey pseudofunctors, which categorify in various precise senses usual Mackey functors (introduced by Dress and for which we refer to [Web00]) aim then to describe properties of adjunctions which happen to hold in the context of the representation theory of finite groups. Bicategories of spans and suitable variations are the right objects where to look for encoding these properties, resulting in universal objects for this theory. Such variations are called *Mackey motives* in this context, in analogy with the universal object for cohomology theories in algebraic geometry.

Just to make some examples of Mackey pseudofunctors across mathematics, one can consider the operation of associating, to a variable groupoid G, the category of representation $\mathrm{Mod}(\Bbbk G)$, the stable module category $\mathrm{Stab}(\Bbbk G)$, the derived category $\mathrm{D}(\Bbbk G)$, or topological objects such as the stable homotopy category of G-spectra $\mathrm{SH}(G)$ and many others. For a wider class of example we refer to section 4 in [BD20]. All of these objects $\mathcal{M}(G)$ enjoy the property of being (contravariantly) functorial in the groupoid, defining hence a pseudofunctor

$$\mathcal{M} \colon \mathsf{gpd}^{\mathsf{op}} \longrightarrow \mathrm{Add}$$

into the 2-category of additive categories, additive functors and natural transformations. Most importantly, they also admit isomorphic left and right adjoints for the restriction

$$\begin{array}{c} \mathcal{M}(G) \\ i_! \left(\dashv \; i^* \downarrow \; \dashv \right) i_* \\ \mathcal{M}(H) \end{array}$$

whenever $i\colon H \hookrightarrow G$ is an inclusion (a faithful functor) of groupoids.

A natural notion of monoidal product is available on these categories in most of the examples, and provides the notion of *Green pseudofunctor*. At a first stage, one can say that a Mackey pseudofunctor $\mathcal{M}$ is a Green pseudofunctor if each $\mathcal{M}(G)$ is actually a *monoidal* additive category, and if this structure is compatible with inductions and restrictions, in an appropriate sense. As conjectured in [Del22], a suitable monoidal structure on the bicategory of Mackey pseudofunctors should give a direct and more elegant definition of a Green pseudofunctor as just being a pseudomonoid with respect to this monoidal structure. This happens to be true and, generalizing the same phenomenon for the 1-categorical case, it's done via the Day convolution product by seeing the bicategory of Mackey pseudofunctors as the bicategory of additive pseudofunctors from the universal category of motives to Add. The main goal of this thesis is to set up the categorical definitions and constructions necessary to verify this conjecture. More in detail, Mackey motives consist of an additive bicategory Mot such that the bicategory of Mackey pseudofunctors Mack is biequivalent to the bicategory of enriched (additive) pseudofunctors

$$\mathsf{PsFun}_{\oplus}(\mathsf{Mot}, \mathrm{Add}).$$

Using this description of the bicategory of Mackey pseudofunctor, we can study a Day convolution structure on it. A necessary step is, however, to define a monoidal structure on the bicategory of motives.

Also, a general theory of bicategories enriched over a monoidal bicategory is then treated in Chapter 1. The main reference for what has been defined and developed at this point is [GS15]. In this work we deal with further fundamental constructions requiring some sort of symmetry, such as the opposite enriched bicategory (Section 1.6) and the tensor product of enriched bicategories (Section 1.7).

Then, the correct notion of Day convolution on the enriched pseudofunctor bicategory $\mathsf{PsFun}_{\oplus}(\mathsf{Mot}, \mathrm{Add})$ requires to set a precise definition and prove elementary properties of *enriched bi(co)ends*. This is done in Chapter 2.

Eventually, in Chapter 3, we prove that the cartesian product of gpd induces a monoidal structure on Mot, allowing to use the machinery of the previous chapters and consider a convolution product on the enriched bicategory of enriched pseudofunctors, which provides a natural notion of tensor product of Mackey pseudofunctors. The generality in which the theory of the first two chapters is developed allows to treat the case of enrichment on the monoidal bicategory of additive categories. Moreover, is moved by the prospect of considering variations of the same problem in which Mackey pseudofunctors take values in different and possibly more complicated bicategories than Add. The examples involve additive derivators and stable (additive) derivators ([Gro13]).

Chapter 1

Enriched bicategories

Throughout this chapter we are going to introduce and work with explicit constructions involving enrichment of bicategories. In the classical (1-categorical) setting one can consider the notion of enriched category over a monoidal category, and this generalizes both the notion of category (when the monoidal category is Set with its cartesian structure) and of monoid in a monoidal category (when the enriched category has just one object). Once precise definitions are settled down, the same happens in the context of bicategories, which are the usual weakening of the concept of Cat-category.

We use the same definition of bicategory enriched over a monoidal bicategory which is used in [GS15], on the other hand, the notion of braiding for a monoidal bicategory also had already been introduced in [Mcc00]. However, what the authors of [GS15] are careful not to do, is to consider constructions for enriched bicategories which require a braiding. The main original result of this chapter will be the construction of the opposite enriched bicategory $\mathcal{C}^{\mathsf{op}}$, for $\mathcal{C}$ a bicategory enriched over a braided monoidal bicategory. This fundamental construction turns out to be unexpectedly involved, but once established will allow us, in the next chapter, to talk about ends and coends in the enriched bicategorical context.

1.1 Monoidal bicategories

In this first section we recall the notion of monoidal bicategory. With a notion of tricategory available (*e.g.* at [GPS95]), one could just say that a monoidal bicategory is a tricategory with one object. That is basically what we are going to do, but in a way that will allow us to appreciate how this notion is at the same time a vertical categorification of the notion of monoidal category. The classical commutative diagrams required for a monoidal category (the two Mac Lane's coherence axioms) are in fact replaced by further structure, namely two 2-isomorphisms, which will then satisfy a new requirement.

Let us first recall the notion of bicategory, following the classical reference [Bén67].

1.1.1 Bicategories

Definition 1.1.1. A *bicategory* $\mathcal{B}$ is the data of

- A class of objects $\mathcal{B}_0$

- For every pair of objects a, b a category $\mathcal{B}(a,b)$ together with composition and unit functors

$$\begin{aligned} \circ = \circ_{a,b,c} &\colon \mathcal{B}(b,c) \times \mathcal{B}(a,b) \longrightarrow \mathcal{B}(a,c), \\ u = u_a &\colon 1 \longrightarrow \mathcal{B}(a,a); \end{aligned}$$

  (By 1 we mean the terminal category with one object and one morphism)

- Natural isomorphisms (called *associator* and *left* and *right unitors*)

$$\begin{array}{ccc} \mathcal{B}(c,d) \times \mathcal{B}(b,c) \times \mathcal{B}(a,b) & \xrightarrow{\mathrm{id}\times\circ} & \mathcal{B}(c,d) \times \mathcal{B}(a,c) \\ {\scriptstyle \circ\times\mathrm{id}}\downarrow & \nearrow\alpha & \downarrow{\scriptstyle\circ} \\ \mathcal{B}(b,d) \times \mathcal{B}(a,b) & \xrightarrow[\circ]{} & \mathcal{B}(a,c) \end{array}$$

$$\begin{array}{ccc} 1 \times \mathcal{B}(a,b) & \overset{\cong}{\searrow} & \\ {\scriptstyle u_b\times\mathrm{id}}\downarrow & \nearrow\lambda & \\ \mathcal{B}(b,b) \times \mathcal{B}(a,b) & \xrightarrow[\circ]{} & \mathcal{B}(a,b) \end{array} \qquad \begin{array}{ccc} \mathcal{B}(a,b) \times 1 & \overset{\cong}{\searrow} & \\ {\scriptstyle \mathrm{id}\times u_a}\downarrow & \nearrow\rho & \\ \mathcal{B}(a,b) \times \mathcal{B}(a,a) & \xrightarrow[\circ]{} & \mathcal{B}(a,b) \end{array}$$

such that for every $a \xrightarrow{k} b \xrightarrow{h} c \xrightarrow{g} d \xrightarrow{f} e$ the following two *associativity* and *identity coherence* axioms hold true:

$$\begin{array}{ccccc} & (f(gh))k & \xrightarrow{\alpha} & f((gh)k) & \\ {\scriptstyle\alpha\circ k}\nearrow & & & & \searrow{\scriptstyle f\circ\alpha} \\ ((fg)h)k & & & & f(g(hk)) \\ & {\scriptstyle\alpha}\searrow & & \nearrow{\scriptstyle\alpha} & \\ & & (fg)(hk) & & \end{array} \tag{AC}$$

commutes in $\mathcal{B}(a,e)$.

$$\begin{array}{ccc} & f(\mathrm{id}g) & \\ {\scriptstyle\alpha}\nearrow & & \searrow{\scriptstyle f\circ\lambda} \\ (f\mathrm{id})g & \xrightarrow[\rho\circ g]{} & fg \end{array} \tag{IC}$$

commutes in $\mathcal{B}(c,e)$.

Remark 1.1.2. Observe that each of the natural isomorphisms α, λ and ρ depends each time on the objects of $\mathcal{B}$ involved, but we avoid to make that explicit for the sake of readability. Moreover, there's an unspoken isomorphism of categories

$$(\mathcal{B}(c,d) \times \mathcal{B}(b,c)) \times \mathcal{B}(a,b) \cong \mathcal{B}(c,d) \times (\mathcal{B}(b,c) \times \mathcal{B}(a,b))$$

in the definition of α, so that the latter should be more precisely read

$$\begin{array}{ccccc}
(\mathcal{B}(c,d)\times\mathcal{B}(b,c))\times\mathcal{B}(a,b) & & \xrightarrow{\cong} & & \mathcal{B}(c,d)\times(\mathcal{B}(b,c)\times\mathcal{B}(a,b)) \\
\Big\downarrow{\scriptstyle \circ\times\mathrm{id}} & & & & \Big\downarrow{\scriptstyle \mathrm{id}\times\circ} \\
\mathcal{B}(b,d)\times\mathcal{B}(a,b) & & \nearrow\alpha & & \mathcal{B}(c,d)\times\mathcal{B}(a,c) \\
 & \searrow{\scriptstyle \circ} & & \swarrow{\scriptstyle \circ} & \\
 & & \mathcal{B}(a,c) & &
\end{array}$$

The natural notion of morphism between bicategories is that of *pseudofunctor*, expressing the fact that since there are no strict equalities of morphisms $(fg)h = f(gh)$, $\mathrm{id}f = f$ or $f\mathrm{id} = f$ in a bicategory, but just structural isomorphisms α,λ and ρ, it's usually too much in real examples to require strict preservation of composition and unit.

Definition 1.1.3. A *pseudofunctor* $F\colon \mathcal{B} \to \mathcal{C}$ between bicategories consists of an object Fa of $\mathcal{C}$ for every object a in $\mathcal{B}$, together with functors

$$F = F_{a,b}\colon \mathcal{B}(a,b) \to \mathcal{C}(Fa, Fb),$$

and natural isomorphisms

$$\begin{array}{ccc}
\mathcal{B}(b,c)\times\mathcal{B}(a,b) & \xrightarrow{\circ} & \mathcal{B}(a,c) \\
\Big\downarrow{\scriptstyle F\times F} & \nearrow\mathsf{fun} & \Big\downarrow{\scriptstyle F} \\
\mathcal{C}(Fb,Fc)\times\mathcal{C}(Fa,Fb)) & \xrightarrow{\circ} & \mathcal{C}(Fa,Fc)
\end{array}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{u_a} & \mathcal{B}(a,a) \\
 & \searrow{\scriptstyle u_{Fa}} \;\; \nearrow\mathsf{un} & \Big\downarrow{\scriptstyle F} \\
 & & \mathcal{C}(Fa,Fa)
\end{array}$$

expressing pseudonaturality and satisfying, for every $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, the following axioms, namely the commutativity of the following diagrams in the respective hom-categories:

$$\begin{array}{ccccc}
(Fh\circ Fg)\circ Ff & \xrightarrow{\mathsf{fun}\circ Ff} & F(hg)\circ Ff & \xrightarrow{\mathsf{fun}} & F((hg)f) \\
\Big\downarrow{\scriptstyle \alpha} & & & & \Big\downarrow{\scriptstyle F(\alpha)} \\
Fh\circ(Fg\circ Ff) & \xrightarrow[Fh\circ\mathsf{fun}]{} & Fh\circ F(gf) & \xrightarrow[\mathsf{fun}]{} & F(h(gf))
\end{array} \tag{PF1}$$

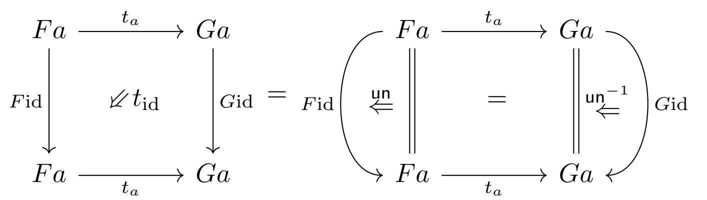
$$\begin{array}{ccc}
\mathrm{id}_{Fb}\circ Ff & \xrightarrow{\mathsf{un}\circ Ff} & F(\mathrm{id}_b)\circ Ff \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \mathsf{fun}} \\
F(f) & \xleftarrow[F(\lambda)]{} & F(\mathrm{id}_b\circ f)
\end{array}
\qquad
\begin{array}{ccc}
Ff\circ\mathrm{id}_{Fa} & \xrightarrow{Ff\circ\mathsf{un}} & Ff\circ F(\mathrm{id}_a) \\
\Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \mathsf{fun}} \\
F(f) & \xleftarrow[F(\rho)]{} & F(f\circ\mathrm{id}_a)
\end{array} \tag{PF2}$$

Remark 1.1.4. Pseudofunctors between two bicategories $\mathcal{B}$ and $\mathcal{C}$ assemble into a bicategory themselves, denoted $\mathsf{PsFun}(\mathcal{B},\mathcal{C})$. A 1-morphism of pseudofunctors $F, G\colon \mathcal{B} \to \mathcal{C}$ is a *pseudonatural transformation.* A 2-morphism is a *modification* (definitions 1.1.5 and 1.1.6 below).

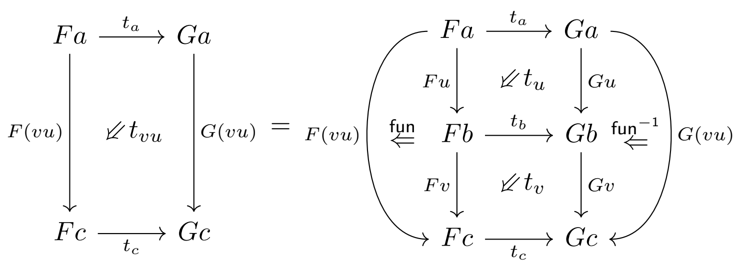
Definition 1.1.5. A *pseudonatural transformation* $t\colon F \Rightarrow G$, consists of a family of 1-morphisms $t_a\colon Fa \to Ga$ and, for every $u\colon a \to b$, an invertible 2-morphism

$$\begin{array}{ccc}
Fa & \xrightarrow{t_a} & Ga \\
\Big\downarrow{\scriptstyle Fu} & \swarrow t_u & \Big\downarrow{\scriptstyle Gu} \\
Fb & \xrightarrow[t_b]{} & Gb
\end{array}$$

such that the following axioms (*unitality*, *functoriality* and *naturality*) are satisfied:

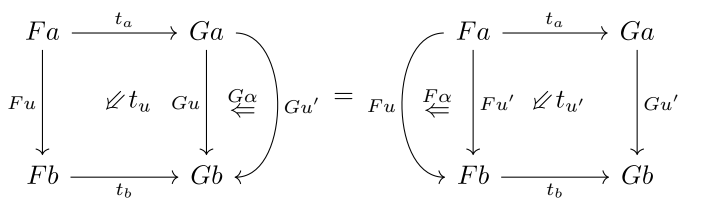


(PTU)



(PTF)

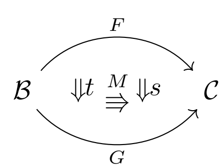
for every composable pair of arrows $a \xrightarrow{u} b \xrightarrow{v} c$, and



(PTN)

for every 2-cell $\alpha\colon u' \Rightarrow u$.

Definition 1.1.6. A *modification* $M\colon t \Rrightarrow s$ between pseudonatural transformations of pseudofunctors $F, G\colon \mathcal{B} \to \mathcal{C}$



consists of a 2-morphism $M_a\colon t_a \Rightarrow s_a$ of $\mathcal{C}$ for every object a of $\mathcal{B}$, such that for every $u\colon a \to b$ the square

$$\begin{array}{ccc} (Gu)t_a & \xrightarrow{\mathrm{id}\circ M_a} & (Gu)s_a \\ {\scriptstyle t_u}\downarrow & & \downarrow{\scriptstyle s_u} \\ t_b(Fu) & \xrightarrow[M_b\circ\mathrm{id}]{} & s_b(Fu) \end{array} \tag{1.1}$$

commutes in the hom category $\mathcal{C}(Fa, Gb)$.

1.1.2 Monoidal structure

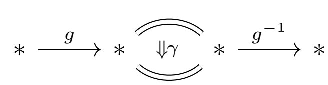
At this point we want to define for a bicategory the appropriate concept of monoidal structure. Part of this structure is just as the usual structure of monoidal category and consists of three families of adjoint equivalences, called a, ℓ, r below, which can be seen at the same time as a generalization of

- The equivalences of categories
$$(\mathcal{C} \times \mathcal{D}) \times \mathcal{E} \simeq \mathcal{C} \times (\mathcal{D} \times \mathcal{E}),$$
$$1 \times \mathcal{C} \simeq \mathcal{C} \simeq \mathcal{C} \times 1.$$
to the case where we no longer consider necessarily the cartesian product of categories but any other $\otimes \colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ on a bicategory which then may not be Cat.

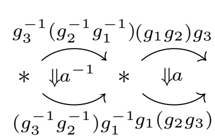
- The structure α, λ, ρ in the definition of bicategory, which include the notion of monoidal category as its one-object case.

What properly concerns the higher structure is nonetheless the fact that coherences required for a usual monoidal category (*i.e.* bicategory with one object), namely the commutativity of the pentagon and of the triangle diagrams in axioms (AC) and (IC), are replaced by further structure, two 2-isomorphisms π and μ fitting those diagrams and satisfying some axioms.

Remark 1.1.7. One of the axiom satisfied by the structure of a monoidal bicategory sheds light on a deep connection between higher categories and cohomology theory, which is known better in its 1-dimensional case. A well known part of the theory of 2-groups, as explained *e.g.* in [BS07], is a classification result. A 2-group $\mathcal{G}$ is given by a bicategory with one object $*$, in which 1-cells are weakly invertible and 2-cells are invertible. To such a $\mathcal{G}$ one can associate a group G of isomorphism classes of 1-cells under the composition and an abelian group A of endomorphism of the identity id_*. Moreover, since π_1 acts on π_2, an action of the first one on the latter, by letting $g \cdot \gamma$ be the whiskering



Eventually, one can associate to this data a function $\alpha \colon G^3 \to A$, given on a triple (g_1, g_2, g_3) by the associator (also called α, as usual)



and this function being a 3-cocycle relies on the coherence axiom for the associator. The classification result states then that the equivalence classes of 2-groups are determined by tuples (G, A, ρ, α) where G is a group, A a G-module, ρ an action of G on A and $\alpha \colon G^3 \to A$ *a 3-cocycle*. The coherence constraint for the associator comes then from the condition of a function $G^3 \to A$ to be a 3-cocycle (historically, at least, but one could say vice versa, depending on the point of view). Then, the correspondence will work similarly one level up, so that the notion of 4-cocycle will determine the constraint for the pentagonator π.

Definition 1.1.8. Let $\mathcal{B}$ be a bicategory. A *monoidal structure* on $\mathcal{B}$ is provided by a tuple $(\mathcal{B}, \otimes, \mathbb{1}, a, \ell, r, \pi, \mu)$, where

- $\otimes \colon \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $\mathbb{1} \colon 1 \to \mathcal{B}$ are pseudofunctors called respectively tensor and identity. We allow us to denote by $\mathbb{1}$ also the object $\mathbb{1}(*)$ in $\mathcal{B}$, the image of the only object in the terminal bicategory 1.

- a, ℓ, r, the *(monoidal) associator, left and right unitors*, are adjoint equivalences in the bicategories $\mathsf{PsFun}(\mathcal{B}^3, \mathcal{B})$ and $\mathsf{PsFun}(\mathcal{B}, \mathcal{B})$

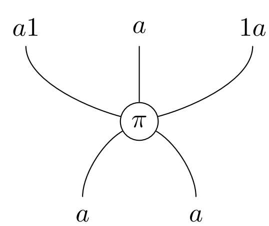
$$\begin{array}{ccc} \mathcal{B} \times \mathcal{B} \times \mathcal{B} & \xrightarrow{\mathrm{id} \times \otimes} & \mathcal{B} \times \mathcal{B} \\ {\scriptstyle \otimes \times \mathrm{id}} \downarrow & \nearrow a & \downarrow {\scriptstyle \otimes} \\ \mathcal{B} \times \mathcal{B} & \xrightarrow[\otimes]{} & \mathcal{B} \end{array}$$

$$\begin{array}{ccc} \mathcal{B} & = & \\ {\scriptstyle \mathbb{1} \times \mathrm{id}} \downarrow & \nearrow \ell & \| \\ \mathcal{B} \times \mathcal{B} & \xrightarrow[\otimes]{} & \mathcal{B} \end{array} \qquad \begin{array}{ccc} \mathcal{B} & = & \\ {\scriptstyle \mathrm{id} \times \mathbb{1}} \downarrow & \nearrow r & \| \\ \mathcal{B} \times \mathcal{B} & \xrightarrow[\otimes]{} & \mathcal{B} \end{array}$$

- π, called *2-associator*, or *pentagonator*, is an invertible modification (2-cell in $\mathsf{PsFun}(\mathcal{B}^4, \mathcal{B})$) with components

$$\begin{array}{ccccc} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{A,B \otimes C,D}} & A \otimes ((B \otimes C) \otimes D) & \\ {\scriptstyle a_{A,B,C} \otimes D} \nearrow & & & & \searrow {\scriptstyle A \otimes a_{B,C,D}} \\ ((A \otimes B) \otimes C) \otimes D & & \Downarrow \pi_{A,B,C,D} & & A \otimes (B \otimes (C \otimes D)) \\ {\scriptstyle a_{A \otimes B,C,D}} \searrow & & & & \nearrow {\scriptstyle a_{A,B,C \otimes D}} \\ & & (A \otimes B) \otimes (C \otimes D) & & \end{array}$$

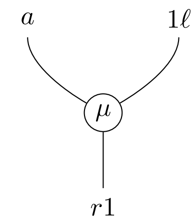
depicted as



- μ, called *2-unitor*, an invertible modification (2-cell in $\mathsf{PsFun}(\mathcal{B}^2, \mathcal{B})$) whose components are

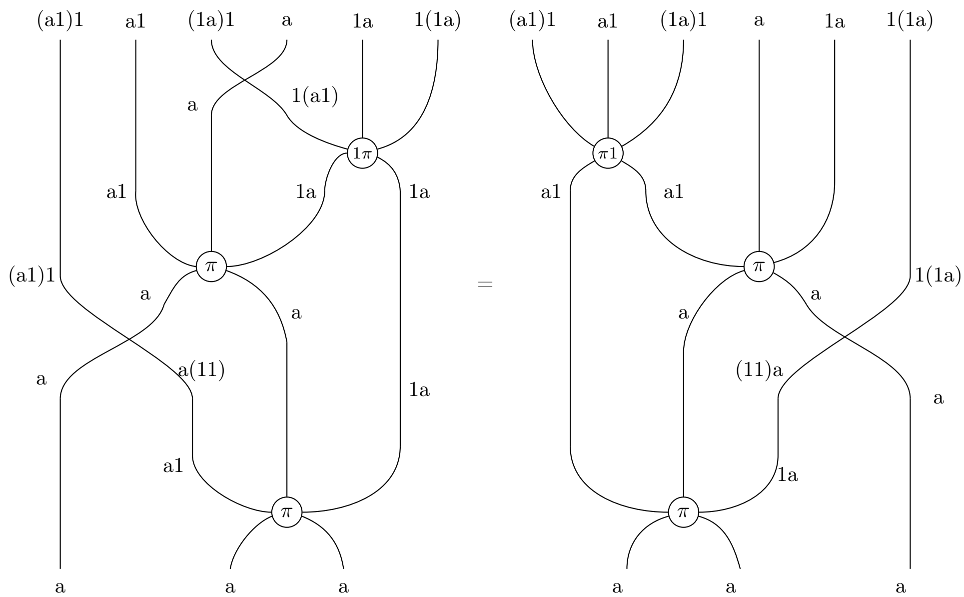
$$\begin{array}{ccccc} & & A \otimes (\mathbb{1} \otimes B) & & \\ & {\scriptstyle a} \nearrow & \Downarrow \mu_{A,B} & \searrow {\scriptstyle A \otimes l_B} & \\ (A \otimes \mathbb{1}) \otimes B & & \xrightarrow[r_A \otimes B]{} & & A \otimes B \end{array}$$

depicted as



These data are subject to the following axioms of Remarks 1.1.10 and Remark **??** below.

Remark 1.1.9. A first axiom to which the data of monoidal bicategory is subject to is the following coherence, the *non-abelian 4-cocycle condition* (see Remark 1.1.7) and is a string diagrammatic version of the one object case in the definition of a tricategory [GPS95]:

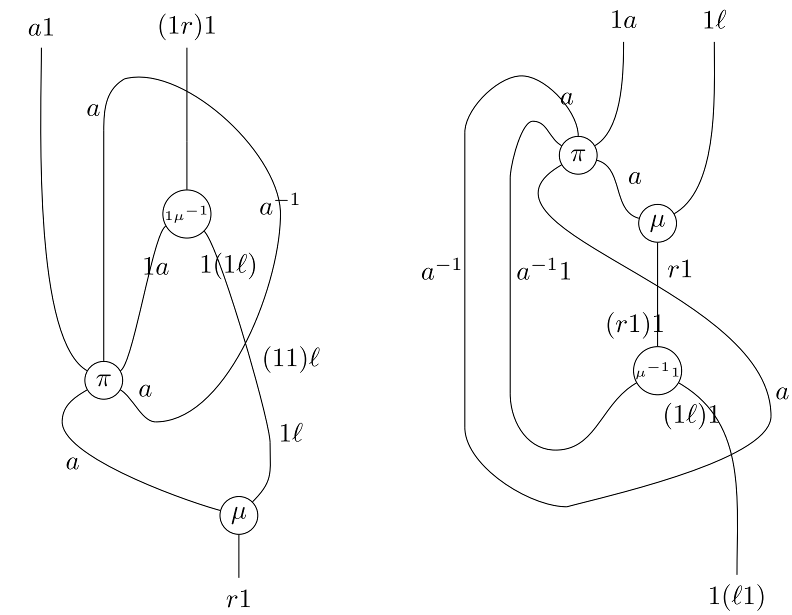


Then, other two axioms are required for the structure of a monoidal bicategory.

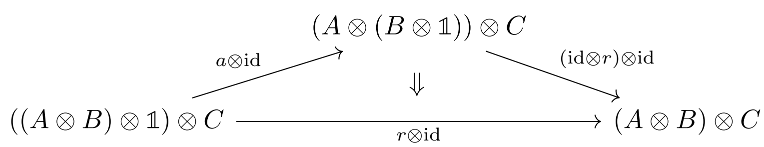
Remark 1.1.10. It is customary (as in [GPS95]) to give the definition of monoidal bicategory considering some further useful structure. Precisely, two more 2-unitors δ and γ are usually given (the right one and the left one), going as

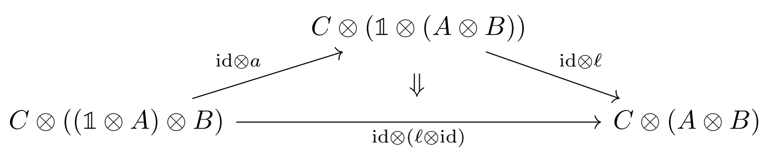
$$\begin{array}{ccc} & \mathbb{1}\otimes(A\otimes B) & \\ {}^{a}\nearrow & \Downarrow\gamma & \searrow^{\ell} \\ (\mathbb{1}\otimes A)\otimes B & \xrightarrow[\ell\otimes B]{} & A\otimes B \end{array} \qquad \begin{array}{ccc} & A\otimes(B\otimes\mathbb{1}) & \\ {}^{a}\nearrow & \Downarrow\delta & \searrow^{A\otimes r} \\ (A\otimes B)\otimes\mathbb{1} & \xrightarrow[r]{} & A\otimes B \end{array}$$

and two more axioms are required. This structure can, though, be defined starting from π and μ as follows. Let us start from the 2-cells

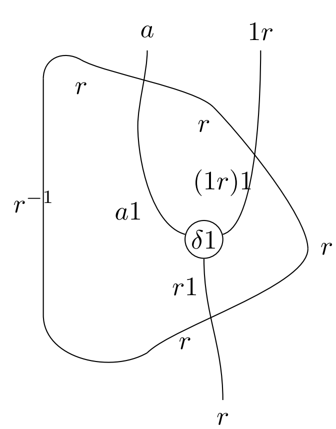


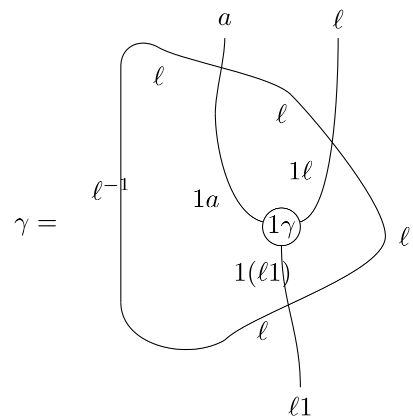
(which are going to be $\delta 1$ and 1γ) and go as



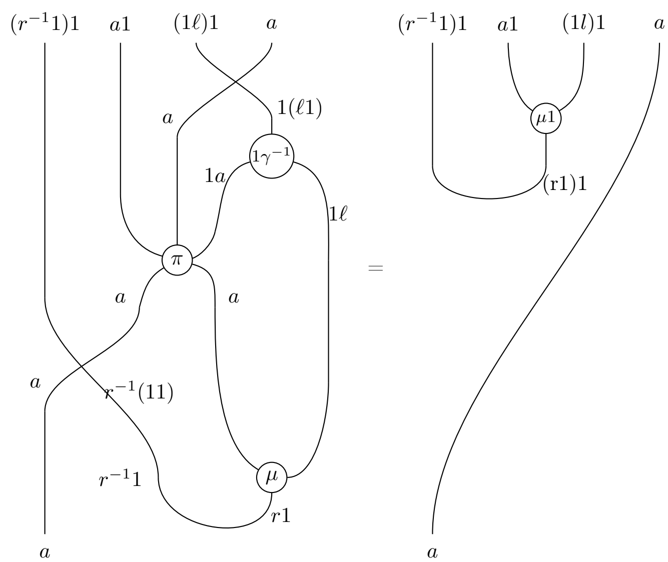


and further compose, once we fix $C = \mathbb{1}$, defining:

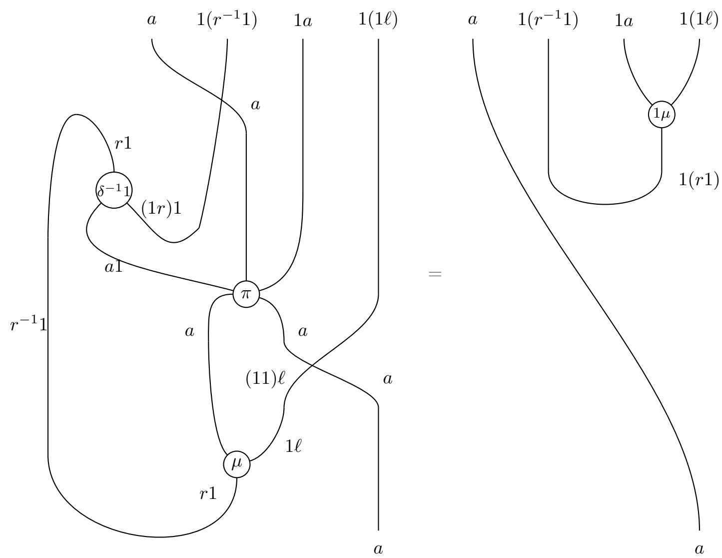
$\delta =$ 



Now, the so called *left and right normalization axioms*, state the following equalities of 2-cells



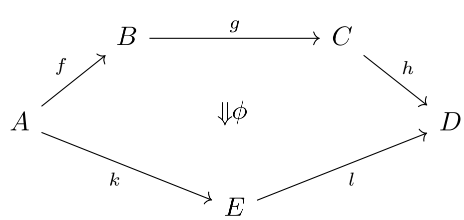
and



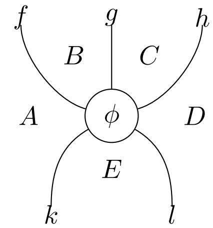
1.2 String diagrams

We assemble in this section our conventions for the use of string diagrams. The fundamental results allowing the use of this notation and of its rules are to be found in [JS91], where Joyal and Street formalized for the first time the theory of string diagrams for monoidal categories.

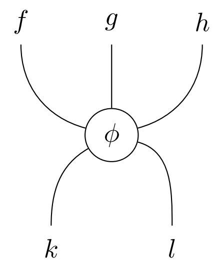
Construction. The following notation assumes that we work in a 2-category, but it can be used for bicategories as justified by the Strictification Theorem for bicategories. Given a 2-cell α, its string diagrammatic representation is constructed via its Poincaré dual. More precisely, we use as convention the following directions: we write and read the 1-cells from left to right, while the direction of 2-cells is from top to bottom. For example, a 2-cell



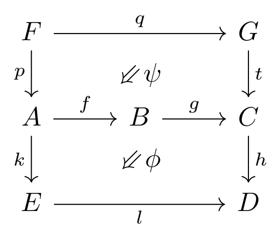
will be represented by placing a labeled dot, or disk, at the center of the 2-cell, while 1-morphisms will be represented by wires delimiting areas associated to each object. Objects, and that is one of among the advantages of this formalism, will usually be implicit. That means ϕ will be represented as



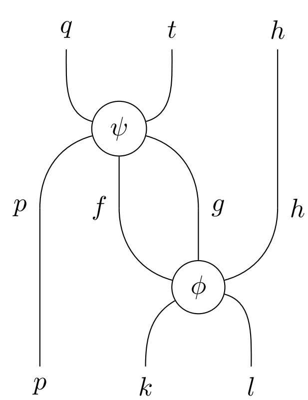
or, as usual, just as



This construction works if, as we said, we orient the direction of 1-cell from left to right and of 2-cell from top to bottom. Please observe that the string diagrammatic representation of a 2-cell will, nonetheless be the same regardless the different orientations in which one can represent the latter. By that, we intend that wires will always be oriented appropriately, following the orientation just explained above. On top and at the bottom there will be the sequences of composed morphisms constituting respectively the domain and the codomain of the 2-cell. The pasting composition of two such cells, for example

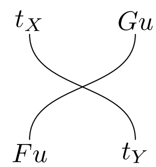


is then represented as a “vertical” composition



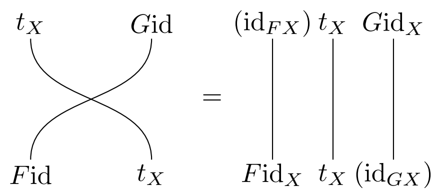
There are some rules for general bicategories. Some of them are just the translation into this formalism of general structure (2-cells) and axioms (equalities of 2-cells) for any pseudonatural transformation or modification.

Remark 1.2.1. The structure and the axioms for a pseudonatural transformation $t\colon F \Rightarrow G$ translate in string diagrams as the data of a 2-cell

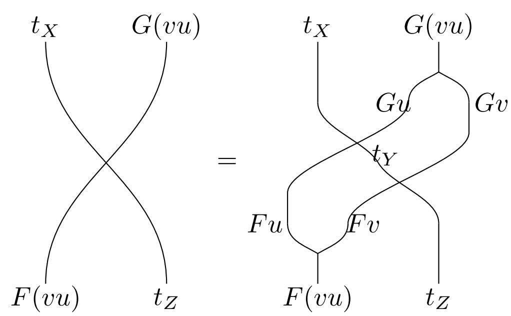


for every $u\colon X \to Y$. The axioms that this structure satisfy are then

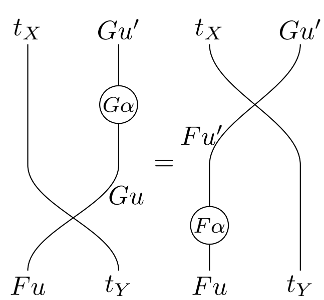
(PTU) For every object X



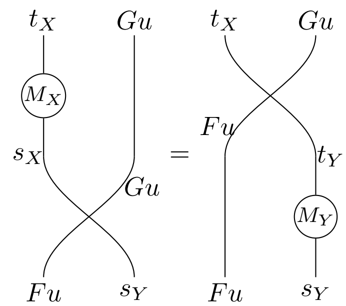
(PTF) For every composable $X \xrightarrow{u} Y \xrightarrow{v} Z$



(PTN) For every $\alpha\colon u' \to u$



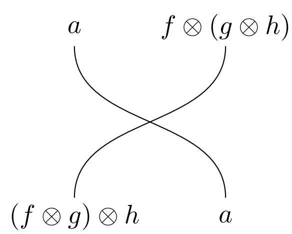
For what concerns modifications, let's say $M_X\colon t_X \Rightarrow s_X$, the axiom translate directly from 1.1 as the identity for every $u\colon X \to Y$.



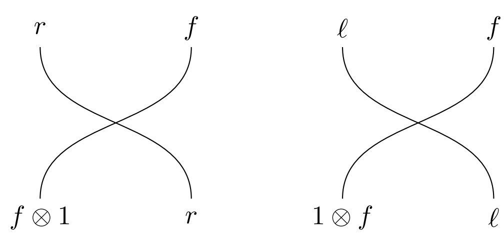
1.2.1 Calculus rules for monoidal bicategories

We write this section in order to explain some useful rules arising with the interplay of the monoidal structure and the string diagrammatic representation.

Remark 1.2.2. Let $\mathcal{B}$ be a monoidal bicategory. Let us consider the rules concerning the monoidal associator and unitors. Whenever $(f, g, h): (A, B, C) \to (A', B', C')$ is an arrow in $\mathcal{B}^3$, one can consider the two tensored morphisms $f \otimes (g \otimes h)$ and $(f \otimes g) \otimes h$. Since a is a pseudonatural transformation, it comes with an invertible 2-cell $a_{f,g,h}$ in $\mathcal{B}$ depicted as

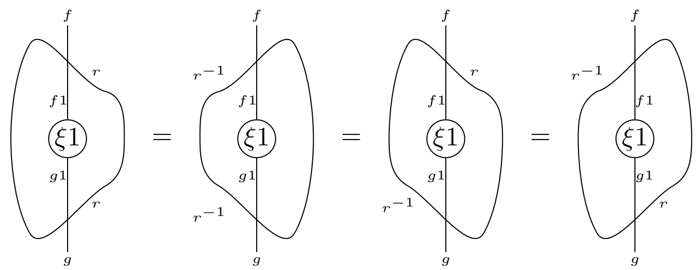


Therefore, by taking this isomorphism, or its inverse, we get a string diagrammatic rule stating that we can shift parenthesis in tensored triples of morphisms by switching its composition with either a or a^{-1}. Crossing with the left and right unitors give, for the same reason, 2-cells

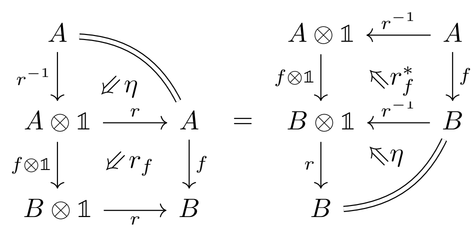


Also, from now on, the tensored morphisms in string diagrams will just be denoted by juxtaposition.

Remark 1.2.3. The argument in Definition 1.1.10 used to define the left and right monoidal unitors γ and δ suggests another rule saying that we can "loop a right (or left) unitor" around a 2-cell whenever every 1-morphism in this cell happens to be the identity of $\mathbb{1}$ in every exterior last (or first) component. This gives a new 2-cell that no longer involves the identity of the tensor unit. Moreover, different ways to loop this r around a 2-cell, such as



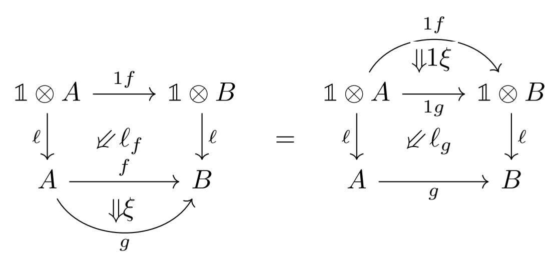
provide the same morphism. This is a consequence of the equality, for every $f\colon A \to B$, of the following 2-cells



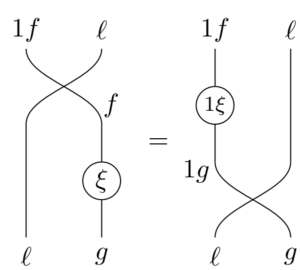
due to the definition of the mate r_f^* and the triangular identity for the adjoint equivalence defining r.

The following remark basically says the converse of the previous one also holds true.

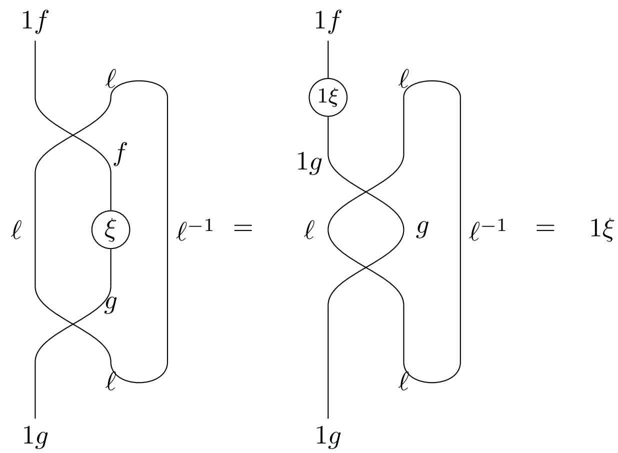
Remark 1.2.4. The operation of looping an r (respectively ℓ) around a 2-cell ξ gives precisely $\xi 1$ (respectively 1ξ). This equality directly comes from the naturality of ℓ as a pseudonatural transformation (axiom (PTN)) saying that



which expresses as string diagram as

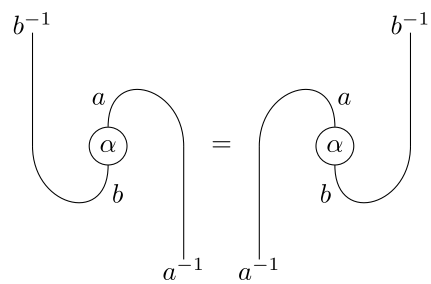


Thus, it's clear that

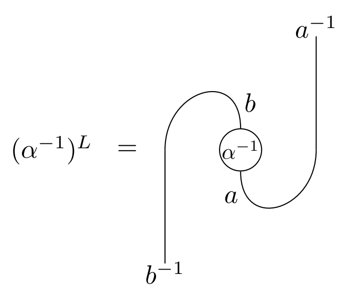


Another useful observation is the following one.

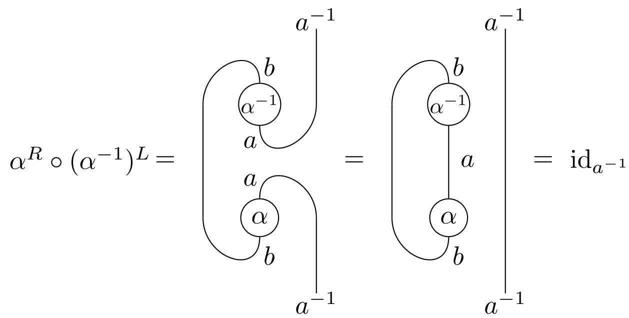
Lemma 1.2.5. *Let a, b be two adjoint equivalences in a bicategory, and suppose them to be isomorphic via $\alpha\colon a \Rightarrow b$. Then, the two right and left mates α^R and α^L of α coincide:*

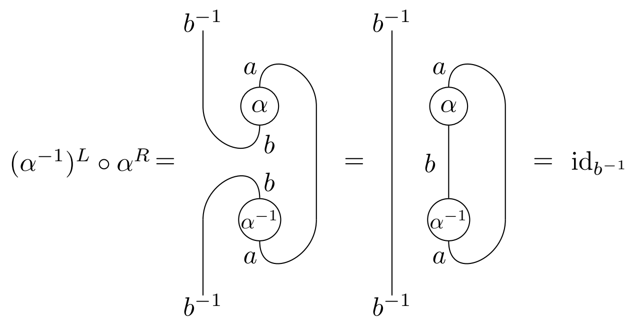


Proof. It suffices to see that both are inverses of the same morphism, which is any of the mates of α^{-1}, let's say the left one (the one in which 1-cells are bent to the left):



One has on one hand that this is inverse to α^R via





On the other hand, the fact that $(\alpha^{-1})^L$ is also inverse to α^L is a plain application of the triangular identities. □

1.3 Enrichments

The idea is to enrich a bicategory over a monoidal bicategory $\mathcal{V}$, giving rise to the notion of $\mathcal{V}$-bicategory $\mathcal{C}$. This is going to be a parallel generalization of both the notion of bicategory (for the case $\mathcal{V} = \mathsf{Cat}$, with its usual cartesian monoidal structure) and of pseudomonoid in $\mathcal{V}$ (when the enriched bicategory has just one object). Just as for the 1-categorical version of the story, an enriched bicategory is *not* a bicategory with some property, *nor* a bicategory with some further structure.

1.3.1 Pseudomonoids

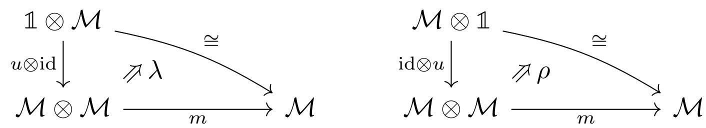
Definition 1.3.1. Let $\mathcal{V}$ be a monoidal bicategory. A *pseudomonoid* $\mathcal{M}$ in $\mathcal{V}$ is the data of

- An object $\mathcal{M}$ of $\mathcal{V}$
- 1-morphisms of $\mathcal{V}$ called multiplication and unit

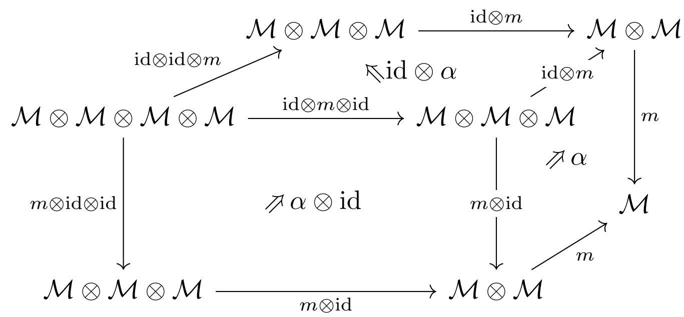
$$\begin{aligned} m &\colon \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M} \\ u &\colon \mathbb{1} \longrightarrow \mathcal{M} \end{aligned}$$

- Invertible 2-morphisms of $\mathcal{V}$

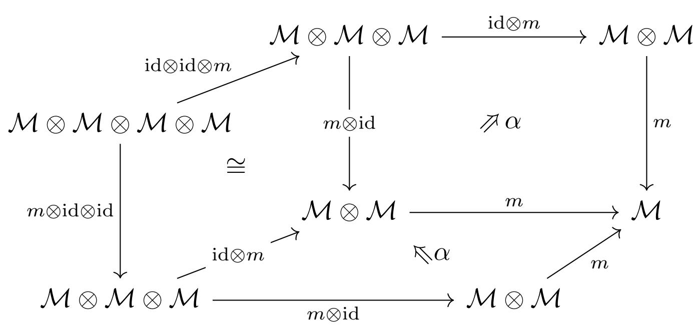
$$\begin{array}{ccc} \mathcal{M} \otimes \mathcal{M} \otimes \mathcal{M} & \xrightarrow{\mathrm{id} \times m} & \mathcal{M} \otimes \mathcal{M} \\ {\scriptstyle m \times \mathrm{id}}\downarrow & \nearrow \alpha & \downarrow{\scriptstyle m} \\ \mathcal{M} \otimes \mathcal{M} & \xrightarrow[m]{} & \mathcal{M} \end{array}$$



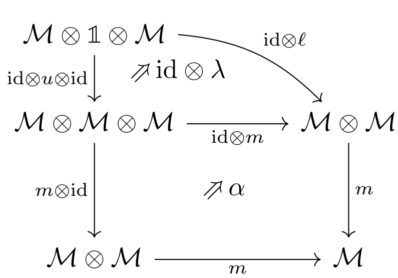
subject to MacLane coherence axioms, that express as equality of pasting diagrams as follows:



(AC)
=



(Observe that the clockwise side of the pentagon for the usual associativity coherence correspond to the first cell, while the counterclockwise to the second one. Also, we require



(IC)
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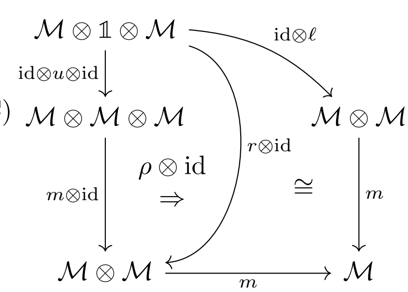
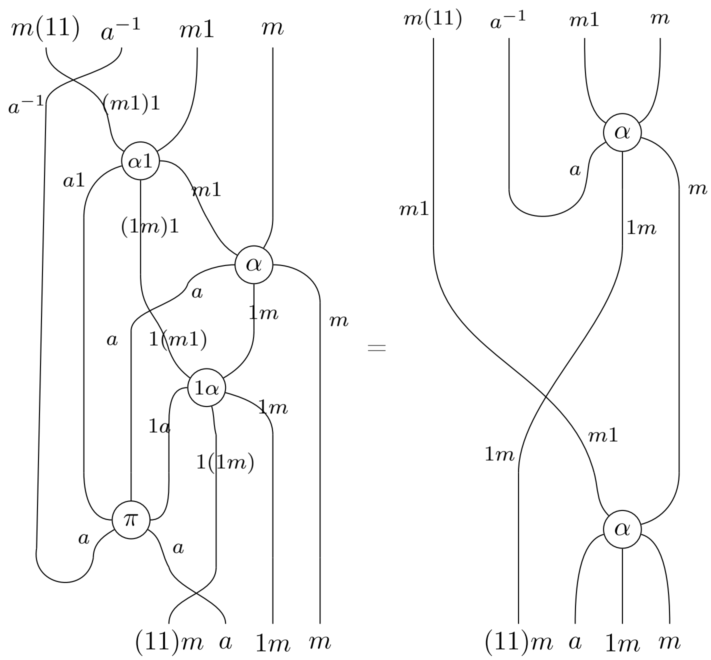
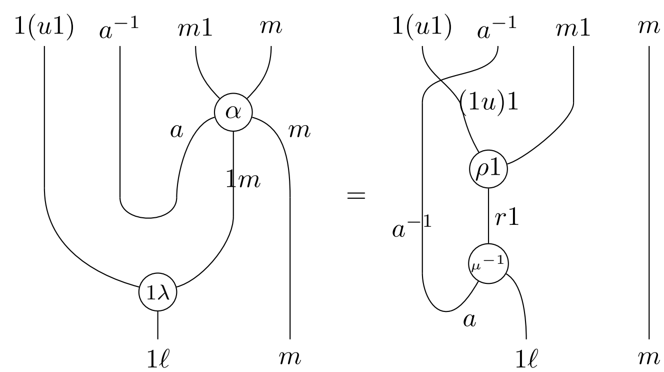


where the first cell corresponds to the clockwise composition of the triangle in the usual identity coherence, and the second one to the counterclockwise.

Remark 1.3.2. It is convenient to translate these coherence axioms as string diagrams. This will make explicit use of every isomorphism from the monoidal structure of the bicategory $\mathcal{V}$, which are usually hidden in the classical diagrammatic version stated above. The axiom (AC) then becomes



while the identity coherence (IC) becomes



1.3.2 $\mathcal{V}$-bicategories

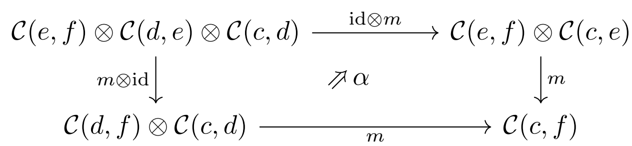
We are then ready to give the definition of a $\mathcal{V}$-bicategory appreciating how the previous concepts of pseudomonoid as well as of bicategory fit into it.

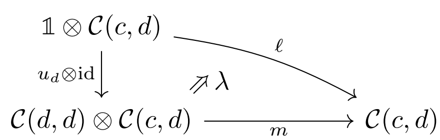
Definition 1.3.3 ([GS15] Definition 3.1)**.** Let $\mathcal{V}$ be a monoidal bicategory. A $\mathcal{V}$*-bicategory* $\mathcal{C}$ is the data of

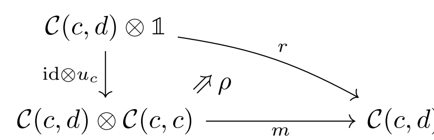
- A class of objects $\mathcal{C}_0$
- For every pair of objects c, d in $\mathcal{C}_0$ an object $\mathcal{C}(c, d)$ in $\mathcal{V}$, called *hom-objects*, together with composition and unit 1-morphisms, for every tiple of objects c, d, e

$$m\colon \mathcal{C}(d, e) \otimes \mathcal{C}(c, d) \longrightarrow \mathcal{C}(c, e)$$
$$u_c\colon \mathbb{1} \longrightarrow \mathcal{C}(c, c)$$

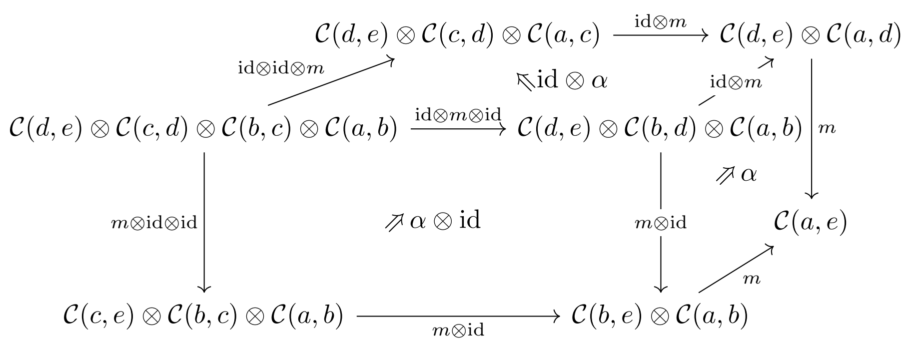
- 2-isomorphisms in $\mathcal{V}$ (called associator and left and right unitors)







subject to the enriched version of coherence axioms (AC) and (IC) below, expressed as identities of 2-cells



(AC)
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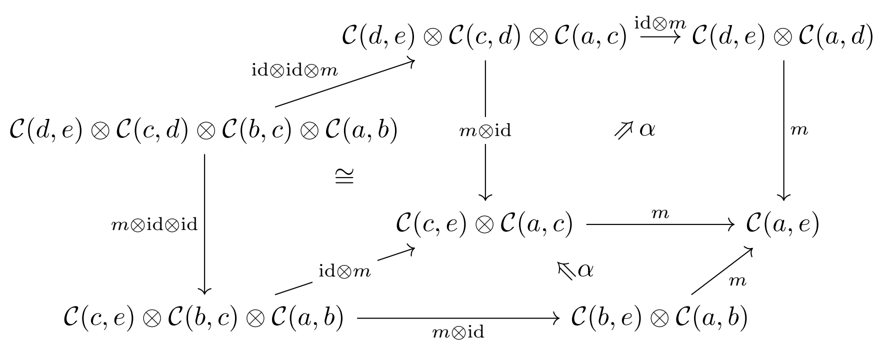

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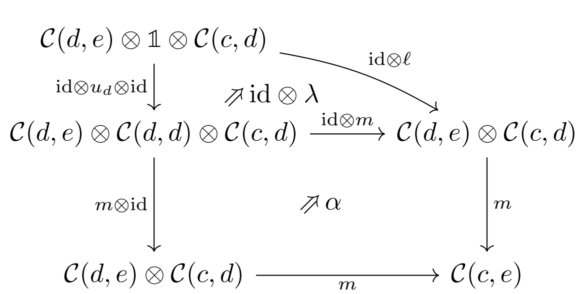

(IC)
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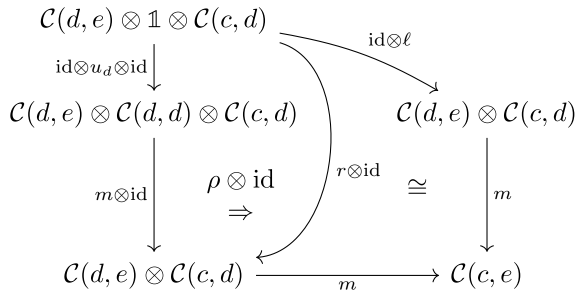


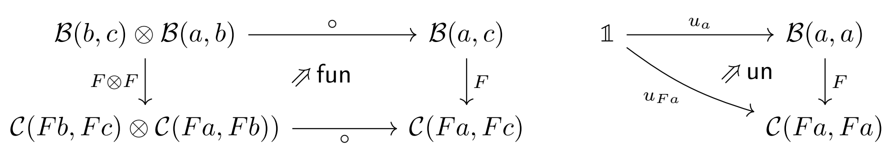
Remark 1.3.4. We wrote in this classical diagrammatic way the axioms (AC) and (IC) for a $\mathcal{V}$-bicategory for sake of clarity, but please note how one could be clearer and be more precise (showing associators) by considering, instead, the string diagrammatic version of the coherence axioms for pseudomonoid of Remark 1.3.2.

Example 1.3.5. There is a *unit $\mathcal{V}$-bicategory* $\mathcal{J}$ consisting of one object $*$, and hom-object $\mathcal{J}(*,*)=\mathbb{1}$. This is precisely the pseudomonoid structure of the tensor unit, provided by $u_*=\mathrm{id}\colon\mathbb{1}\to\mathcal{J}(*,*)$ and composition $m\colon\mathbb{1}\otimes\mathbb{1}\to\mathbb{1}$ equivalently give by the left or the right monoidal unitor for the monoidal unit.

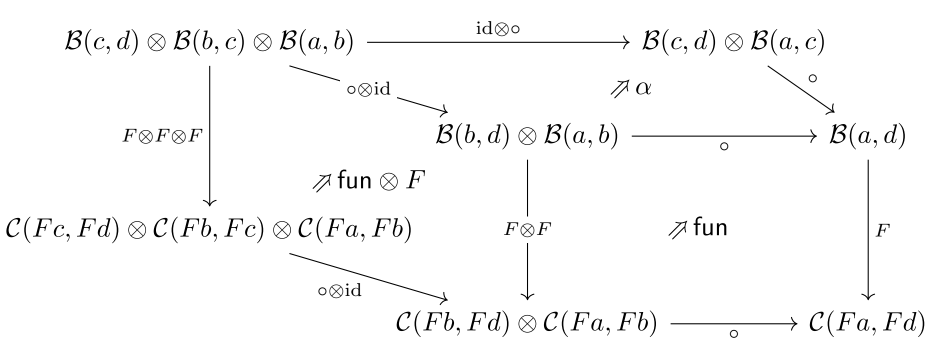
One should define at this point also a notion of morphism of $\mathcal{V}$-bicategories, by generalizing the notion of pseudofunctor.

1.3.3 $\mathcal{V}$-pseudofunctors

Definition 1.3.6. Let $\mathcal{V}$ be a monoidal bicategory. A *$\mathcal{V}$-pseudofunctor* $F\colon\mathcal{B}\to\mathcal{C}$ between $\mathcal{V}$-bicategories is the data of a map on objects $F_0\colon\mathcal{B}_0\to\mathcal{C}_0$ together with, for every pair of objects a,b in $\mathcal{B}$, 1-morphisms of $\mathcal{V}$ still denoted by $F=F_{a,b}\colon\mathcal{B}(a,b)\to\mathcal{C}(Fa,Fb)$ and 2-isomorphisms



subject to the axioms analogue to (PF1) and (PF2) in the definition of pseudofunctor, which express as equality of pasting diagrams:



(PF1)
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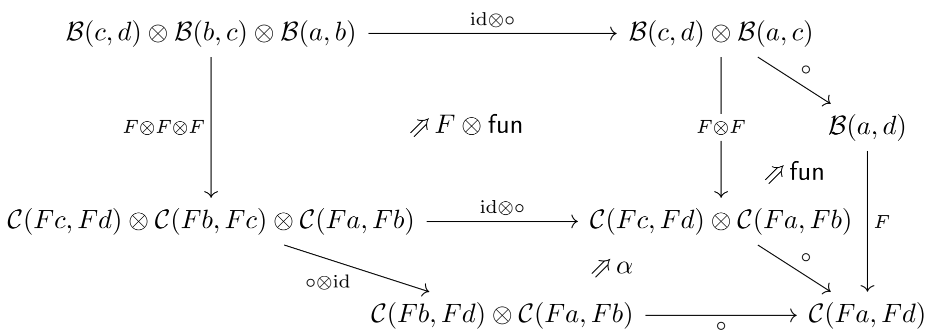
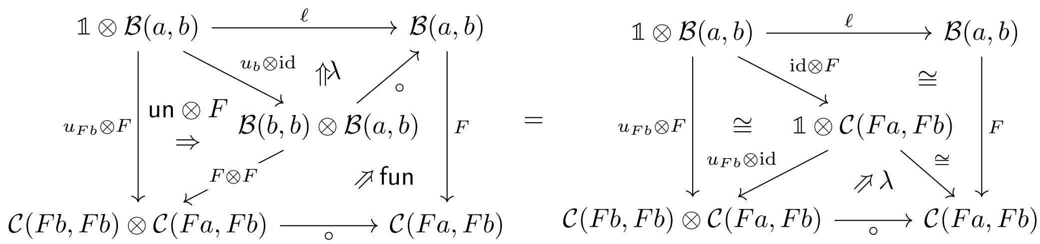
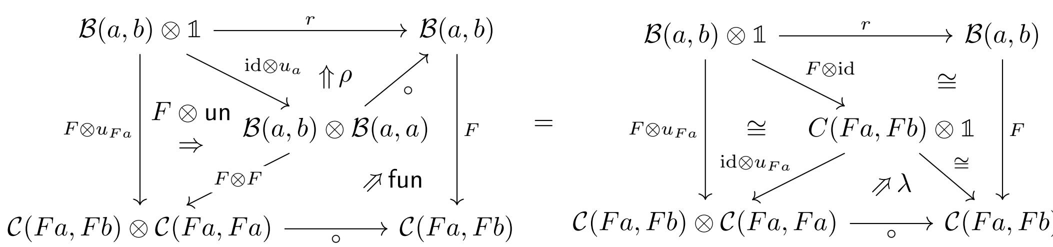


(PF1)

and



(PF2)



We invite the reader to look at [GS15] for a string diagrammatic version.

Remark 1.3.7. The point in translating notions to the enriched context is that the hom-objects $\mathcal{B}(a, b)$ are no longer categories, but objects in $\mathcal{V}$, and hence we need to reformulate axioms without referring to its objects. This basic obstacle, which obviously is fundamentally linked to enrichments even in the 1-categorical setting, can be dealt with by using the following bicategorical vocabulary. By a *1-morphism* $f : c \to d$ in a $\mathcal{V}$-bicategory $\mathcal{C}$ we mean a 1-morphism in $\mathcal{V}$ of the form

$$f : \mathbb{1} \to \mathcal{C}(c, d).$$

This perfectly consolidated trick in enriched category theory (see [Kel05]) carries a translation of the notion of composition: given $f : c \to d$ and $g : d \to e$, we have then a composition defining gf:

$$\mathbb{1} \xrightarrow{\simeq} \mathbb{1} \otimes \mathbb{1} \xrightarrow{g \otimes f} \mathcal{C}(d, e) \otimes \mathcal{C}(c, d) \xrightarrow{\circ} \mathcal{C}(c, e).$$

Here, the isomorphism $\mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$ is $r_{\mathbb{1}}^{-1} \cong \ell_{\mathbb{1}}^{-1}$ (see Lemma 2.1 [GS15]). That means, the composition is only defined up to an isomorphism. We can moreover talk about *2-morphisms* between 1-morphisms in $\mathcal{C}$, and they're just defined to be the 2-morphisms in $\mathcal{V}$ between them. They obviously compose vertically, while horizontally, for

$$c \underset{f'}{\overset{f}{\Downarrow\alpha}} d \underset{g'}{\overset{g}{\Downarrow\beta}} e$$

a pair of 2-morphisms in $\mathcal{C}$, we get $\beta * \alpha$ as the whiskered 2-cell

$$\mathbb{1} \simeq \mathbb{1} \otimes \mathbb{1} \underset{g' \otimes f'}{\overset{g \otimes f}{\overset{\beta\otimes\alpha}{\Downarrow}}} \mathcal{C}(d,e) \otimes \mathcal{C}(c,d) \xrightarrow{\ \circ\ } \mathcal{C}(c,e).$$

1.3.4 $\mathcal{V}$-pseudonatural transformations

The dictionary between enriched and classical morphisms won't allows us to define *verbatim* a *$\mathcal{V}$-pseudonatural transformation* as a usual pseudonatural transformation. The reason is explained in the following example, and consists of a problem shared with the usual 1-categorical version of enriched category theory.

Example 1.3.8. Let us call a *naive* pseudonatural transformation between two enriched pseudofunctors $F, G \colon \mathcal{B} \to \mathcal{C}$ the data of $t_B \colon \mathbb{1} \to \mathcal{C}(FB, GB)$ and, for every $u \colon \mathbb{1} \to \mathcal{B}(A, B)$ an invertible 2-morphism

$$\begin{array}{ccccc} & & \mathcal{C}(GA,GB) \otimes \mathcal{C}(FA,GA) & & \\ & \overset{Gu \otimes t_A}{\nearrow} & & \overset{\circ}{\searrow} & \\ \mathbb{1} \cong \mathbb{1} \otimes \mathbb{1} & & \Downarrow t_u & & \mathcal{C}(FA,GB) \\ & \underset{t_B \otimes Fu}{\searrow} & & \underset{\circ}{\nearrow} & \\ & & \mathcal{C}(FB,GB) \otimes \mathcal{C}(FB,FB) & & \end{array}$$

such that axioms (PTN), (PTF) and (PTU) of Definition 1.1.5 hold true.

Consider then the cartesian monoidal bicategory Cat $\times$ Cat, as well as the Cat $\times$ Cat-bicategory $\mathcal{C}$ with two objects A, B and the only non-trivial hom-objects $\mathcal{B}(A,B) = (T, \varnothing)$, while $\mathcal{B}(X,X) = \mathbb{1} = (T,T)$ for $X = A, B$, with T being the terminal category. Then, consider the Cat $\times$ Cat-pseudofunctor

$$F \colon \mathcal{B} \longrightarrow \text{Cat} \times \text{Cat}$$

mapping the two objects onto a fixed category $\mathcal{C}$, and defined on hom-objects as

$$u_{\mathcal{C},\mathcal{C}} \colon \mathcal{B}(X,X) = \mathbb{1} \longrightarrow \text{Cat} \times \text{Cat}((\mathcal{C},\mathcal{C}),(\mathcal{C},\mathcal{C}))$$
$$(u_{\mathcal{C}}, \varnothing) \colon \mathcal{B}(A,B) = (T,\varnothing) \longrightarrow \text{Cat} \times \text{Cat}((\mathcal{C},\mathcal{C}),(\mathcal{C},\mathcal{C}))$$

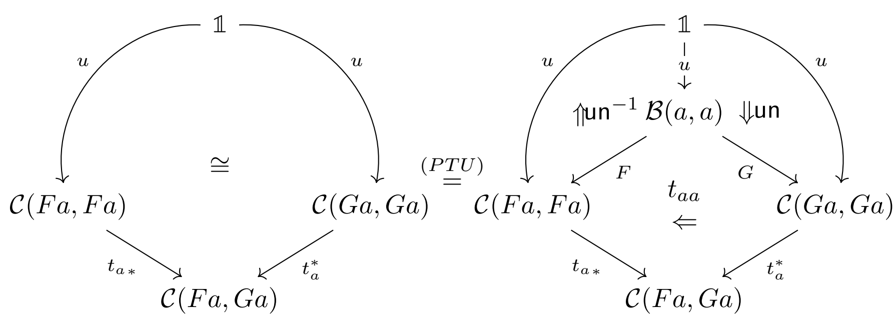
Then, one would definitely want enriched pseudonatural transformations $F \Rightarrow F$ to exist, because, more precisely, one would want them to correspond to pairs of endo-pseudonatural transformations of the pseudofunctors $\mathcal{B} \to$ Cat induced by post-composition of F with the canonical projections. However, no naive one can be found since no pair of functors goes $\mathbb{1} = (T,T) \to \mathcal{B}(A,B) = (T,\varnothing)$, hence no enriched morphism exists from A to B.

Therefore, the definition of $\mathcal{V}$-pseudonatural transformation will not be the naive translation, but rather the following:

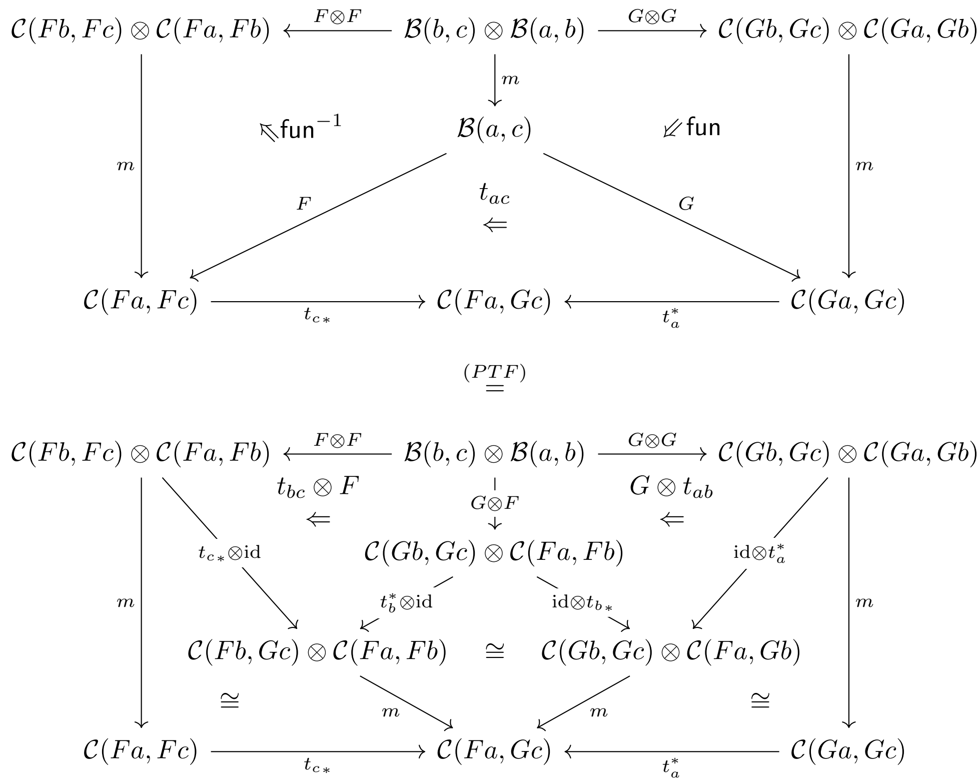
Definition 1.3.9. Let $F, G \colon \mathcal{B} \to \mathcal{C}$ be $\mathcal{V}$-pseudofunctors. Then, a *$\mathcal{V}$-pseudonatural transformation* $t \colon F \Rightarrow G$ is the data of a family of morphisms $t_a \colon \mathbb{1} \to \mathcal{C}(Fa, Ga)$ and, for every pair of objects a, b, a 2-isomorphism

$$\begin{array}{ccc} \mathcal{B}(a,b) & \xrightarrow{\;G\;} & \mathcal{C}(Ga,Gb) \\ {\scriptstyle F}\big\downarrow & \swarrow t_{ab} & \big\downarrow{\scriptstyle {t_a}^*} \\ \mathcal{C}(Fa,Fb) & \xrightarrow[t_{b*}]{} & \mathcal{C}(Fa,Gb) \end{array}$$

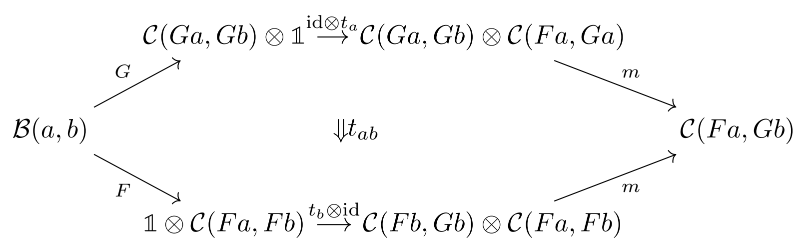
(see Remark 1.3.10 for the definition of $t_a^* and t_{b*}$) satisfying the unitality axiom (PTU):



and the functoriality axiom (PTF);



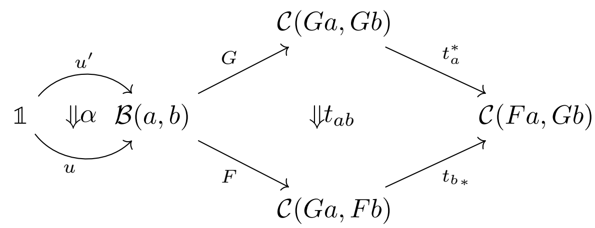
Remark 1.3.10. First, observe that the 2-dimensional structure of an enriched pseudonatural transformation expands as



(1.2)

Also, it is interesting to observe how this definition boils down to the definition of pseudonatural transformation in the non-enriched setting. The structure is provided by considering, for every $u\colon a \to b$, the morphism t_u given by whiskering t_{ab} with $u\colon \mathbb{1} \to \mathcal{B}(a,b)$. Unitality and functoriality axioms for a pseudonatural transformations evidently correspond to the same (PTU) and (PTF) given in the definition of enriched pseudonatural transformation.

What is remarkable is that the naturality axiom in the definition of pseudonatural transformation is now included in the property of each 2-cell, since it just boils down to the well-definedness of the horizontal composition of two 2-cells

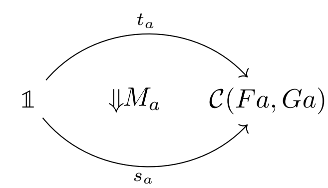


Remark 1.3.11. A natural thing for the $\mathcal{V}$-pseudofunctors between $\mathcal{V}$-bicategories $\mathcal{C}$ and $\mathcal{D}$ would be to assemble into a $\mathcal{V}$-bicategory $\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C},\mathcal{D})$, promoting then the previously introduced notion of $\mathcal{V}$-pseudonatural transformation into an *object*

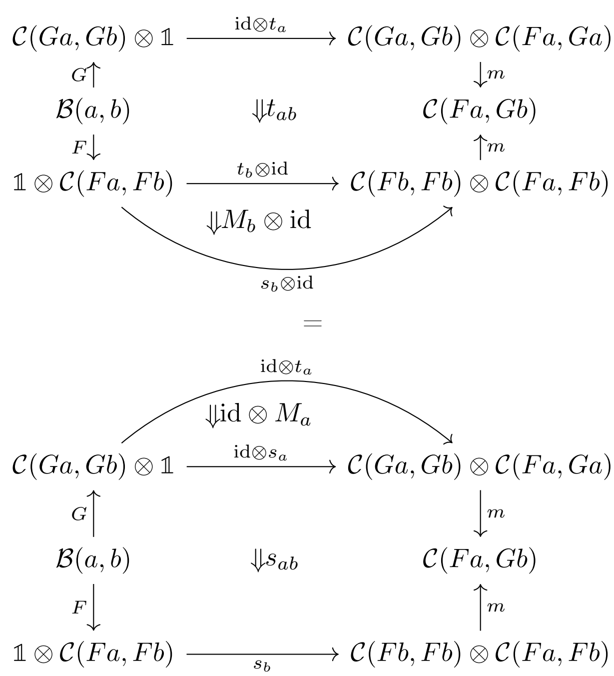
$$\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C},\mathcal{D})(F,G)$$

of $\mathcal{V}$, whenever $F,G\colon \mathcal{C}\to\mathcal{D}$ are $\mathcal{V}$-pseudofunctors. But in order to do so, we need the theory of bi(co)ends (see Proposition 2.3.1 and Definition 2.3.3). So far, $\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C},\mathcal{D})$ is just a bicategory, with $\mathcal{V}$-pseudonatural transformation and $\mathcal{V}$-*modifications* as 1 and 2-cells:

Definition 1.3.12. Let $F,G\colon \mathcal{B}\to\mathcal{C}$ be $\mathcal{V}$-pseudofunctors and $s,t\colon F\Rightarrow G$ be $\mathcal{V}$-pseudonatural transformations between them. A $\mathcal{V}$-*modification* M from t to s (denoted $M\colon t\Rrightarrow s$) is the data of a family of 2-cells in $\mathcal{V}$



such that the following equality of 2-cells holds:



equationally written as $M_b 1 \circ t_{ab} = s_{ab} \circ 1 M_a$.

Remark 1.3.13. As argued in Section 4 of [GS15], the structure defined so far defines the *tricategory* of $\mathcal{V}$-bicategories, whenever $\mathcal{V}$ is a monoidal bicategory.

1.3.5 Closed monoidal bicategories

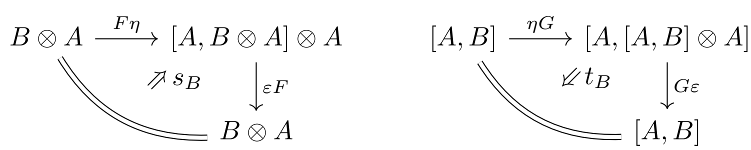
We want to define the notion of closed monoidal bicategory. It will follows that such a monoidal bicategory $\mathcal{V}$ is also canonically a $\mathcal{V}$-bicategory. Definition A.0.3 of pseudoadjunction given in Appendix A will be crucial.

Definition 1.3.14. A monoidal bicategory $\mathcal{V}$ is said to be *left closed* if for every object $A \in \mathcal{V}$ the pseudofunctor $- \otimes A \colon \mathcal{V} \to \mathcal{V}$ is part of a pseudoadjunction $- \otimes A \dashv [A, -]$.

Let's take some time in order to make explicit the involved structure explicit. To have a closed monoidal bicategory means then to have a monoidal bicategory as in Definition 1.1.8 together with, for every object A, two pseudonatural transformations

$$\eta^A \colon \mathrm{id} \Longrightarrow [A, - \otimes A]$$
$$\varepsilon^A \colon [A, -] \otimes A \Longrightarrow \mathrm{id}$$

plus modifications s, t having components



satisfying the swallowtail equations (A.3) and (A.4).

Proposition 1.3.15. *A closed monoidal bicategory $\mathcal{V}$ has a canonical structure of $\mathcal{V}$-bicategory.*

Proof. The hom-objects are defined as $\mathcal{V}(A, B) = [A, B]$. The composition morphism

$$m \colon [B, C] \otimes [A, B] \longrightarrow [A, C]$$

is defined to be the adjoint under $- \otimes A \dashv [A, -]$ of the composition

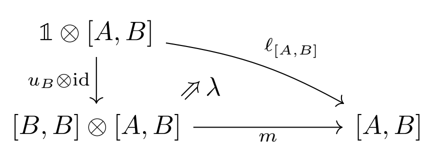
$$([B, C] \otimes [A, B]) \otimes A \xrightarrow{a} [B, C] \otimes ([A, B] \otimes A) \xrightarrow{\mathrm{id} \otimes \varepsilon^A_B} [B, C] \otimes B \xrightarrow{\varepsilon^B_C} C$$

The unit morphism $\mathbb{1} \to [A, A]$ is the adjoint of the left monoidal unitor

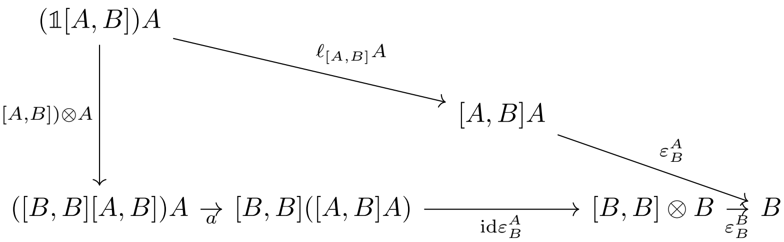
$$\ell_A \colon \mathbb{1} \otimes A \to A.$$

It remains to identify the higher structure α, λ, ρ and to show that it satisfies the axioms (AC) and (IC).

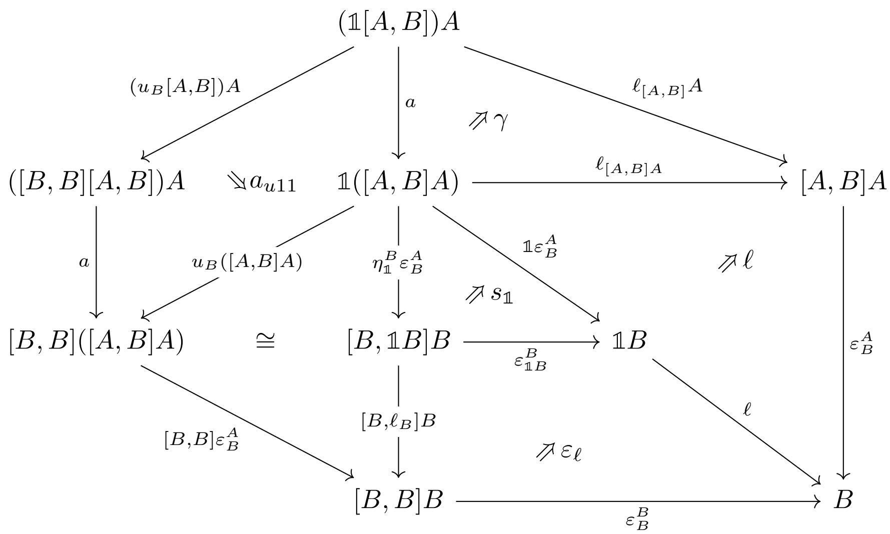
In order to find λ, of the form



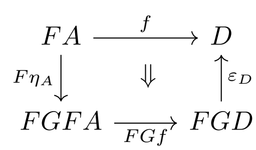
we can first transpose the diagram, finding (from now on we avoid to explicit the tensor sign for sake of space and readability)



and then expand it, in order to fill it with naturality isomorphisms. Then, by transposing that back, we can define the result to be the desired 2-cell λ.

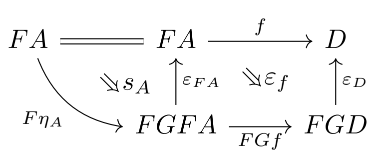


The unlabeled 2-isomorphism comes here directly from the definition of $u_B = [B, \ell_B]\eta_{\mathbb{1}}^B$ and the tensoring of the left and the right 2-unitor from the bicategory structure of $\mathcal{V}$. The right 2-unitor is given indeed by a perfectly symmetric construction, and for what concerns the 2-associator we first observe that for every pseudoadjunction $F \dashv G$ and every map $f\colon FA \to D$ there's a 2-isomorphism

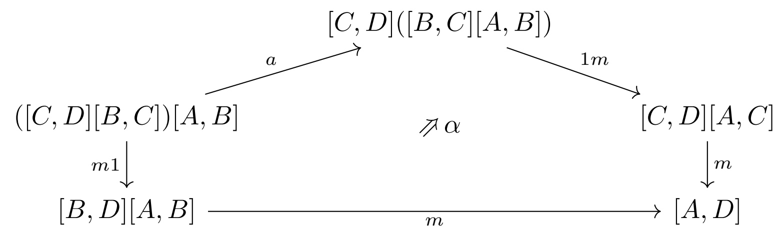


(1.3)

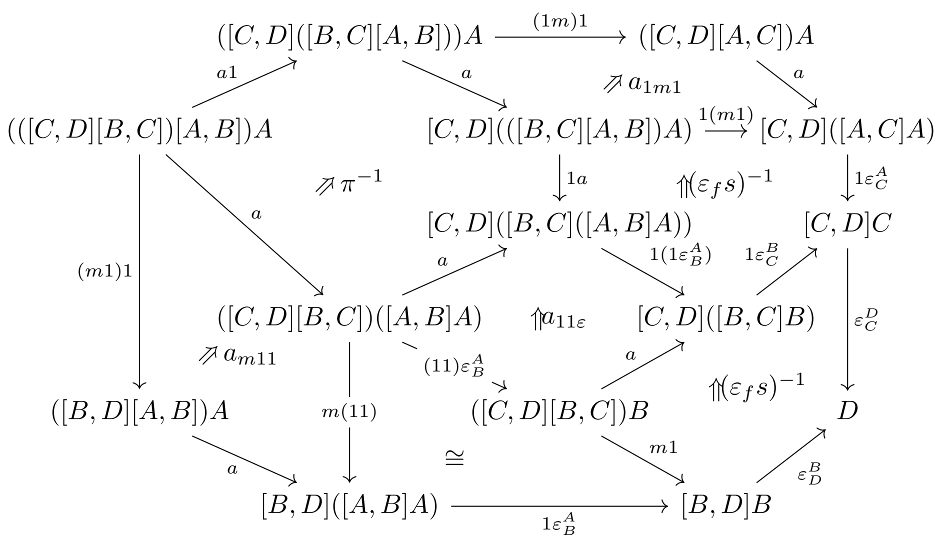
defined by the composition



In other terms, the 2-isomorphism above is indeed an isomorphism $f \Rightarrow \varepsilon_D \circ F(\overline{f})$, for the transpose $\overline{f}$ being $Gf\eta_A$. The 2-associator α can thus to be defined as a 2-cell of the form



corresponding, via the adjunction $- \otimes A \dashv [A, -]$, to



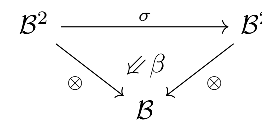
for the morphism f as in notation of (1.3) being the compositions which transposes to m. The verification of associativity and identity coherence axioms for this structure is left to the (very) willing reader. □

1.4 Braided monoidal bicategories

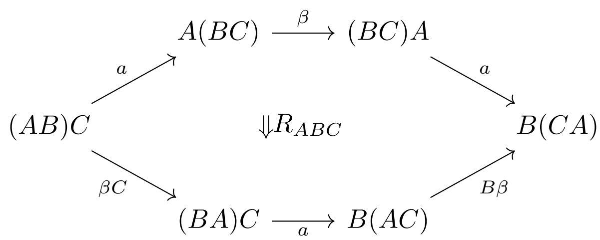
Braided monoidal categories are categories equipped with an isomorphism $A \otimes B \to B \otimes A$ for every pair of objects A, B. The requirements that this map satisfy (the so called hexagon identities), require that if we have three tensored objects $A \otimes B \otimes C$, then swapping the first two ($B \otimes A \otimes C$) and then the resulting last two ($B \otimes C \otimes A$), then we find the same thing as directly swapping A with $B \otimes C$. Same thing (the second hexagon identity) for the symmetric operation starting with C. Even if the operation described may sound more like a commutative triangle (two operations equal one operation), there are of course associators involved, hence the hexagonal shape.

In the bicategorical context, with no surprise, these hexagons are replaced by two 2-isomorphism, satisfying (and that is maybe more surprising) a sizable set of fairly non-trivial axioms.

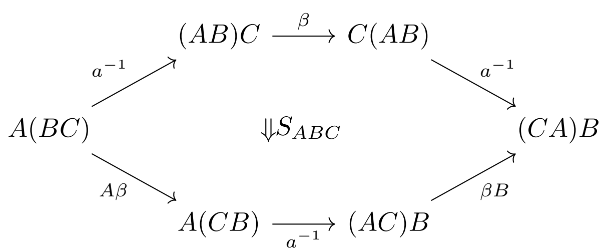
Definition 1.4.1. Let $(\mathcal{B}, \otimes, \mathbb{1}, a, l, r, \pi, \mu)$ be a monoidal bicategory. A *braided structure* on it is the data of a pseudonatural equivalence



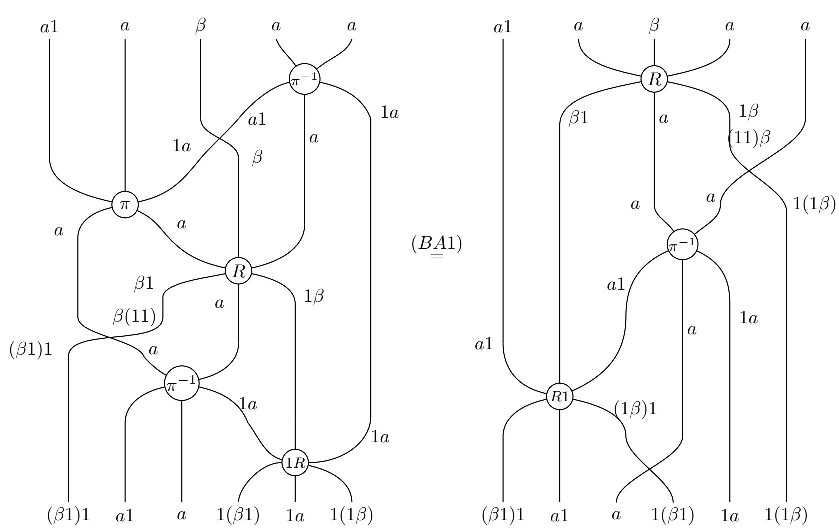
together with invertible modifications (2-cells in $\mathsf{PsFun}(\mathcal{B}^3, \mathcal{B})$) called R and S having components



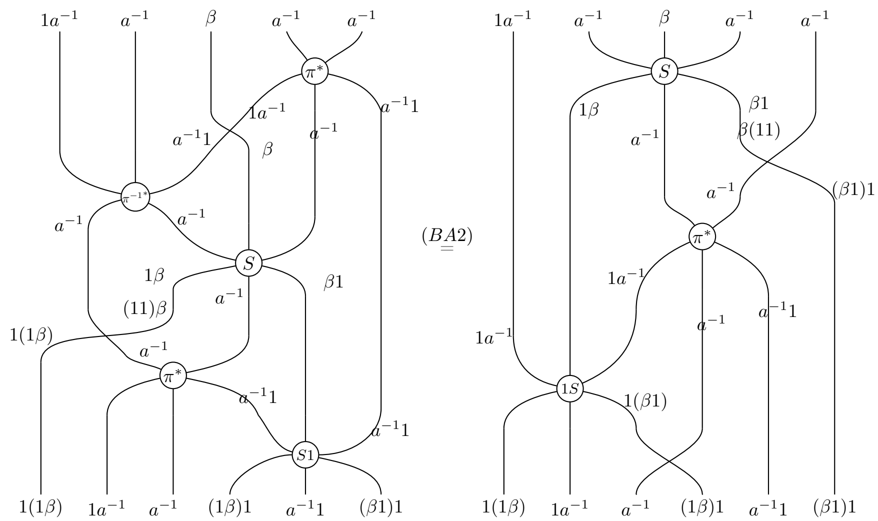
and



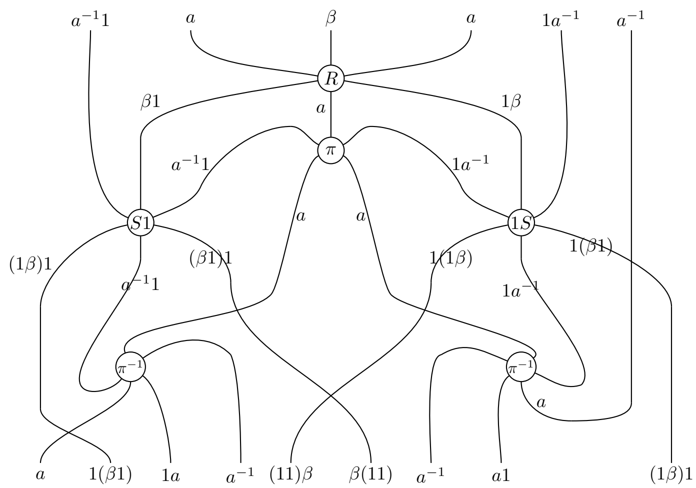
satisfying the following four *braiding axioms* (see [Mcc00] and [Gur11b] for the pasting diagrammatic form):



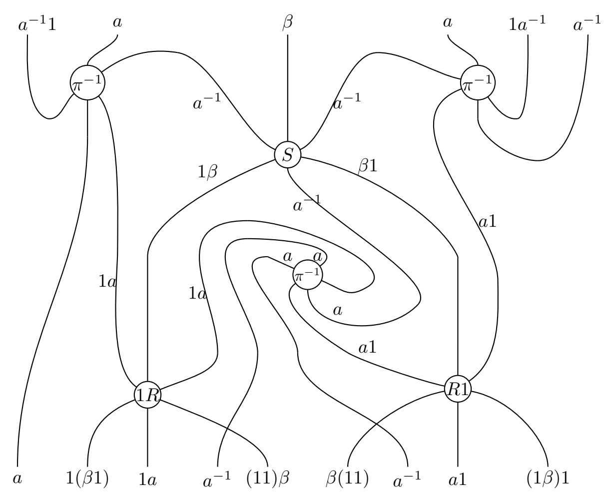
The second one, perfectly symmetric and involving S, instead of R



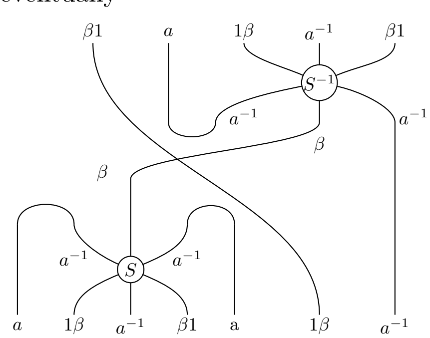
The third one



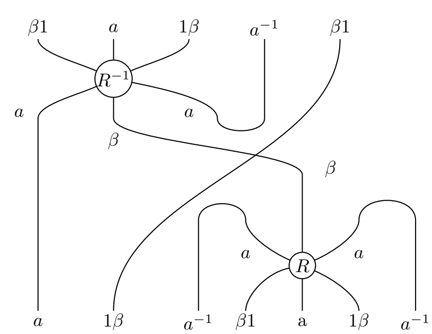
(BA3)



and eventually



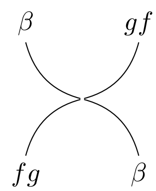
(BA4)



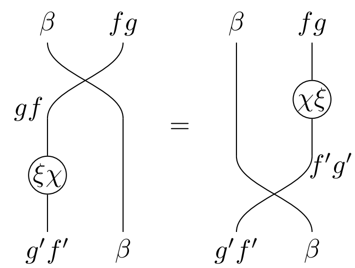
Remark 1.4.2. Observe that we have a string diagrammatic rule telling us that we can switch the order of a tensored pair of morphisms by switching its composition with β. This is again (similarly to what happens with the associator a, see Remark 1.2.2) a direct translation of the naturality 2-isomorphisms

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & \swArrow \beta_{f,g} & \downarrow{\scriptstyle g\otimes f} \\ A' \otimes B' & \xrightarrow[\beta]{} & B' \otimes A' \end{array}$$

and again this rule is used in stating the axioms above without explicitly label the corresponding 2-cell



Moreover, naturality of β (axiom (PTN)) gives us the equality



Remark 1.4.3. A further structure which is considered in the literature and which we are not going to use throughout this work is the notion of *sylleptic* structure, which consists of an invertible morphism from the identity to β^2 satisfying another bunch of axioms (see Remark 3.5.11). Full details about that can be found in [Mcc00]. An even further axiom for a syllepsis defines the notion of *symmetric monoidal bicategory.*

Proposition 1.4.4. *In any braided monoidal bicategory there are isomorphisms*

$$\begin{array}{ccc} A \otimes \mathbb{1} & \xrightarrow{\beta_{A,\mathbb{1}}} & \mathbb{1} \otimes A \\ & {\scriptstyle r}\searrow \ \swArrow \ \swarrow{\scriptstyle \ell} & \\ & A & \end{array} \qquad \begin{array}{ccc} \mathbb{1} \otimes A & \xrightarrow{\beta_{\mathbb{1},A}} & A \otimes \mathbb{1} \\ & {\scriptstyle \ell}\searrow \ \swArrow \ \swarrow{\scriptstyle r} & \\ & A & \end{array}$$

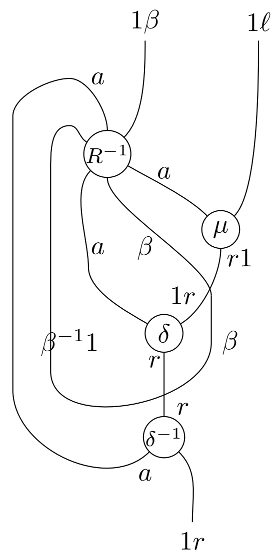
relating β, r, ℓ *and respectively called* $K_{\ell,r}$ *and* $K_{r,\ell}$.

Proof. Let us define the first one, from $\ell\beta$ to r, and let us do that by using Remark 1.2.3, and hence by defining a morphism of sort

$$\begin{array}{ccc} C \otimes (A \otimes \mathbb{1}) & \xrightarrow{1\beta} & C \otimes (\mathbb{1} \otimes A) \\ & {\scriptstyle 1r}\searrow \ \swArrow \ \swarrow{\scriptstyle 1\ell} & \\ & C \otimes A & \end{array}$$

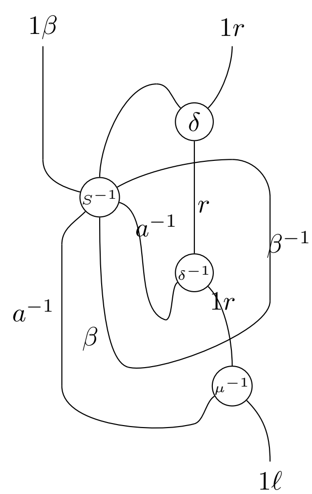
and then looping an ℓ around once fixed $C = \mathbb{1}$. In string diagram, this last morphism is given by

$\mathrm{id} \otimes K_{\ell,r} =$



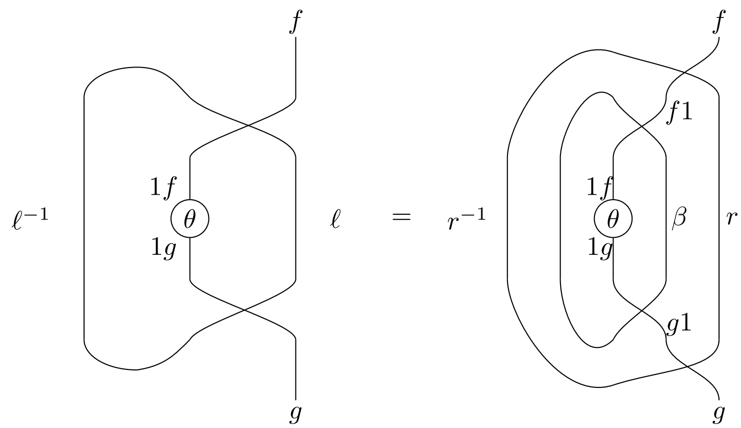
The second morphism $K_{r,l} \colon r\beta \Rightarrow \ell$ is similarly defined starting from

$\mathrm{id} \otimes K_{r,l} =$

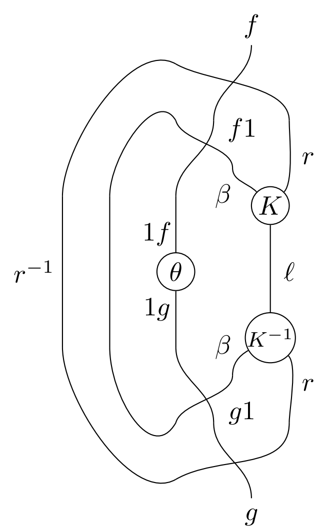


and then looping an ℓ around that, too. □

Lemma 1.4.5. *Let $f, g \colon A \to B$ be two morphisms in a monoidal bicategory and consider the morphisms* $\mathrm{id} \otimes f, \mathrm{id} \otimes f \colon \mathbb{1} \otimes A \to \mathbb{1} \otimes B$. *For every 2-cell* $\theta \colon \mathrm{id} \otimes f \Rightarrow \mathrm{id} \otimes g$, *there is an equality of 2-cells*



Proof. The argument consists of inserting in the right hand side the isomorphism $K_{r,\ell}\colon r\beta \Rightarrow \ell$, together with its inverse, in order to find



and then by letting them slide - by the modification property of K - over f (the morphism K) and under g (the morphism K^{-1}), in order to cancel them out on the other side. □

1.5 Strictification results

We dedicate this section to recollect the state of art of strictification results for monoidal and braided monoidal bicategories. The underlying reasons for this theory is that unlike what happens for a bicategory, which is always biequivalent to a 2-category, the same does not hold for a tricategory, not even for the one-object case, the one of monoidal bicategories. For the latter, a notion of product of 2-categories known as the Gray product is fundamental. This is a tool particularly well suited for dealing with coherence, since it is a product on the category of strict 2-categories (hence fairly manageable in practice), but with respect to the usual cartesian product introduces a flexibility allowing to detect non-strict structures.

1.5.1 Gray tensor product

Given two 2-categories $\mathcal{C}$ and $\mathcal{D}$, their cartesian product $\mathcal{C} \times \mathcal{D}$ is a construction defining a left 2-adjoint $- \times \mathcal{D}$ to the 2-functor $2\mathrm{Fun}(\mathcal{D}, -)$. There is a well known construction replacing the cartesian product and allowing the introduction of some weakening. This notion was first introduce by Gray in [Gra74]. If we call the category of 2-categories and 2-functors 2Cat, one possible definition of the Gray product is as the 2-category $\mathcal{C} \otimes_G \mathcal{D}$ such that there is a natural isomorphism of sets

$$2\mathrm{Cat}(\mathcal{C} \otimes_G \mathcal{D}, \mathcal{E}) \cong 2\mathrm{Cat}(\mathcal{C}, 2\mathrm{Fun}^{\mathrm{Ps}}(\mathcal{D}, \mathcal{E})),$$

where $2\mathrm{Fun}^{\mathrm{Ps}}(\mathcal{D}, \mathcal{E})$ is the 2-category whose objects are 2-functors $\mathcal{D} \to \mathcal{E}$, 1-cells are *pseudo*natural transformations, and 2-cells are modifications between them. The Gray product defines a symmetric monoidal structure on 2Cat. By unwinding this definition, we discover that any 2-functor

$$F\colon \mathcal{C} \otimes_G \mathcal{D} \longrightarrow \mathcal{E}$$

is essentially given by pseudonatural transformations $F(f,-)\colon F(C,-) \Rightarrow F(C',-)$ with components $F(f,-)_D\colon F(C,D) \to F(C',D)$, that is the data of invertible 2-cells

$$\begin{array}{ccc} F(C,D) & \xrightarrow{F(f,-)_D} & F(C',D) \\ {\scriptstyle F(C,g)}\downarrow & \overset{\sim}{\swarrow} & \downarrow{\scriptstyle F(C,g)} \\ F(C,D') & \xrightarrow[F(f,-)_{D'}]{} & F(C',D') \end{array}$$

for every $f\colon C \to C'$ and $g\colon D \to D'$ satisfying the unitality, functoriality and naturality axioms. Therefore, that is the structure that we get when we define a *semi-strict monoidal 2-category*, also known as a *Gray monoid*. This definition is given in [Cra98], and before by Baez and Neuchl in (a first version of) [BN20].

Definition 1.5.1. A *semi-strict monoidal* structure on a 2-category $\mathcal{C}$ is the data of an object $\mathbb{1}$ in $\mathcal{C}$ and a 2-functor

$$- \otimes -\colon \mathcal{C} \otimes_G \mathcal{C} \to \mathcal{C}$$

such that the strict associativity and unitality constraint hold true, that is

$$- \otimes (- \otimes -) = (- \otimes -) \otimes -,$$

$$\mathbb{1} \otimes - = -, \quad - \otimes \mathbb{1} = -.$$

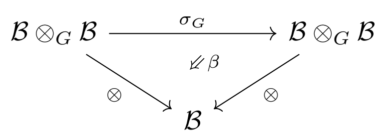
Remark 1.5.2. Semi-strict monoidal 2-categories are a special case of monoidal bicategories. This can be checked at [Gur06], Section 5, where the notion of *cubical pseudofunctor* is treated. Cubical pseudofunctor are precisely those pseudofunctors between 2-categories $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that (Gurski shows) correspond precisely to strict 2-functor $\mathcal{C} \otimes_G \mathcal{D} \to \mathcal{E}$. If we consider then the tensor product 2-functor $\mathcal{C} \otimes_G \mathcal{C} \to \mathcal{C}$ defining a semi-strict monoidal 2-category, we get that this is a pseudofunctor (a cubical one, in particular), and hence it equips the 2-category $\mathcal{C}$ with the structure of a monoidal bicategory.

The point, roughly speaking, is that the replacement of the cartesian product of 2-categories by the Gray product (reason for the adjective *semi-strict*) introduces a weakening that allows us to find for any monoidal bicategory a monoidal biequivalence onto a semi-strict monoidal 2-category. This result is a special (one object) case of the general coherence theorem for tricategories which can be found in [GG09]:

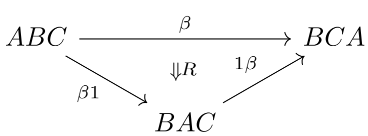
Theorem 1.5.3 (Coherence for tricategories)**.** *Any monoidal bicategory is monoidally biequivalent to a semi-strict monoidal bicategory*

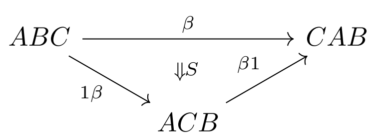
Let us come to the braided case. Crans in [Cra98] also defines a braided structure on a semi-strict monoidal 2-category. This structure and its axioms are the ones given at Definition 1.4.1: the pseudonatural equivalence β, modifications R, S and the four braiding axioms, applied to a semi-strict monoidal 2-category, which is in particular a monoidal bicategory. Moreover, some further compatibility of R and S are demanded (see below). Let us call σ_G the symmetry for 2-categories under the Gray tensor product.

Definition 1.5.4. Let $\mathcal{B}$ be a semi-strict monoidal bicategory. A braided structure on $\mathcal{B}$ consists of a pseudonatural equivalence of 2-functors

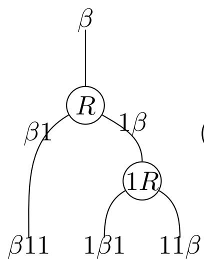


and two modifications

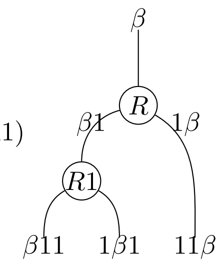


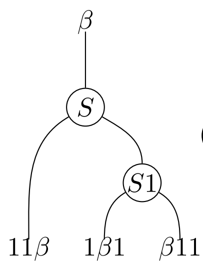


satisfying the following strict version of the four braiding axioms:

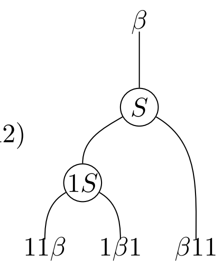


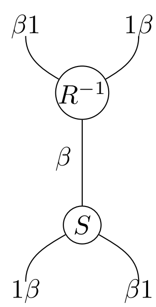
$\overset{(BA1)}{=}$



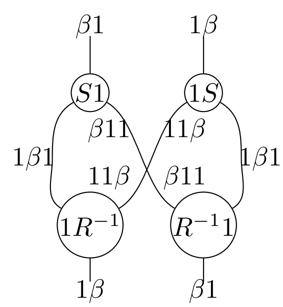


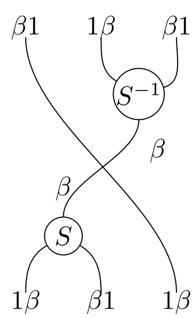
$\overset{(BA2)}{=}$



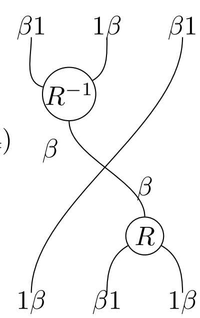


$\overset{(BA3)}{=}$



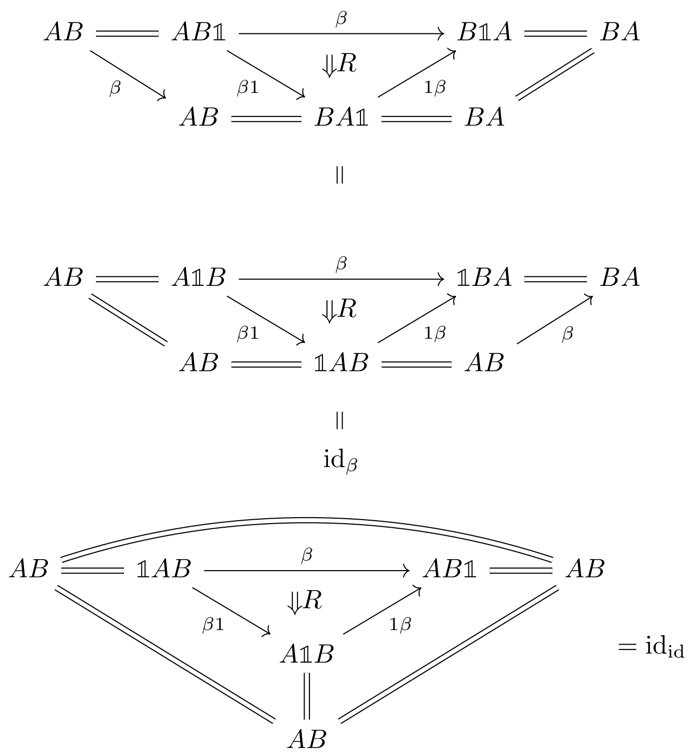


$\overset{(BA4)}{=}$



Moreover, β, R, S are required to satisfy the following constraints: two commutative squares for every object A

and the following identities of 2-cells (where squares commutes by the two previous conditions)



Similarly, three more conditions for S.

A fundamental result for the rest of the work is then the following result, for which we refer to Theorem 27 of [Gur11b].

Theorem 1.5.5. *Any braided monoidal bicategory is braided monoidally equivalent to a semi-strict braided monoidal 2-category.*

In the literature, other use of this strictification results, which allows to heavily simplify computations, can be found at [DX24].

1.6 The enriched opposite bicategory

The braiding on the monoidal bicategory $\mathcal{V}$ is fundamental for most of our constructions. In this section we define the structure of $\mathcal{V}$-bicategory $\mathcal{C}^{\mathsf{op}}$, for a given $\mathcal{V}$-bicategory $\mathcal{C}$. The braiding is, as usual, necessary for defining the composition.

Definition 1.6.1. Let $\mathcal{V}$ be a braided monoidal bicategory. For any $\mathcal{V}$-bicategory $\mathcal{C}$, there's a $\mathcal{V}$-bicategory $\mathcal{C}^{\mathsf{op}}$ called *enriched opposite bicategory* defined as having the same objects of $\mathcal{C}$ and the hom-objects $\mathcal{C}^{\mathsf{op}}(c,d) = \mathcal{C}(d,c)$. The composition

$$\underline{c}\colon \mathcal{C}^{\mathsf{op}}(d,e) \otimes \mathcal{C}^{\mathsf{op}}(c,d) \longrightarrow \mathcal{C}^{\mathsf{op}}(c,e)$$

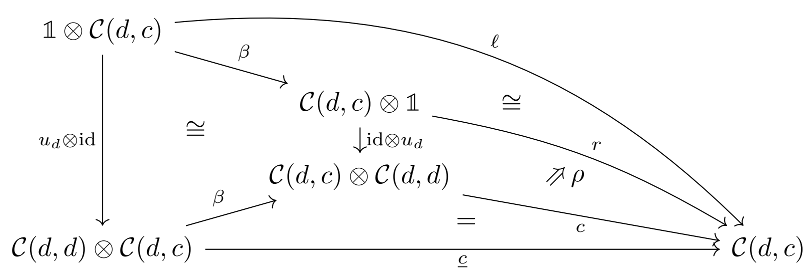
is defined as the composite

$$\mathcal{C}(e,d) \otimes \mathcal{C}(d,c) \xrightarrow{\beta} \mathcal{C}(d,c) \otimes \mathcal{C}(e,d) \xrightarrow{c} \mathcal{C}(e,c),$$

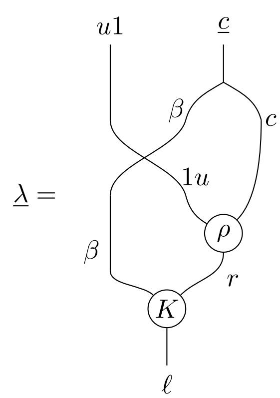
and the unit is the very same morphism as that of $\mathcal{C}$

$$u_c\colon \mathbb{1} \longrightarrow \mathcal{C}^{\mathsf{op}}(c,c) = \mathcal{C}(c,c).$$

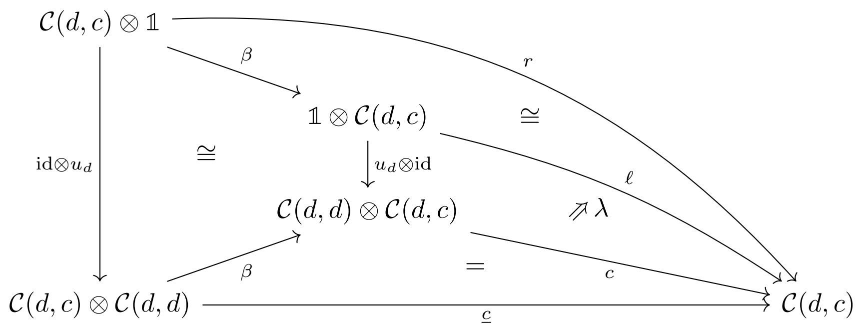
The left associator $\underline{\lambda}$ is defined in terms of the right associator ρ for $\mathcal{C}$ as the following composite 2-cell:



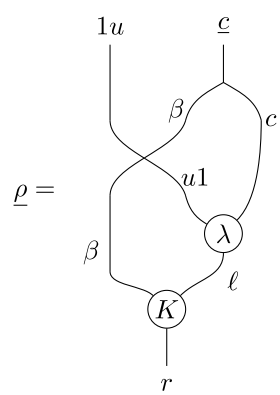
The above isomorphisms are the naturality of β and the morphism $K_{r,\ell}$ of Proposition 1.4.4, so that as a string diagram we have



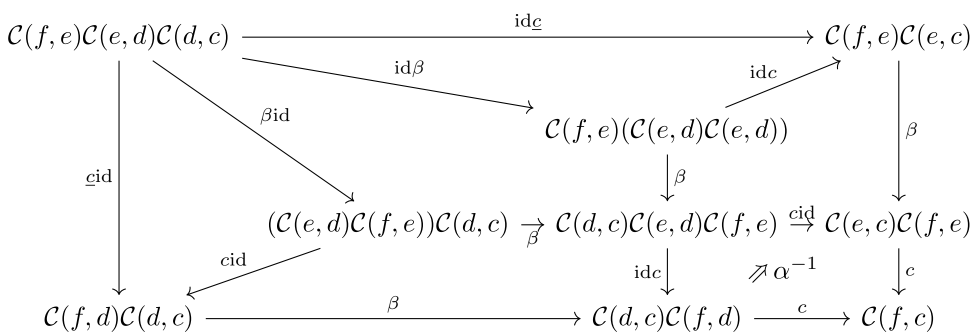
Analogously, it works for the right unitor



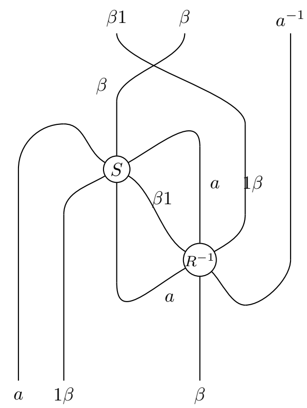
This last morphism has, then, a string-diagrammatic representation as



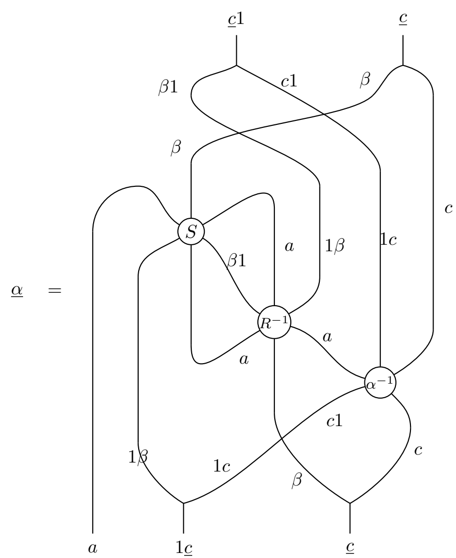
For what concerns the associator, we can define it like the following pasted diagram



Here, triangles commute by definition, the two areas on the bottom and on the right side can be filled with the naturality isomorphism for β, while for the remaining square we have (remembering how we are omitting the monoidal associators in the above diagram) the 2-cell



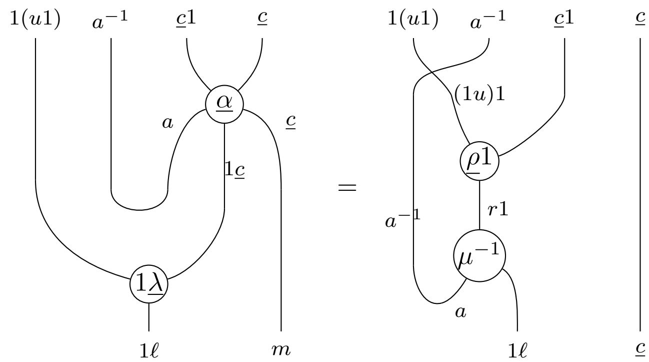
Therefore, we have



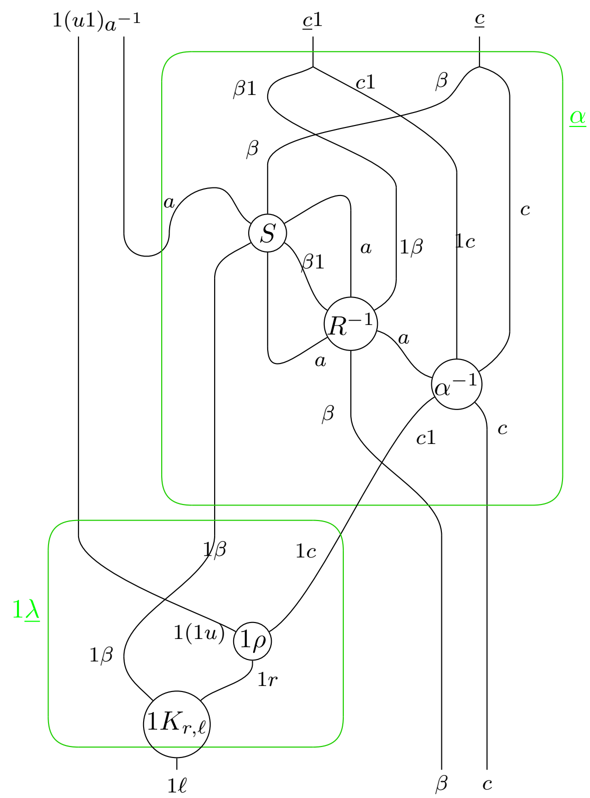
We need now to verify the coherence axioms for the $\mathcal{V}$-bicategory $\mathcal{C}^{\mathsf{op}}$. This apparently trivial task requires quite a lot of effort if made without assuming strictification. We nonetheless provide here a complete proof, since it was drawn up prior to our understanding of the power of the strictification theorems, as well as to provide, to anyone who wished, the complete structure that appears in the case of non-strict monoidal bicategories. See Remark 1.7.2.

Proposition 1.6.2. *Let $\mathcal{C}$ be a $\mathcal{V}$-bicategory. The morphisms $u, \underline{c}$ and $\underline{\alpha}, \underline{\rho}, \underline{\lambda}$ of Definition 1.6.1 define $\mathcal{C}^{\mathsf{op}}$ as a $\mathcal{V}$-bicategory.*

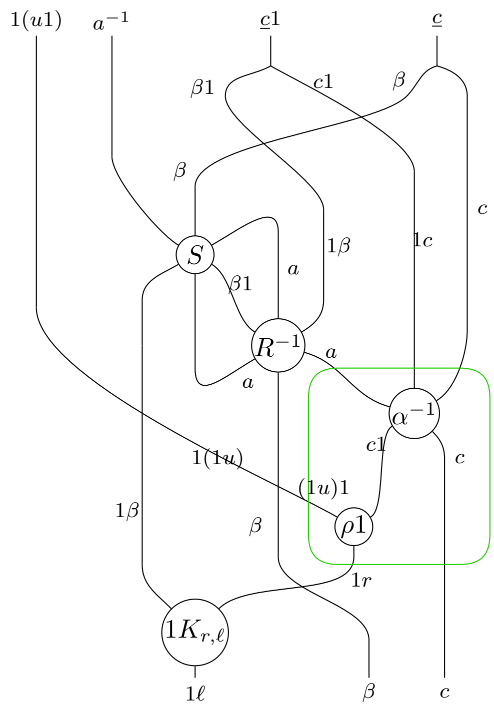
Proof. It has to be proved that the two axioms for a $\mathcal{V}$-bicategory hold true. Let's start with the identity coherence (IC), saying that we should have equalities in $\mathcal{V}$ of the two 2-cells



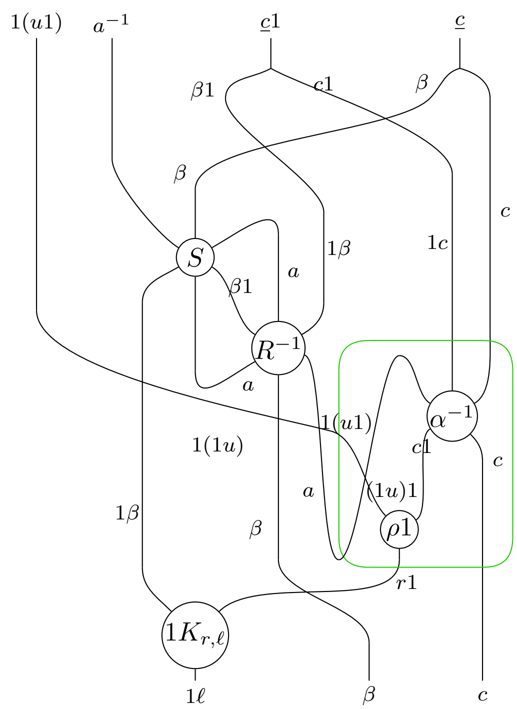
Let us expand the right hand side, in order to find



Thus, let us shift 1ρ over β, turning it into $\rho 1$

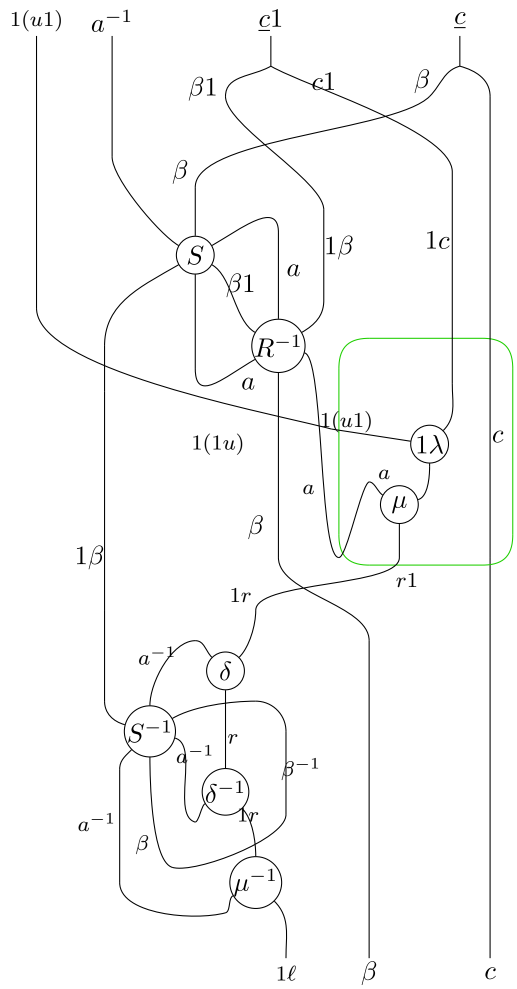


Now, in the highlighted part we observe that we can stretch a

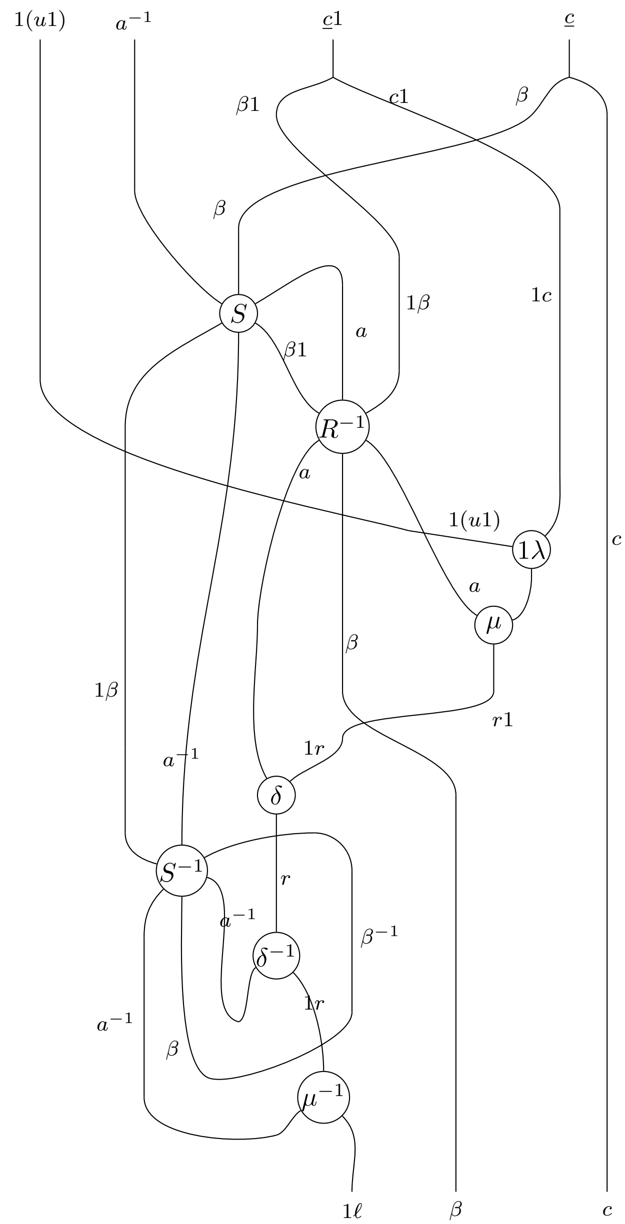


Here, we observe that we can use (IC) for $\mathcal{C}$, and at the same time expand $1K$ (defined at

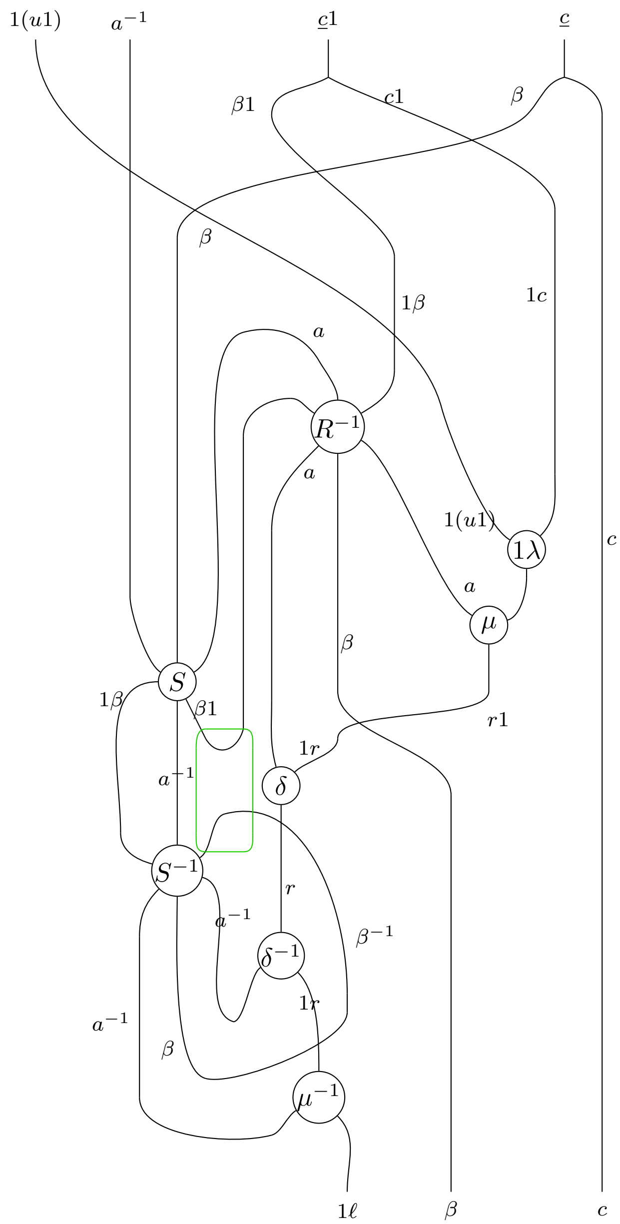
Proposition 1.4.4), finding



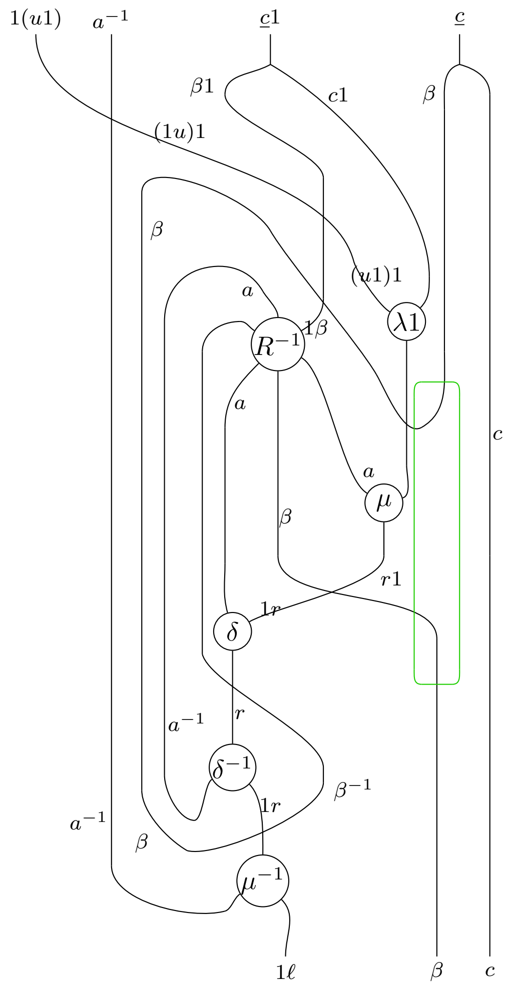
Now, we are allowed to make the counit and the unit for a to meet, finding



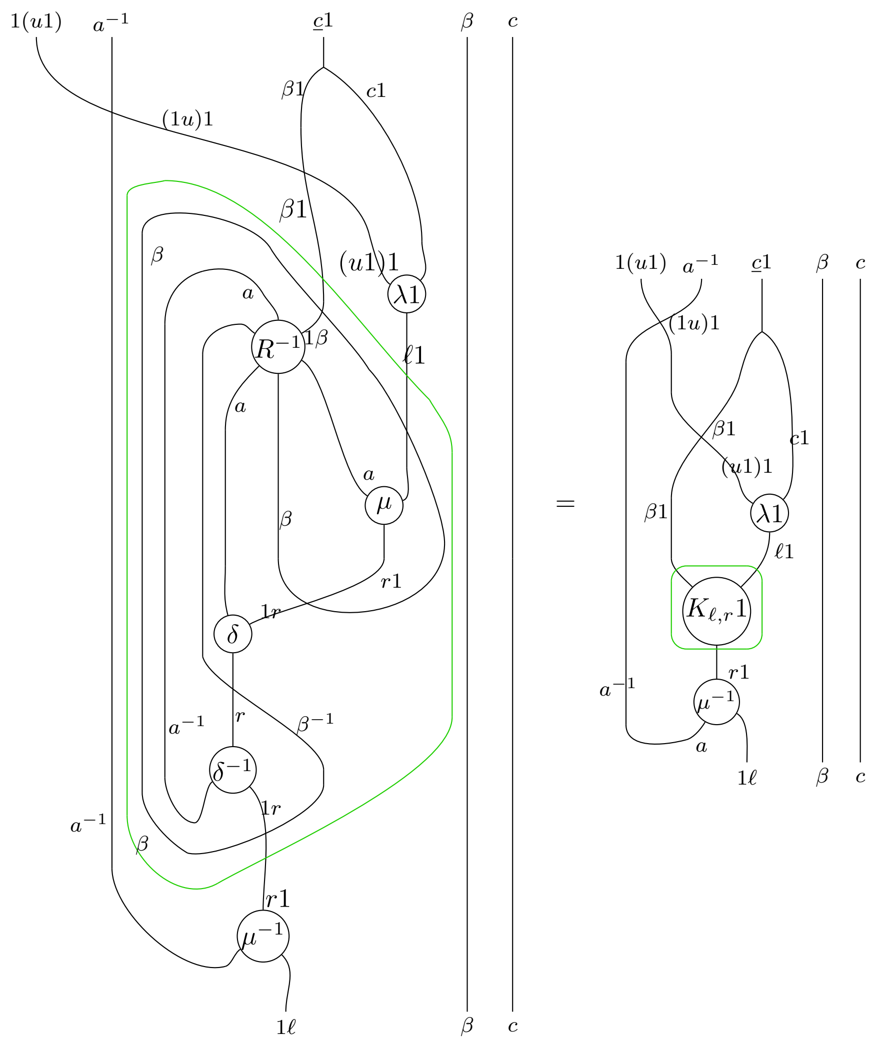
Then, observe that, always by the modification property, we can make the whole S and R^{-1} to pass under $1(u1)$.



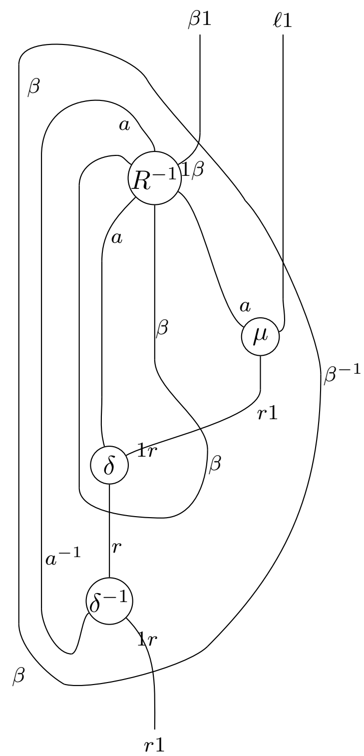
Hence, we can cancel out S and its inverse, by linking $\beta 1$'s in the highlighted part. Moreover, we can make 1λ pass over β, turning it into $\lambda 1$.



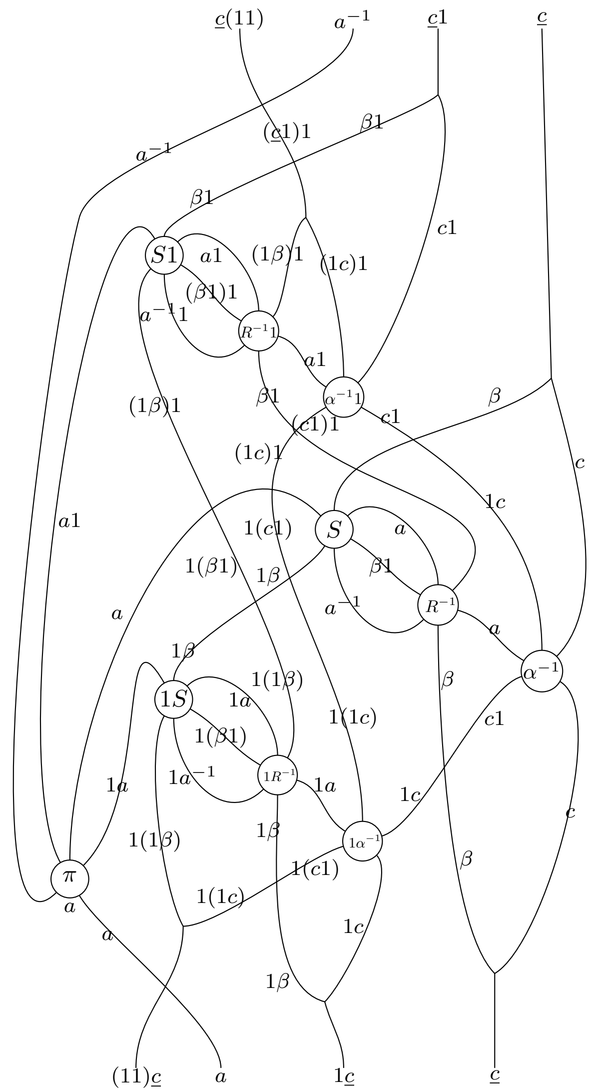
Now we can link β's, and eventually confront the result with the initial right hand side of the (IC), which then become



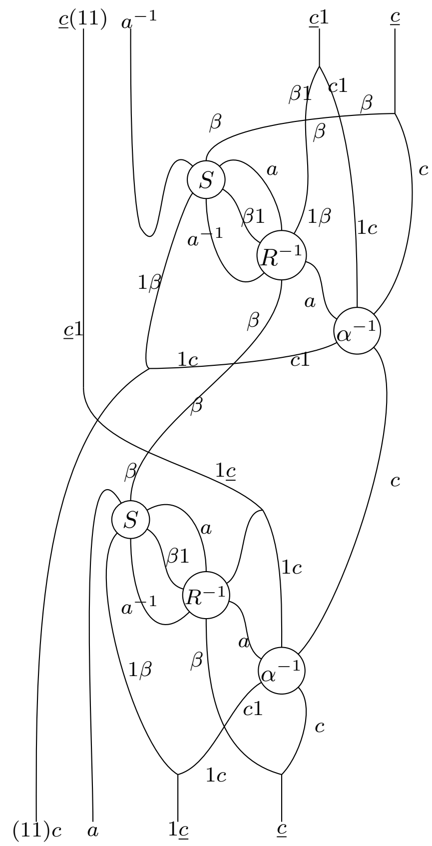
This equality happens eventually to be true whether the highlighted part is equal to $K1$. This can be seen by linking β's, turning it into



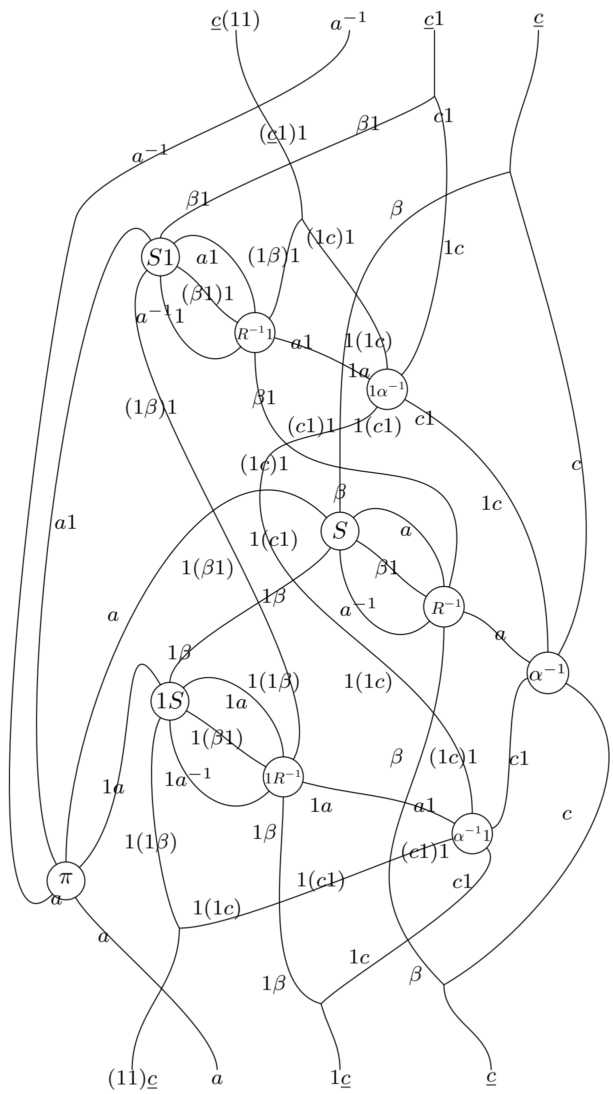
which is precisely $1K$ (Proposition 1.4.4) with a loop of β around, namely $K1$. This proves the identity coherence. Now, let us focus on the associativity coherence. Let us start with the left hand side of the axiom below. We can recognize the three components $\underline{\alpha}1$, $\underline{\alpha}$ and $1\underline{\alpha}$, together with π.



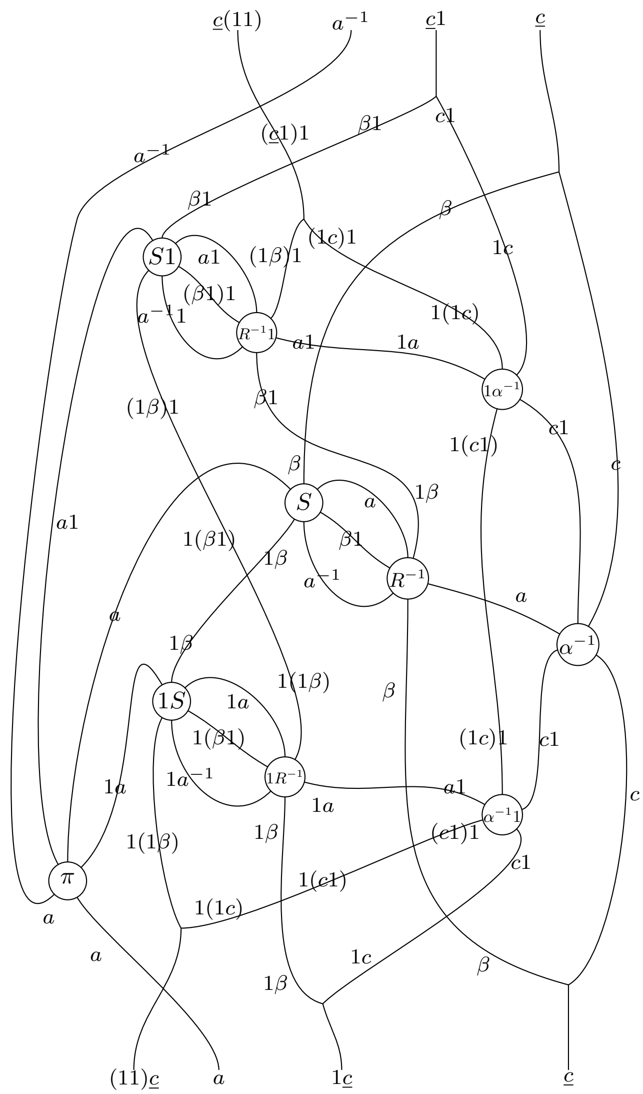
The associativity coherence axiom states that this is equal to the right hand side in the diagram below



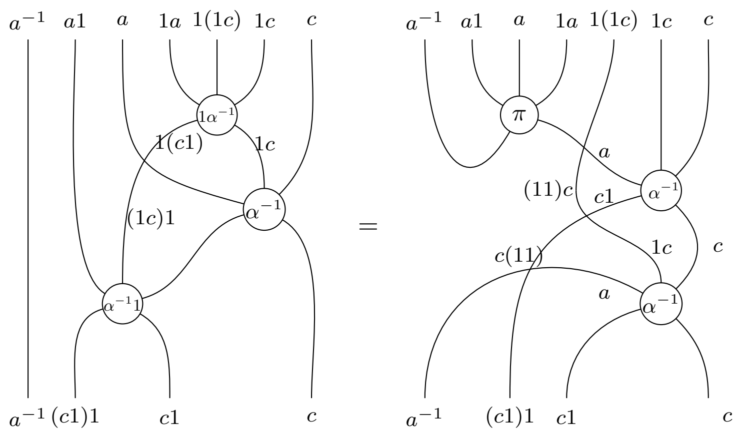
Where, we can recognize the two $\underline{\alpha}$. Let us slide then, in the left hand side, $1\alpha^{-1}$ and $\alpha^{-1}1$ over the β's they have at their right. This will turn them respectively in $\alpha^{-1}1$ and $1\alpha^{-1}$.



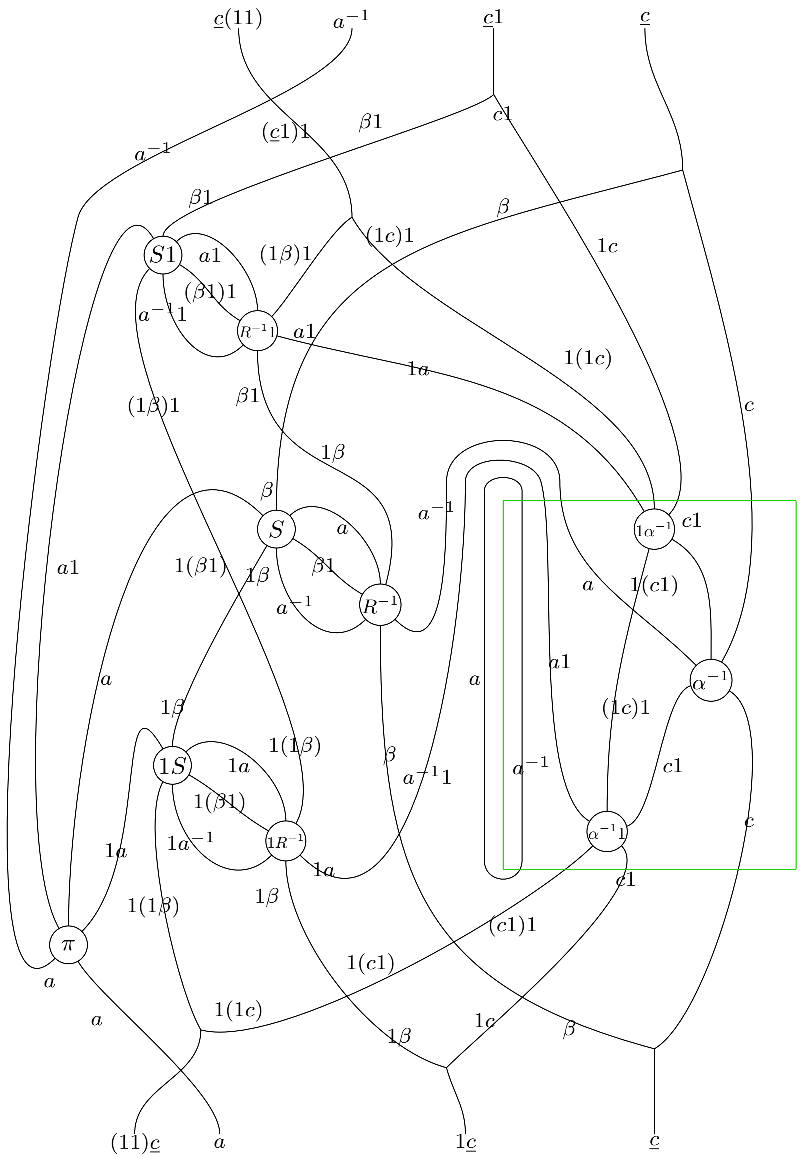
Now, by modification property of the composite $R^{-1}S$, we can let the $1(c1)$ wire out of $1\alpha^{-1}$ (and entering in $\alpha^{-1}1$ as $(1c)1$) pass of the right. We find then



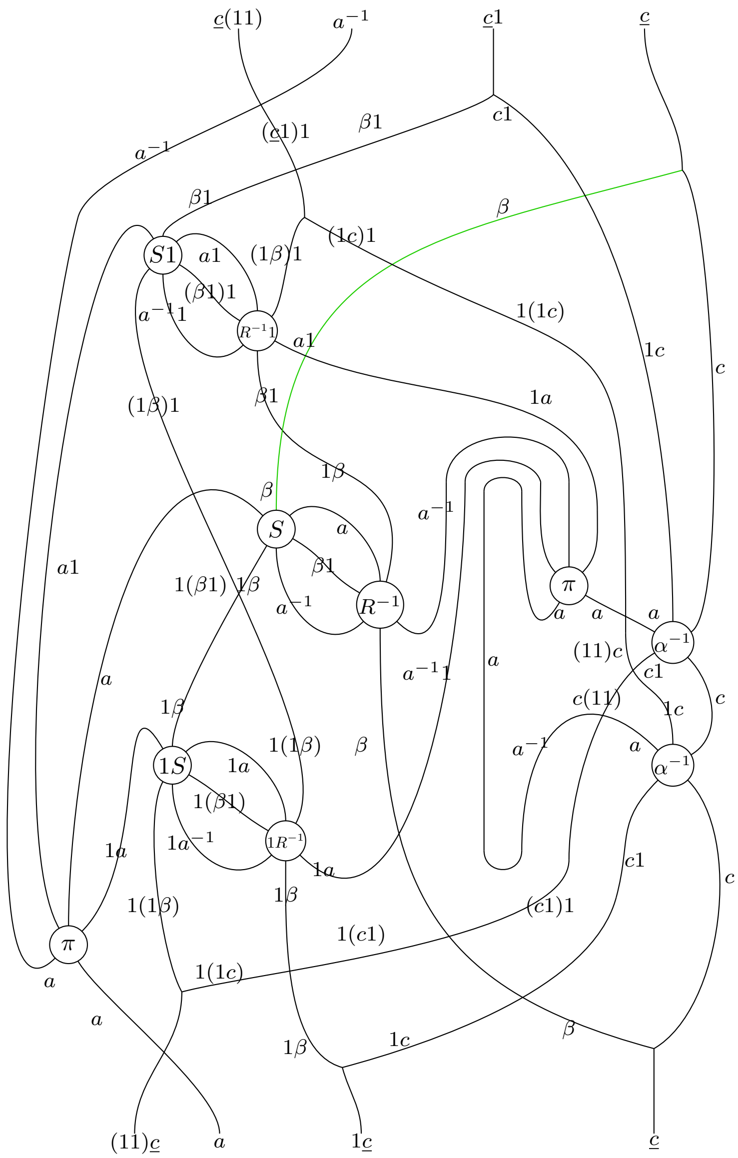
At this point, we want to use axiom (AC) for the $\mathcal{V}$-bicategory $\mathcal{C}$, in its form



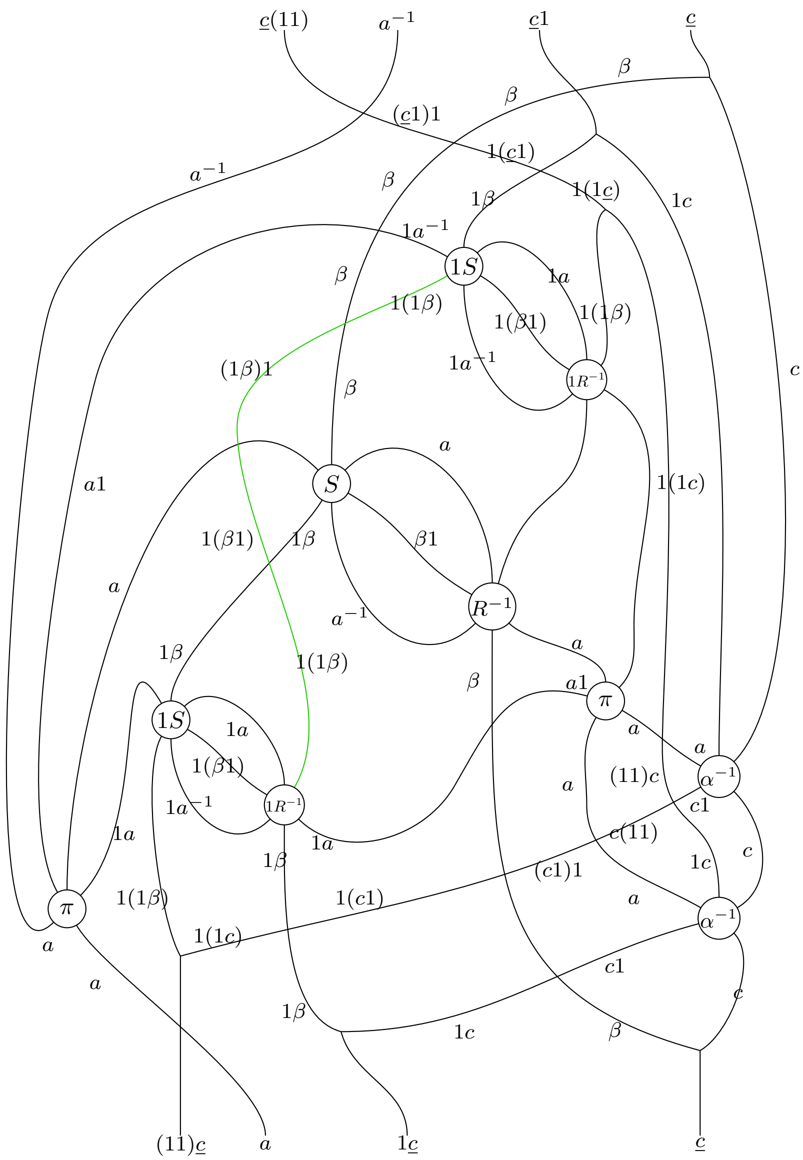
In order to do so, we need to make the a^{-1} on the left to appear. Therefore, we introduce the two isomorphisms $\mathrm{id} \cong a^{-1}a$ and $a^{-1}a \cong \mathrm{id}$ as follows.



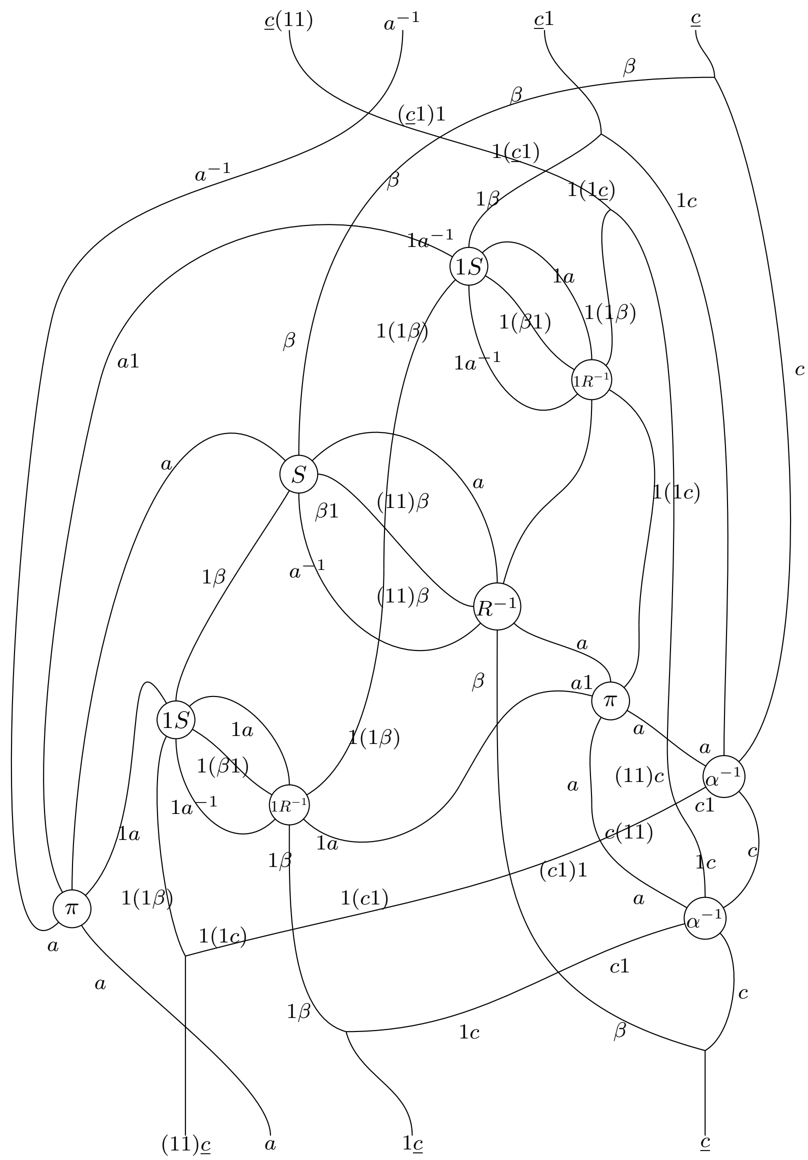
Then we apply the associativity coherence axiom for $\mathcal{C}$ and find



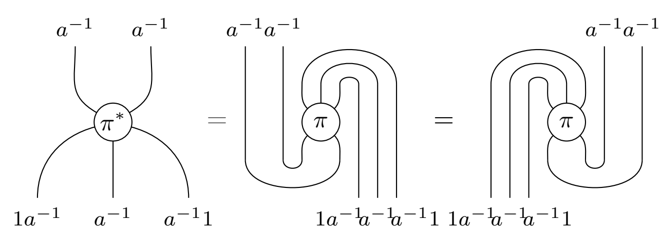
At this point, let us rearrange the components of the 2-cell, by simplifying the resulting counit-unit of a. Moreover, by the naturality axiom for β, let us slide the highlighted β over the composite of $R^{-1}1$ and $S1$, which become then $1R^{-1}$ after $1S$.



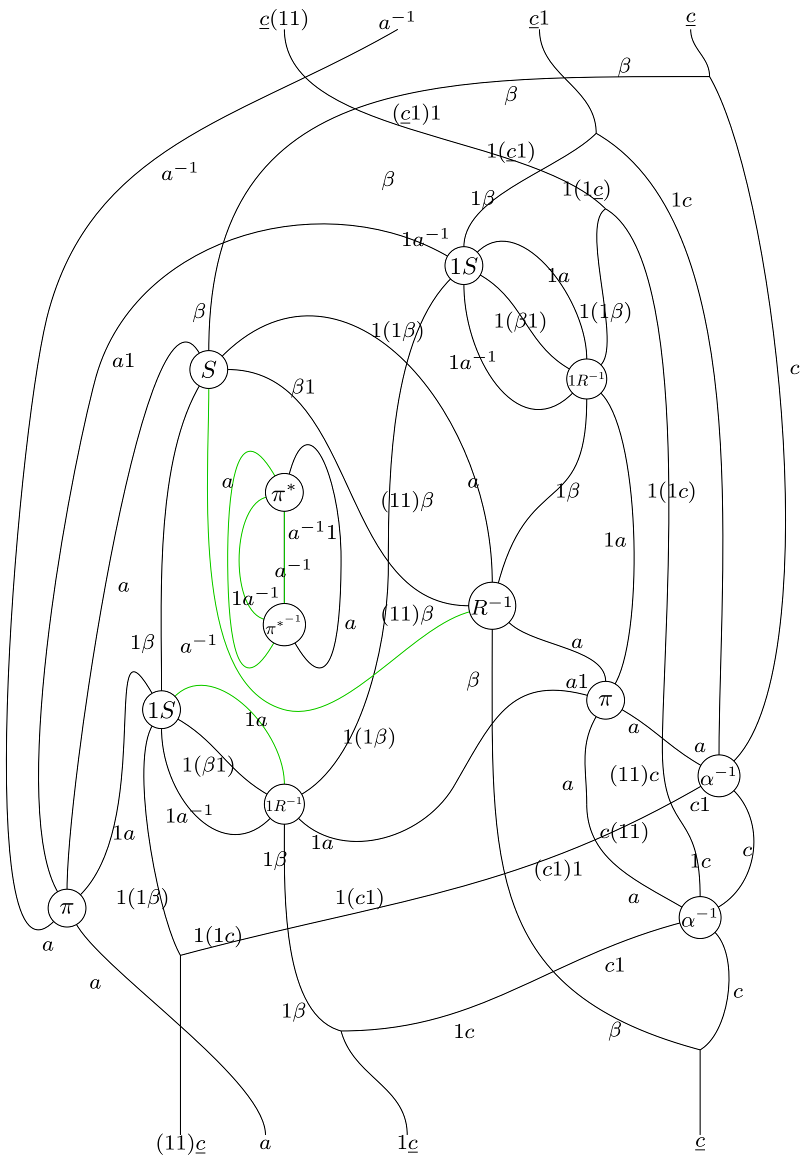
Then, by the modification property, we can slide the various combination of tensor pseudo-functors applied to the highlighted 1-cell under S, and find



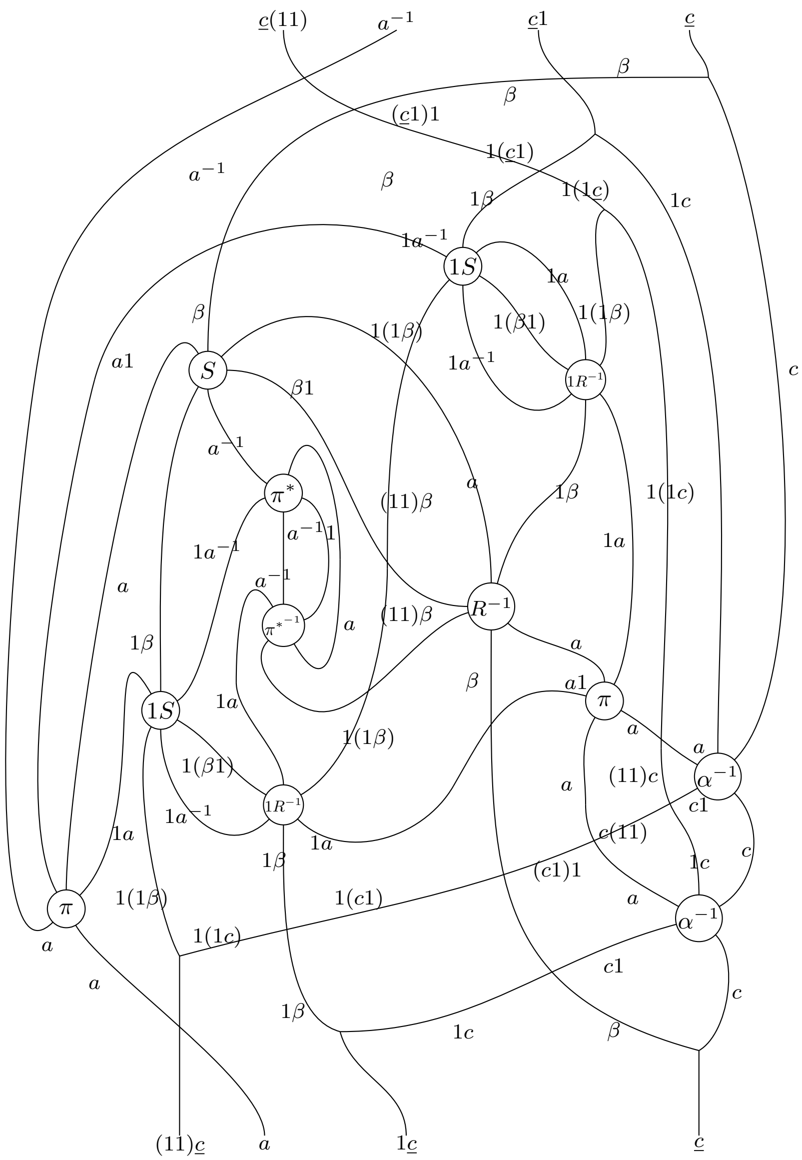
At this point, we aim to use the braiding axiom BA2. In order to do so, we need first to insert an identity in its form $\pi^* \circ \pi^{*-1}$, for π^* the mate



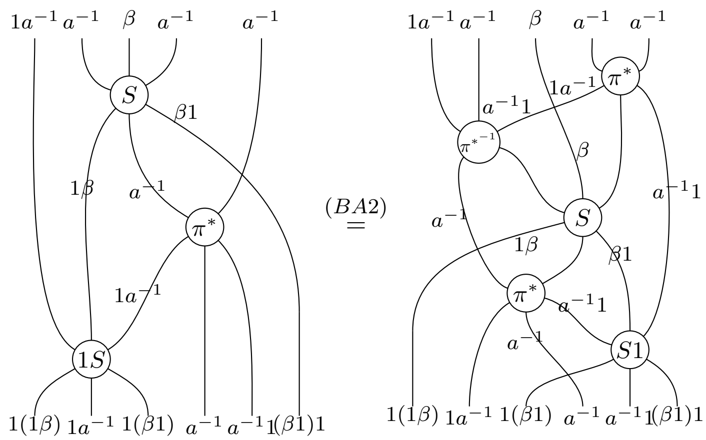
Observe that the second equality is a general fact for every invertible 2-cell between adjoint equivalences (Lemma 1.2.5). We get then the 2-cell



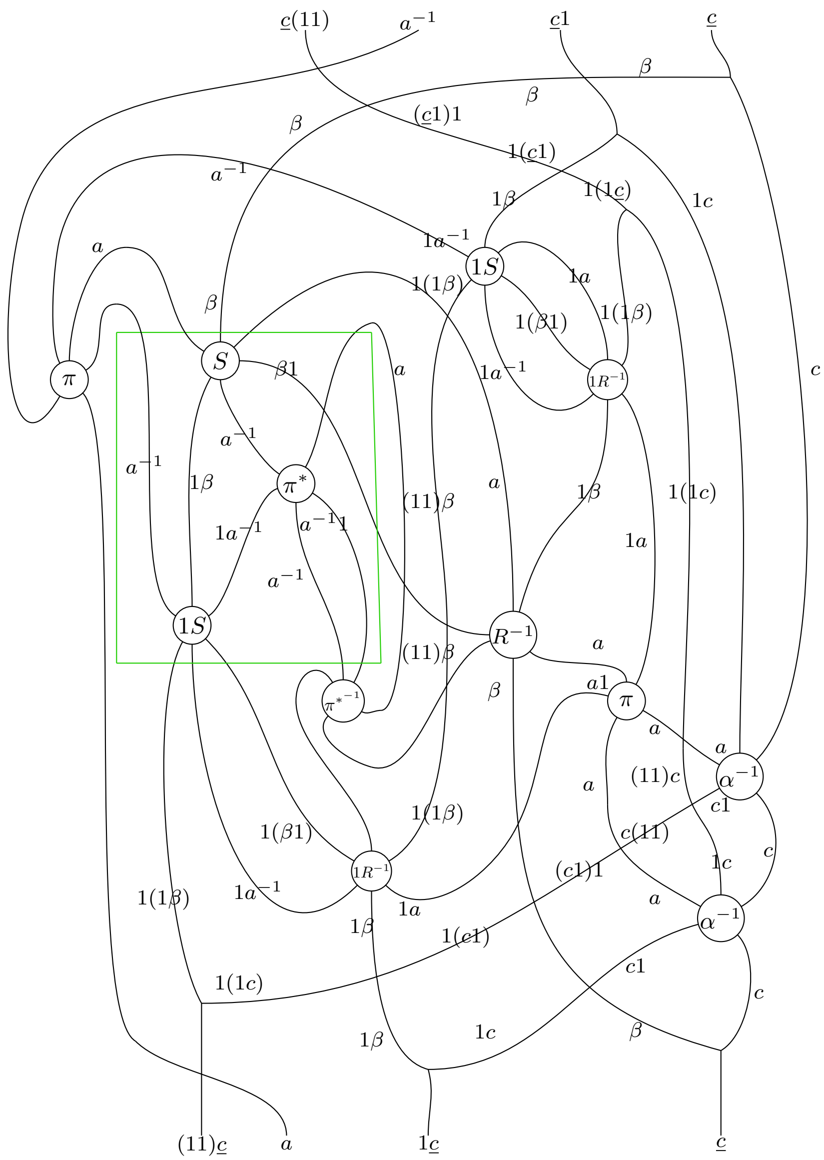
Observe that we can now link the pair of highlighted 1-cells, in order to get



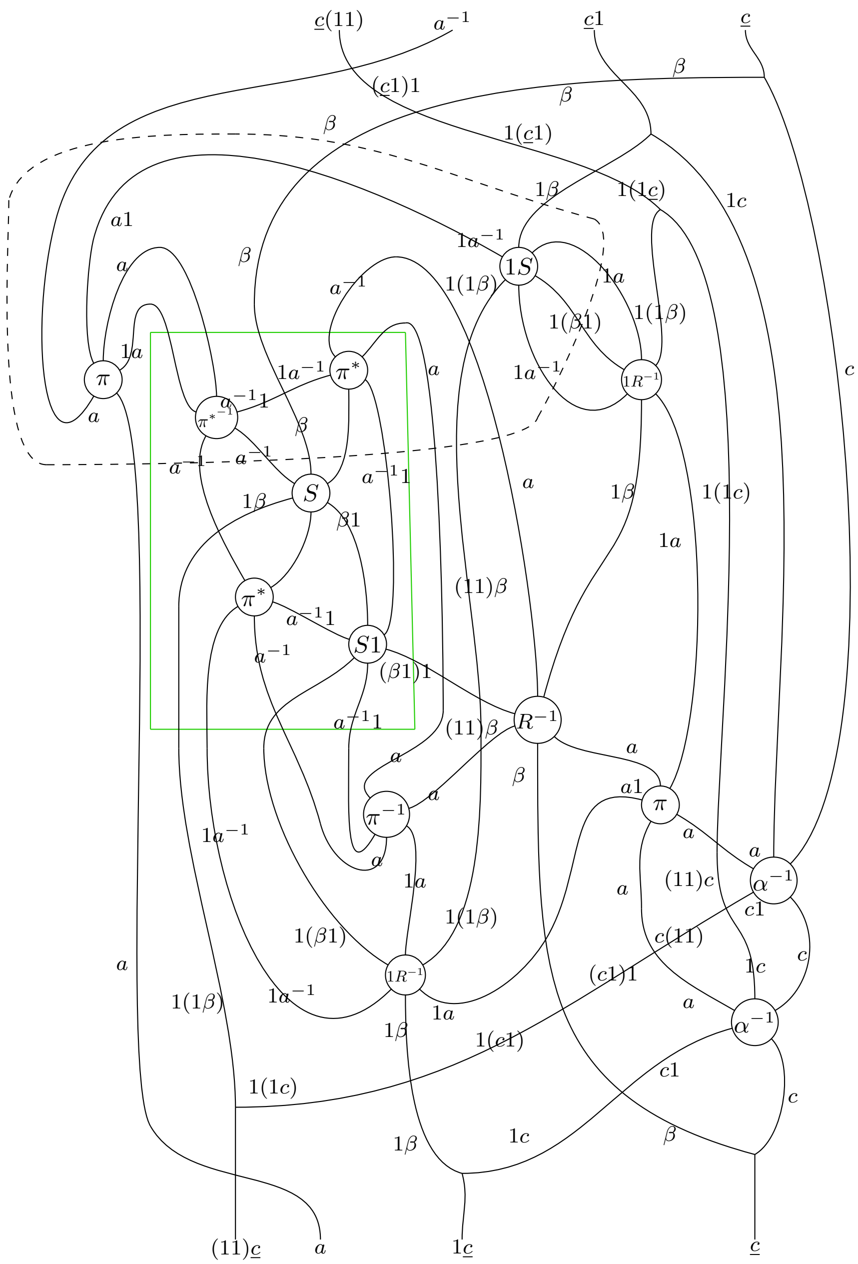
Then, we are going to rearrange the the terms, introducing also an overlap of the two highlighted morphisms. This will allow us to recognize the left hand side of BA2 in the following form:



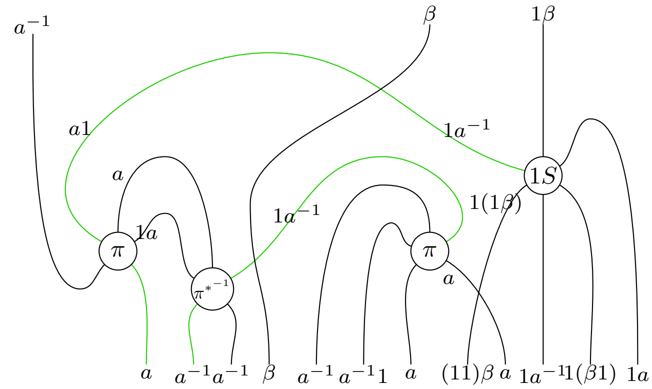
The rearranged 2-cell has then the following form



and via the above braiding axiom it becomes equal to the following. Let us, at the same time, rewrite the non-involved π^* in terms of π.



The next step is to simplify the upper left π and $\pi^{*^{-1}}$. Let us focus on the part enclosed by the dashed line of the diagram involving them, together with the π^* and $1S$ at their right. We can isolate this part of the diagram below and work on it separately, finding



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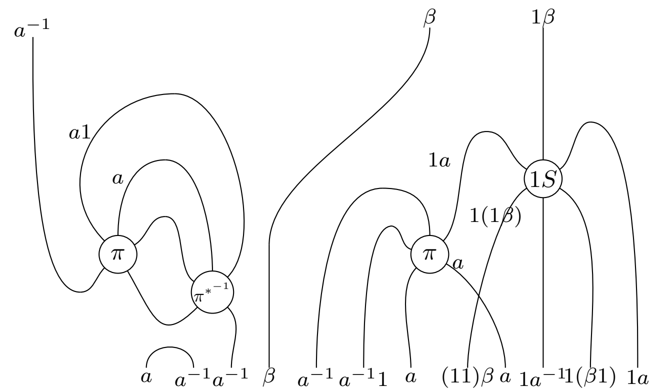

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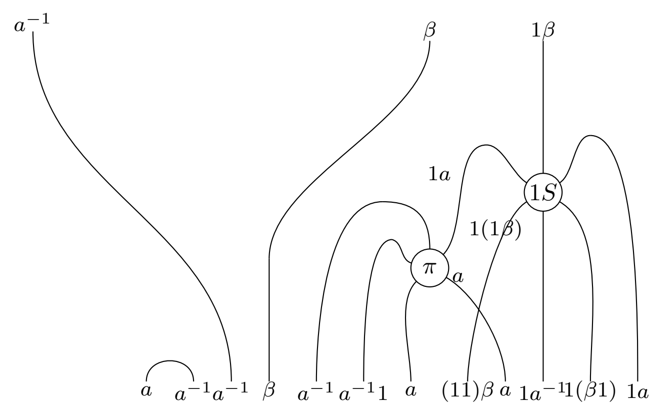
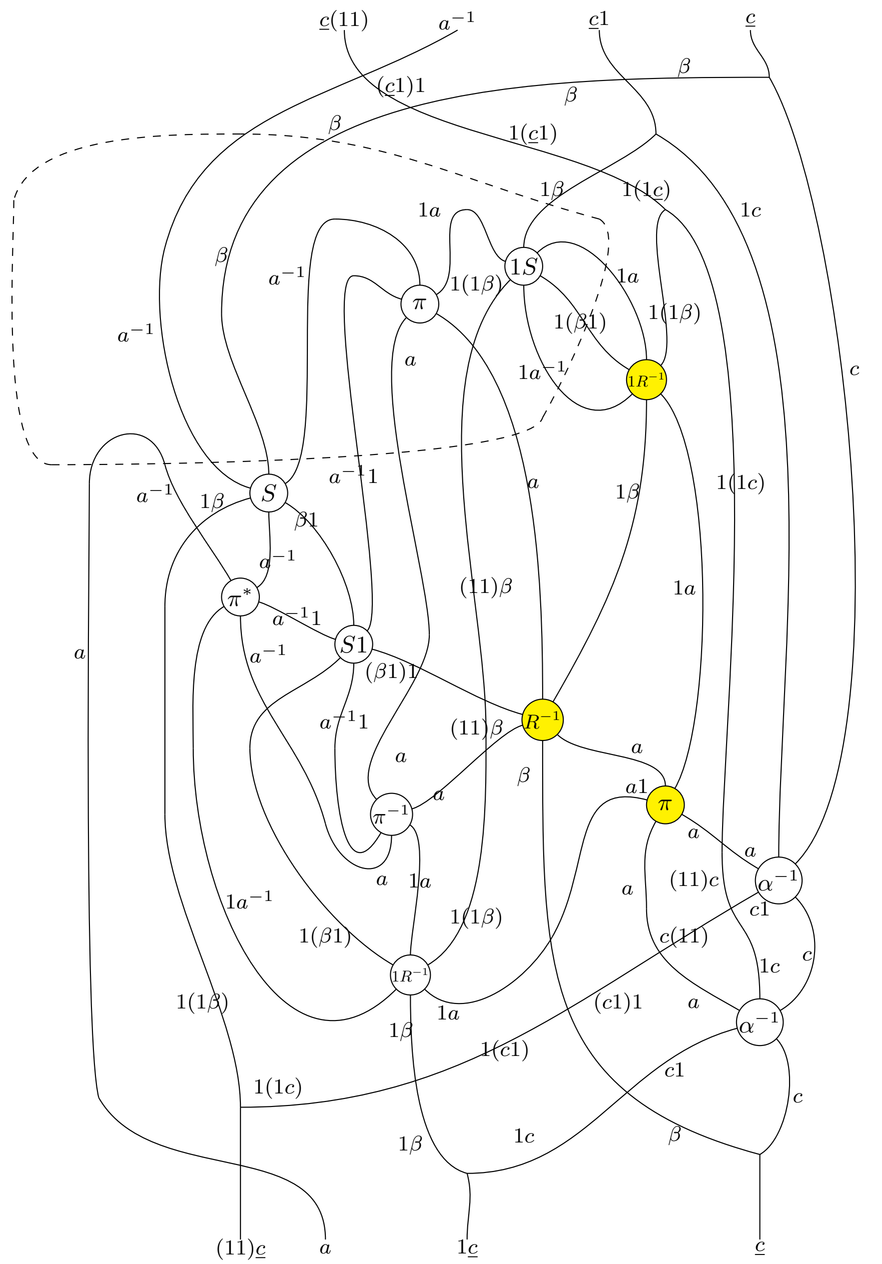
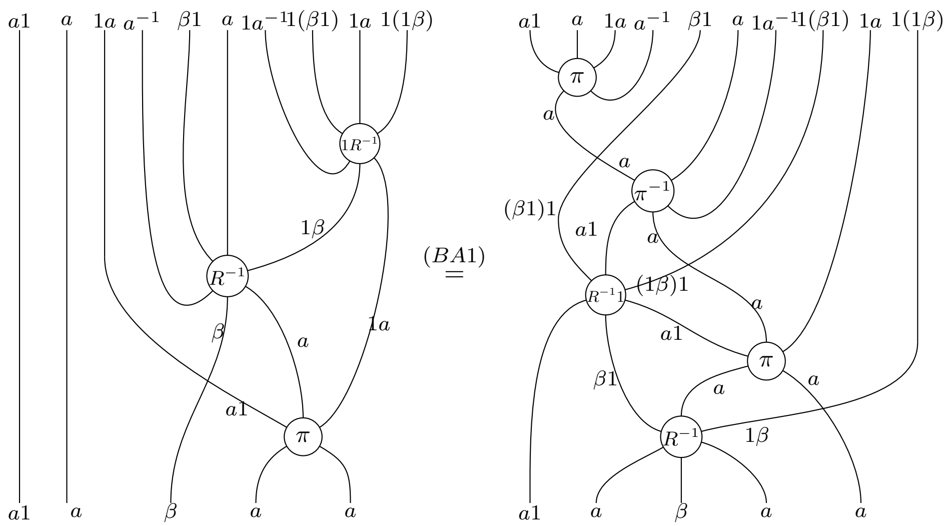


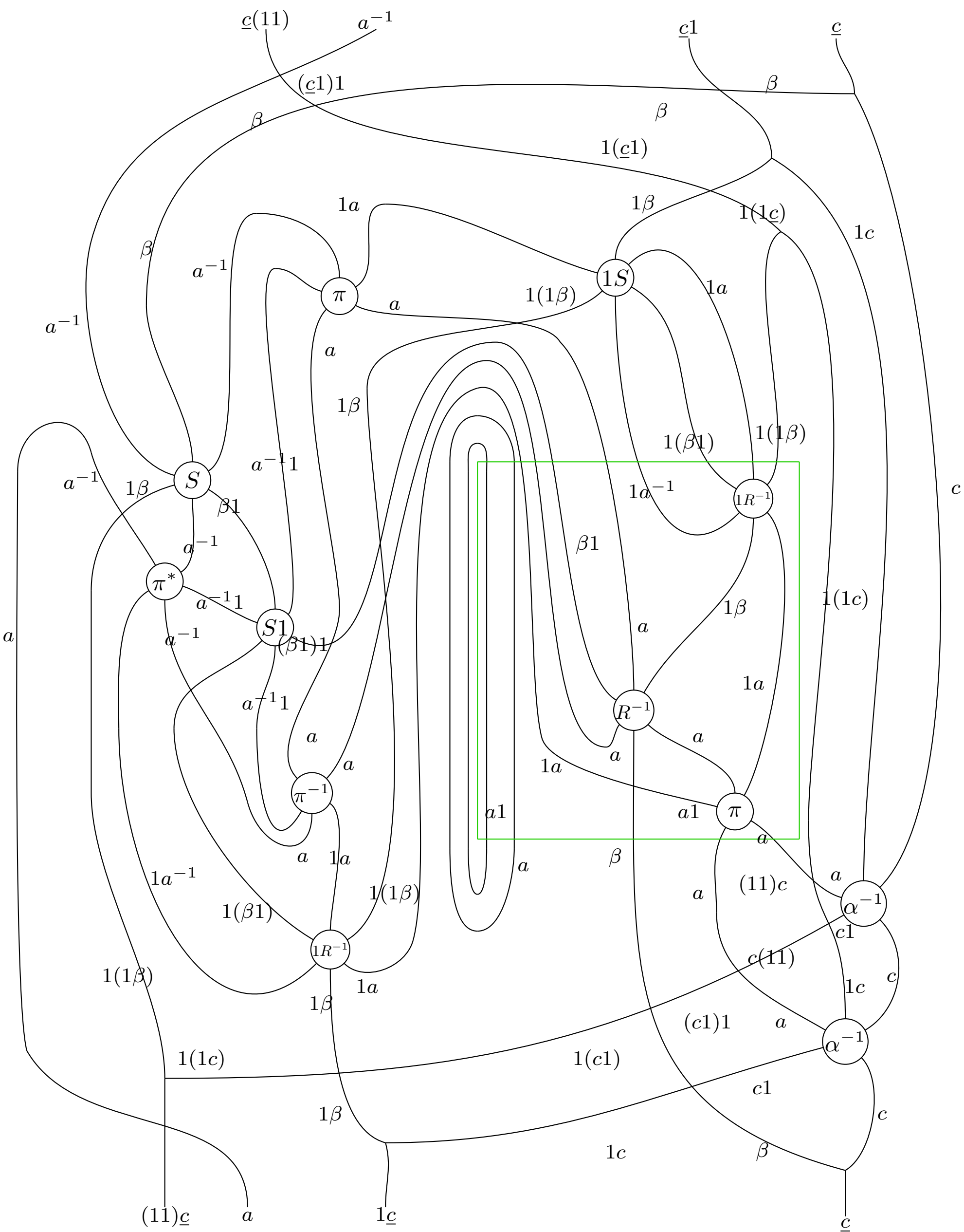
If we replace then the last result in the dashed area, we find then



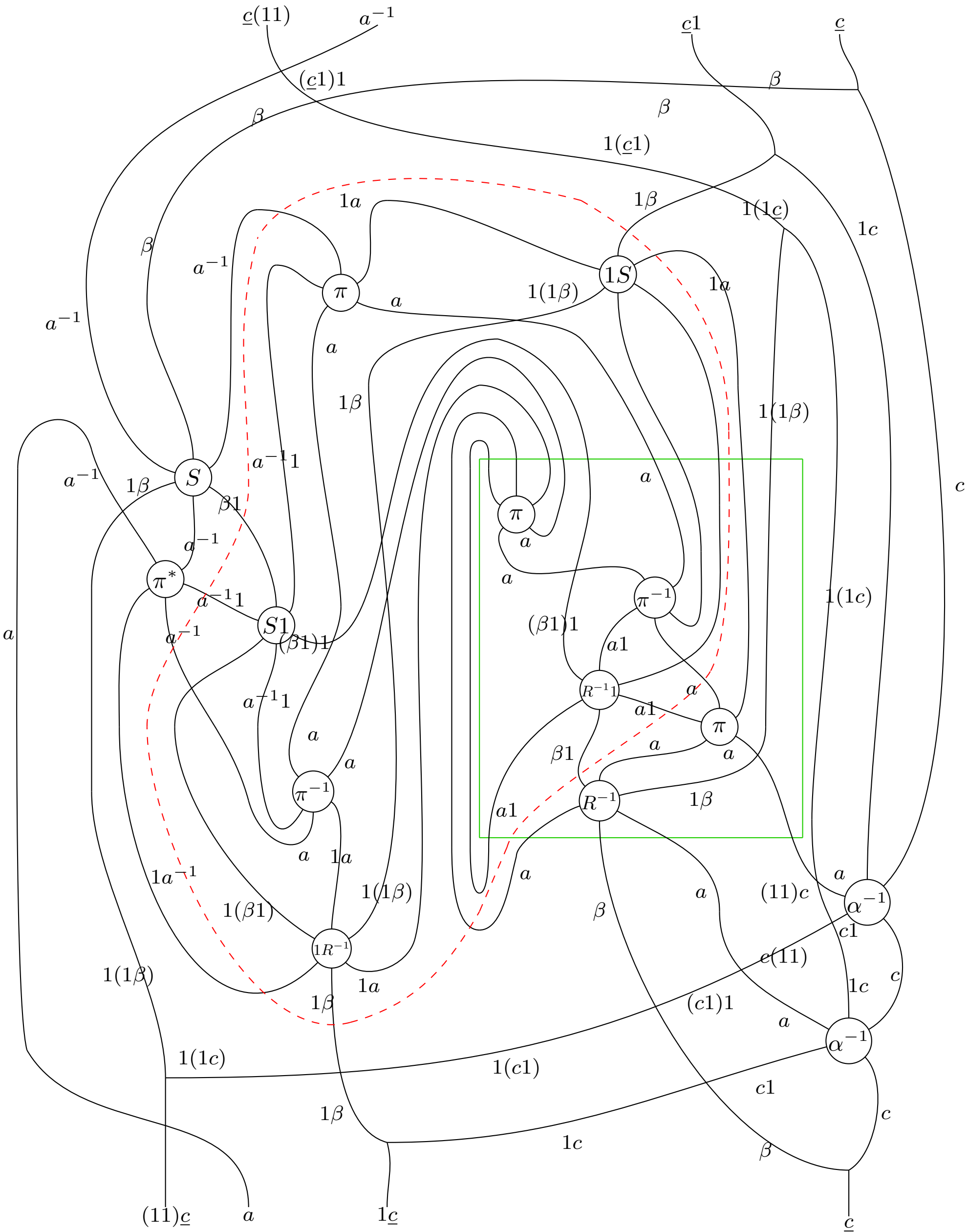
The next step is the use of axiom BA1 in its following form, where we can recognize the highlighted nodes in its left hand side:



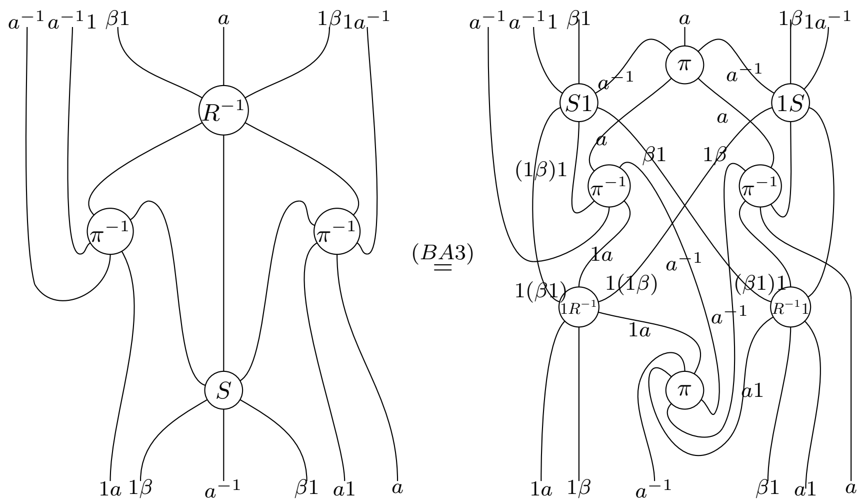
In order to do so, we introduce two identities in the form of nested loops, since we need $a1$ and a to appear. Also, we rearrange some strings, which is always possible by the triangular identities.



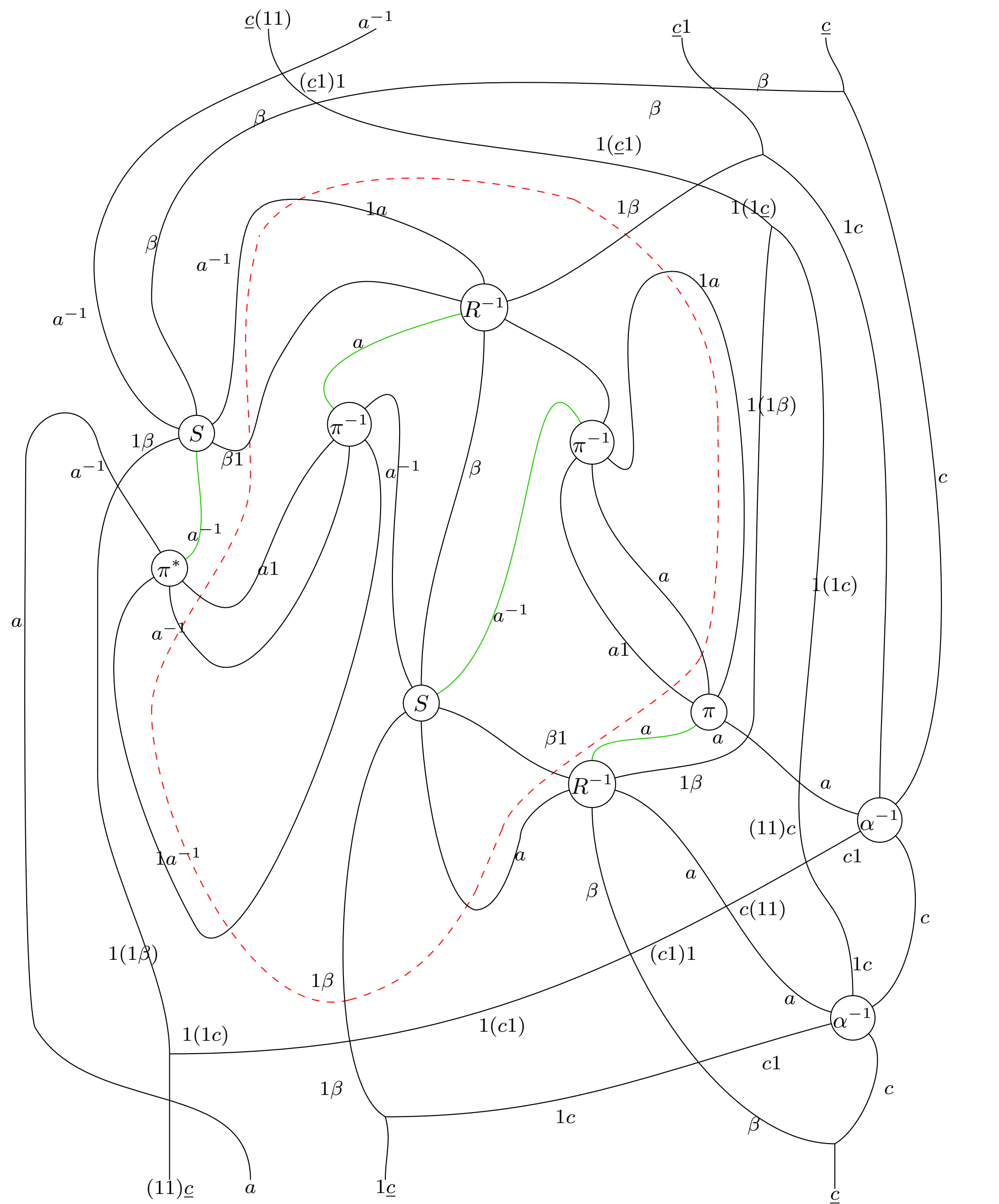
The diagram then becomes



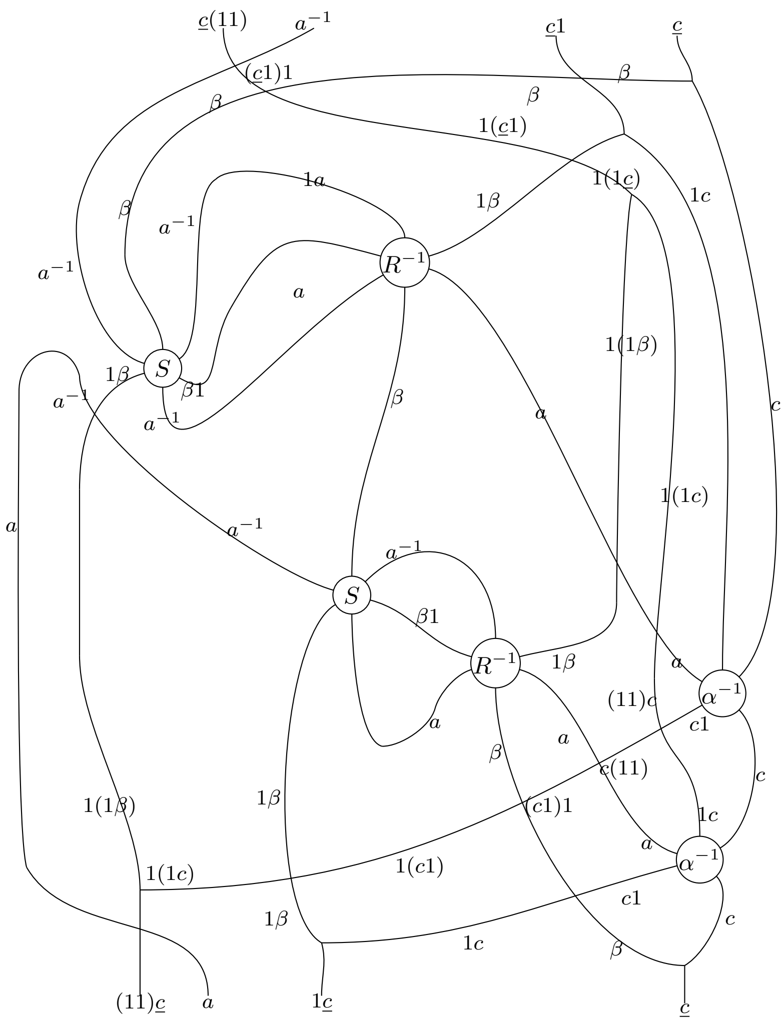
Please observe that there is at this stage, if we exclude, together with the α, α^{-1}, the two cells S and R^{-1}, a certain symmetry that may recall us of the right hand side of axiom BA3 in its following form



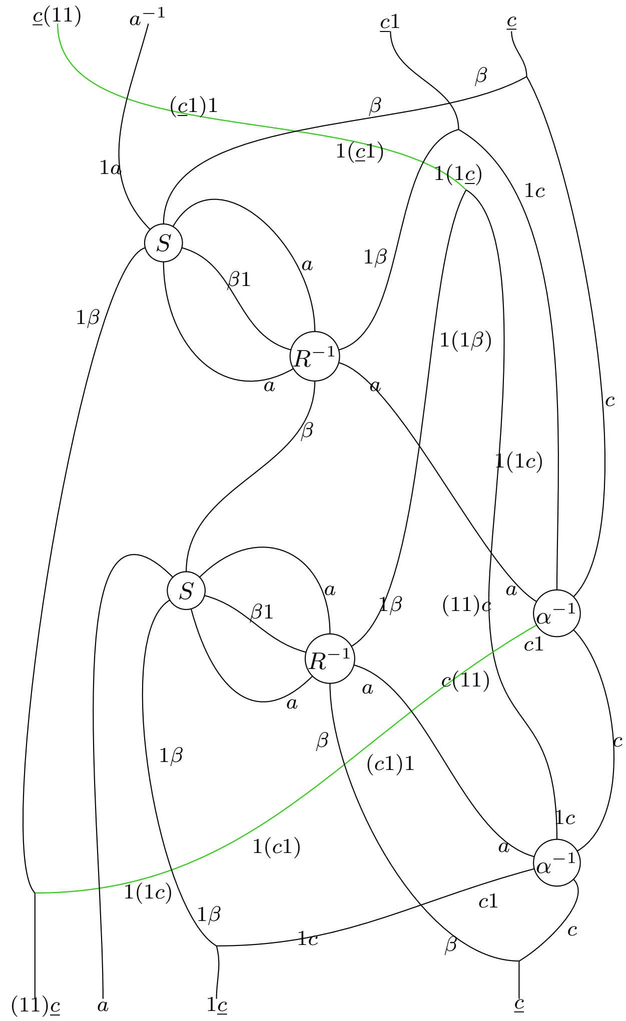
Indeed, the area delimited by the dashed line is precisely this term on the right, which can then be replaced by the much easier left hand side, giving:



Now, once the highlighted 1-morphisms are linked between them in pairs, all of the instances of π simplify two by two, since are each linked to its inverse, and what remains is



Eventually, with minor adjustments, this can be written as



and if we slide the upper and lower highlighted 1-morphisms respectively under and over the composites $R^{-1}S$, what we find is indeed the original right hand side of axiom (AC). $\square$

1.7 Tensor product of $\mathcal{V}$-bicategories

The second example of constructions allowed by the braiding is the tensor product of $\mathcal{V}$-bicategories.

Definition 1.7.1. Le $\mathcal{V}$ be a braided monoidal bicategory and $\mathcal{B}, \mathcal{C}$ be two $\mathcal{V}$-bicategories. Their product $\mathcal{B} \otimes \mathcal{C}$ is defined as having objects the class product $\mathsf{Ob}(\mathcal{B}) \times \mathsf{Ob}(\mathcal{C})$, and as hom-objects $\mathcal{B} \otimes \mathcal{C}((b, c), (b', c'))$ the monoidal product $\mathcal{B}(b, b') \otimes \mathcal{C}(c, c')$. The rest of the structure is defined in the proof of the Theorem 1.7.3 below.

Remark 1.7.2. In the following we will occasionally assume the strictification result for monoidal bicategories recalled in Section 1.5. In general, not every braided monoidal bicategory is biequivalent to a strict one, but it is to a semi-strict braided monoidal 2-category.

This allows us to suppose the monoidal structure a, ℓ, r to be identity 1-cells, and simplifies our computation a lot.

Theorem 1.7.3. *Let $\mathcal{V}$ be a braided monoidal bicategory, and let $\mathcal{B}, \mathcal{C}$ be $\mathcal{V}$-bicategories. Then, the tensor product of $\mathcal{V}$-bicategories $\mathcal{B} \otimes \mathcal{C}$ has a structure of a $\mathcal{V}$-bicategory.*

Proof. The structure is defined as follows: the unit is

$$\underline{u}\colon \mathbb{1} \xrightarrow{r_{\mathbb{1}}^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{u \otimes u} \mathcal{B}(b,b) \otimes \mathcal{C}(c,c).$$

The choice of $r_{\mathbb{1}}$ instead of $\ell_{\mathbb{1}}$ is simply conventional, the two are in fact isomorphic (Lemma 2.1 [GS15]). The composition is

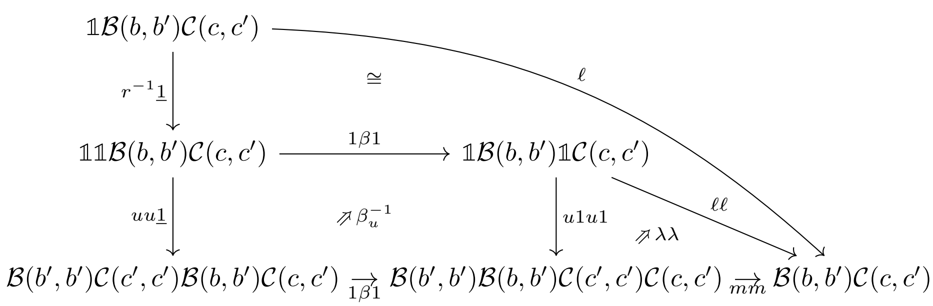
$$\underline{m}\colon \mathcal{B}(b',b'')\mathcal{C}(c',c'')\mathcal{B}(b,b')\mathcal{C}(c,c') \xrightarrow{1\beta 1} \mathcal{B}(b',b'')\mathcal{B}(b,b')\mathcal{C}(c',c'')\mathcal{C}(c,c') \xrightarrow{mm} \mathcal{B}(b,b'')\mathcal{C}(c,c'')$$

As said, this way of writing hides the presence of associators. More precisely, the morphism $\underline{m}$ is, without semi-strictification, defined to be the composite

$$\xrightarrow{a^{-1}} \quad \xrightarrow{a1} \quad \xrightarrow{(1\beta)1} \quad \xrightarrow{a^{-1}1} \quad \xrightarrow{a} \tag{1.4}$$

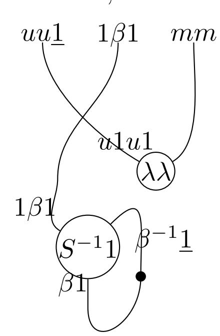
Let us come to the associator and the unitors. We will use an underlined 1 for the identity of a tensor product of two objects. This will be a coherent choice, since so far we used underlining for the new structure $(\underline{u}, \underline{m})$, and here, the identity of a hom-object of the tensor product of two $\mathcal{V}$-bicategories will in fact be a pair of tensored identities.

The left unitor is going to be a 2-cell of the form

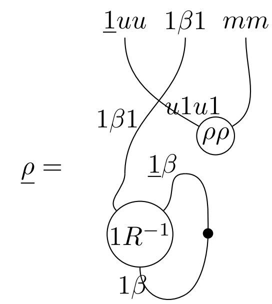


written, as a string diagram in the strict version, as

$\underline{\lambda} \;=$



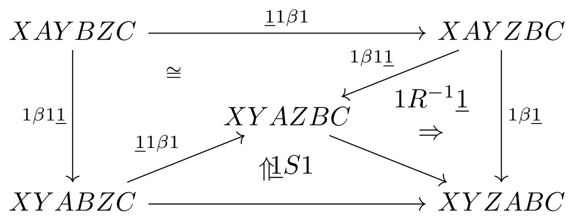
with the bullet standing for the appropriate monoidal unitor. In strict terms, the right unitor has the following definition as a string diagram



The associator $\underline{\alpha}$ is given by the following 2-cell.

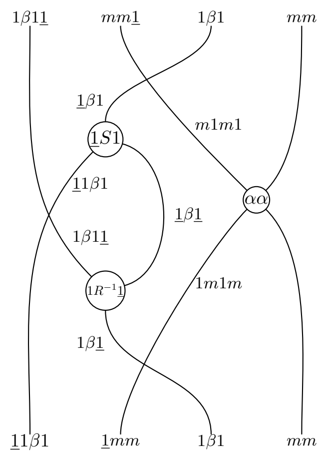
$$
\begin{array}{ccccc}
\mathcal{B}(c,d)\mathcal{C}(z,w)\mathcal{B}(b,c)\mathcal{C}(y,z)\mathcal{B}(a,b)\mathcal{C}(x,y) & \xrightarrow{\underline{1}1\beta 1} & \mathcal{B}(c,d)\mathcal{C}(z,w)\mathcal{B}(b,c)\mathcal{B}(a,b)\mathcal{C}(y,z)\mathcal{C}(x,y) & \xrightarrow{\underline{1}mm} & \mathcal{B}(c,d)\mathcal{C}(z,w)\mathcal{B}(a,c)\mathcal{C}(x,z) \\
\downarrow{\scriptstyle 1\beta 1\underline{1}} & \cong & \downarrow{\scriptstyle 1\beta\underline{1}} & \nearrow \beta_m & \downarrow{\scriptstyle 1\beta 1} \\
\mathcal{B}(c,d)\mathcal{B}(b,c)\mathcal{C}(z,w)\mathcal{C}(y,z)\mathcal{B}(a,b)\mathcal{C}(x,y) & \xrightarrow{\underline{1}\beta 1} & \mathcal{B}(c,d)\mathcal{B}(b,c)\mathcal{B}(a,b)\mathcal{C}(z,w)\mathcal{C}(y,z)\mathcal{C}(x,y) & \xrightarrow{1m1m} & \mathcal{B}(c,d)\mathcal{B}(a,c)\mathcal{C}(z,w)\mathcal{C}(x,z) \\
\downarrow{\scriptstyle mm\underline{1}} & \nearrow \beta_m^{-1} & \downarrow{\scriptstyle m1m1} & \nearrow \alpha\alpha & \downarrow{\scriptstyle mm} \\
\mathcal{B}(b,d)\mathcal{C}(y,w)\mathcal{B}(a,b)\mathcal{C}(x,y) & \xrightarrow{1\beta 1} & \mathcal{B}(b,d)\mathcal{B}(a,b)\mathcal{C}(y,w)\mathcal{C}(x,y) & \xrightarrow{mm} & \mathcal{B}(a,d)\mathcal{C}(x,w)
\end{array}
$$

The isomorphism on the top left corner (the one without a name) is given by the following. Let us use general letters replacing the hom-objects:

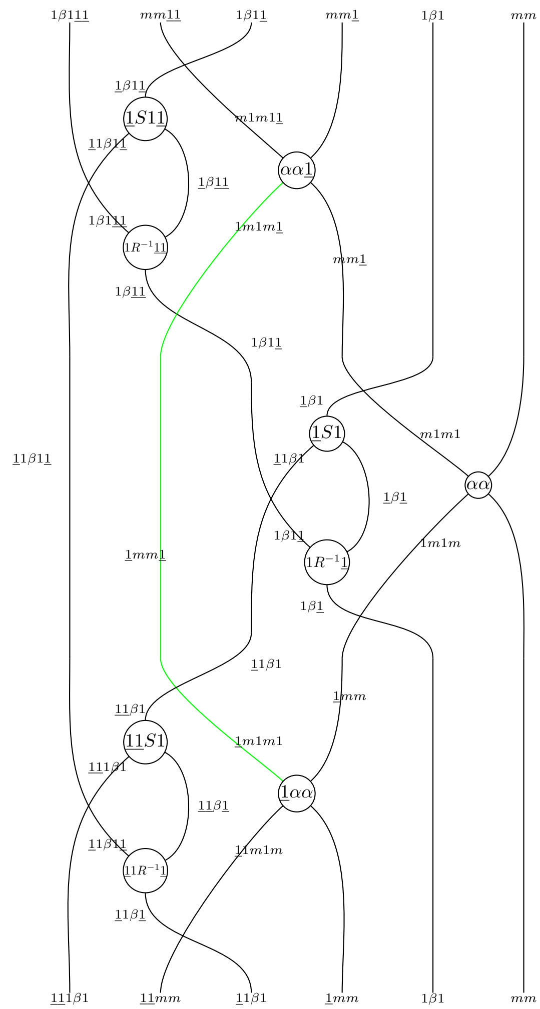


That means, in terms of string diagrams, we can pack it in the following:

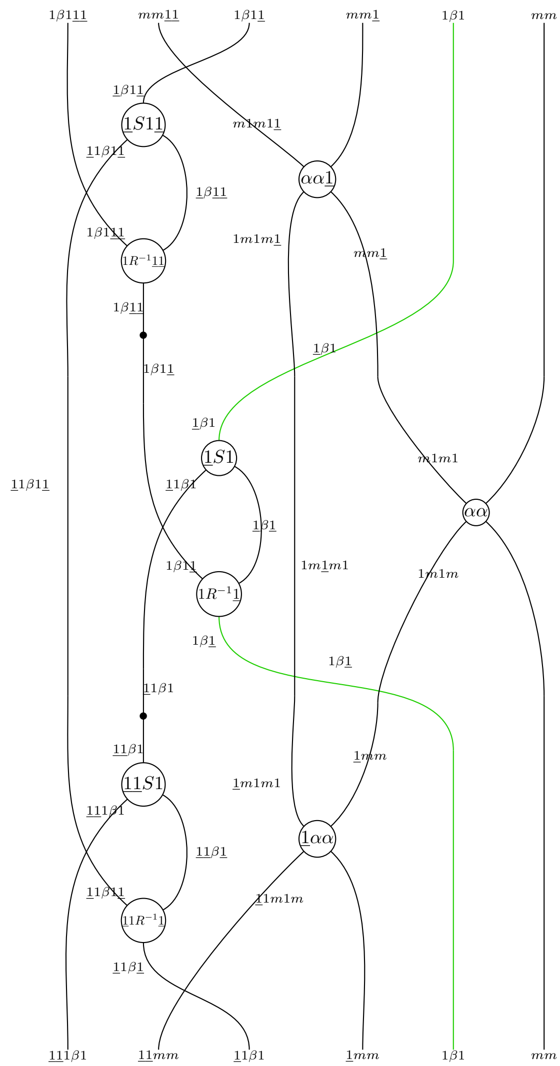
$\underline{\alpha} =$



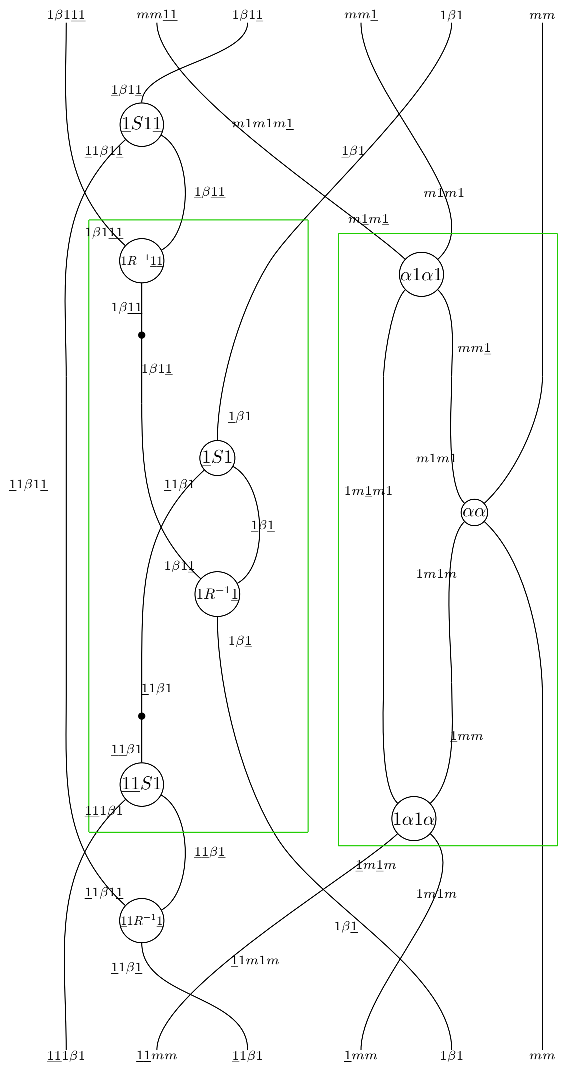
This finishes to set up the definition of the structure of $\mathcal{V}$-bicategory for $\mathcal{B} \otimes \mathcal{C}$. Let us now prove the axioms, starting with (AC). The left hand side is



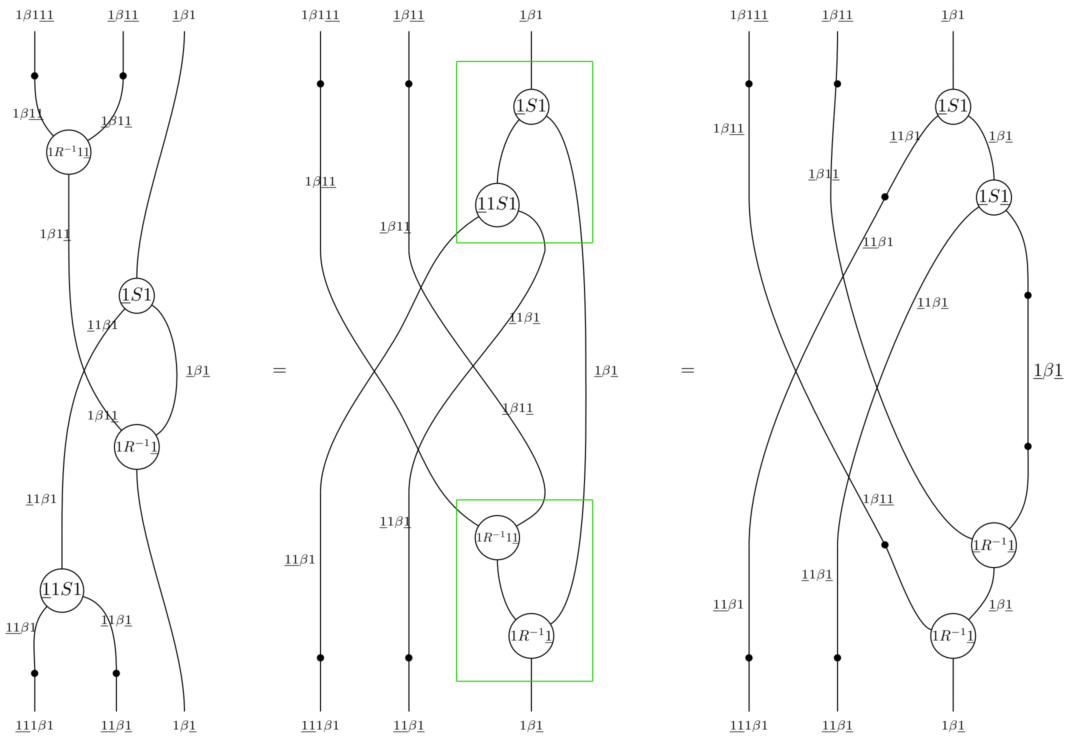
Let us consider the highlighted string $\underline{1}mm\underline{1}$ and slide it past the 2-cell on the right by using the modification property. The aim is to assemble all the instances of α, allowing a simultaneous use of axioms (AC) for $\mathcal{B}$ and $\mathcal{C}$. The above 2-cell becomes



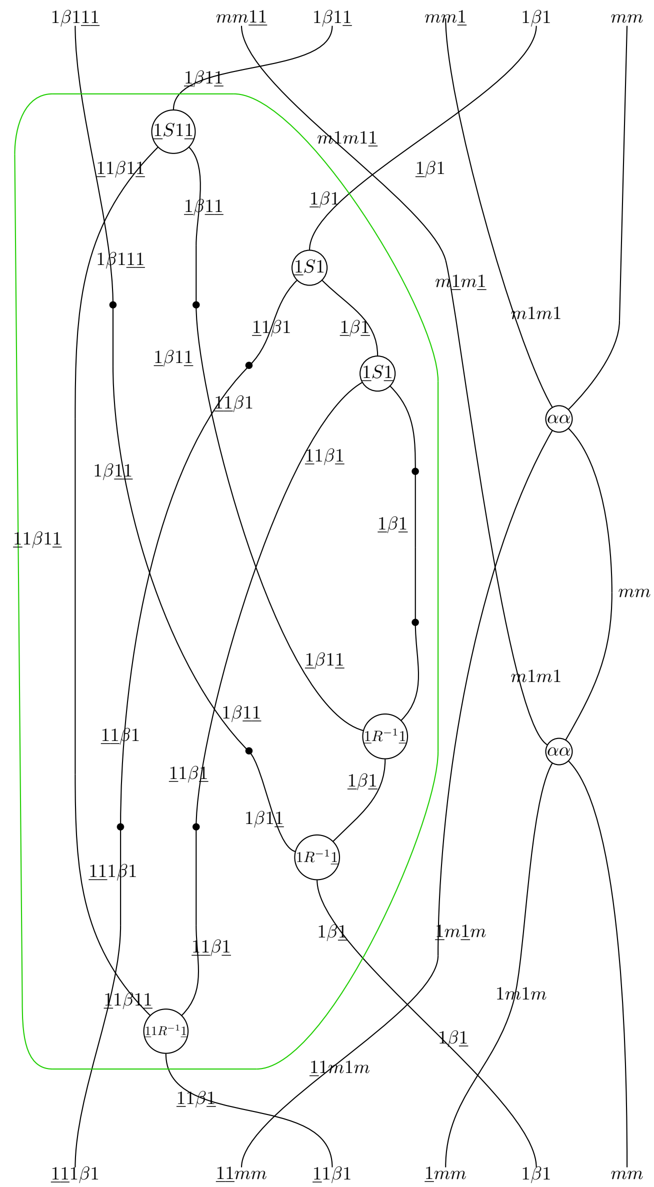
With the bullet here indicating the evident unitors. Then, we can slide $\underline{1}\beta 1$ over $\alpha\alpha\underline{1}$, and $1\beta\underline{1}$ under $\underline{1}\alpha\alpha$ by naturality of β, and find



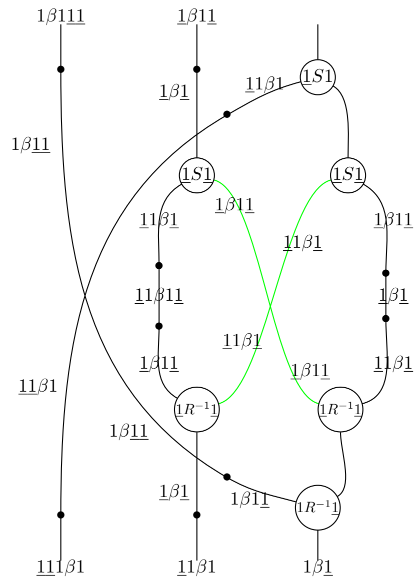
Now, we see that we can apply on the right the axioms of associativity coherence. Also, let us focus on the left highlighted area, which, by sliding the unitors, and applying axioms (BA2) and (BA1), become



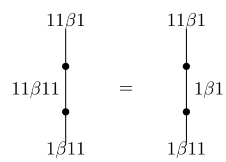
Then let us replace this, together with the other term of associativity coherence. This gives



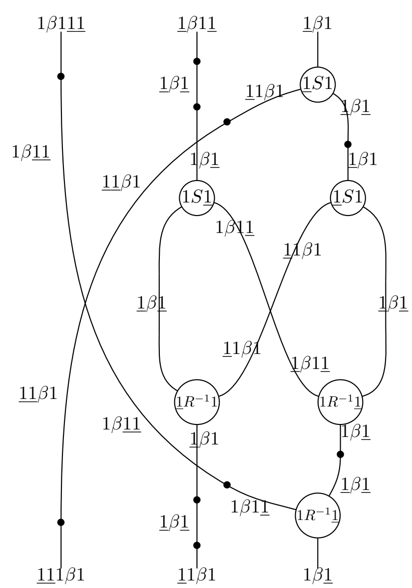
Then, let us focus on the highlighted part. We can introduce unitors on the external left string, in order to push it on top of $\underline{1}S1\underline{1}$ and at the bottom of $\underline{1}1R^{-1}\underline{1}$, we get



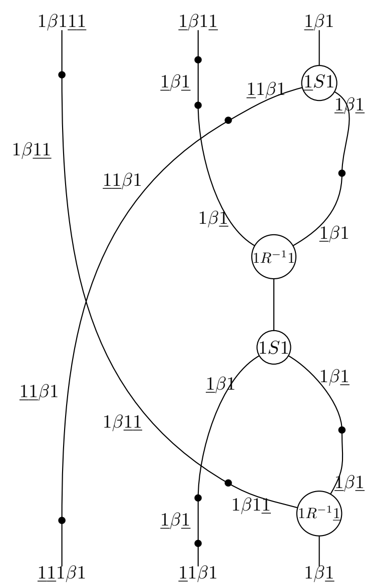
Then, we can first observe that there's an equality



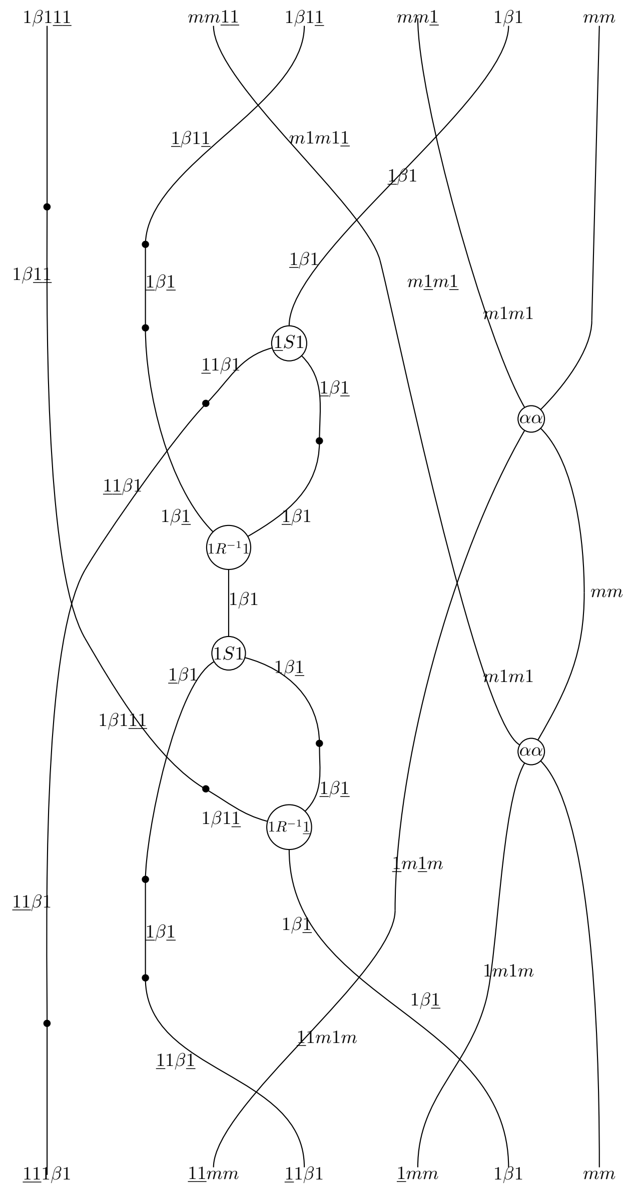
so that we can replace $\underline{1}1\beta1\underline{1}$ by $\underline{1}\beta\underline{1}$. Let us then insert two unitors on each of the two highlighted strings, and push them on top of the two $\underline{1}S\underline{1}$ and at the bottom of the two $\underline{1}R^{-1}\underline{1}$. What we get is then



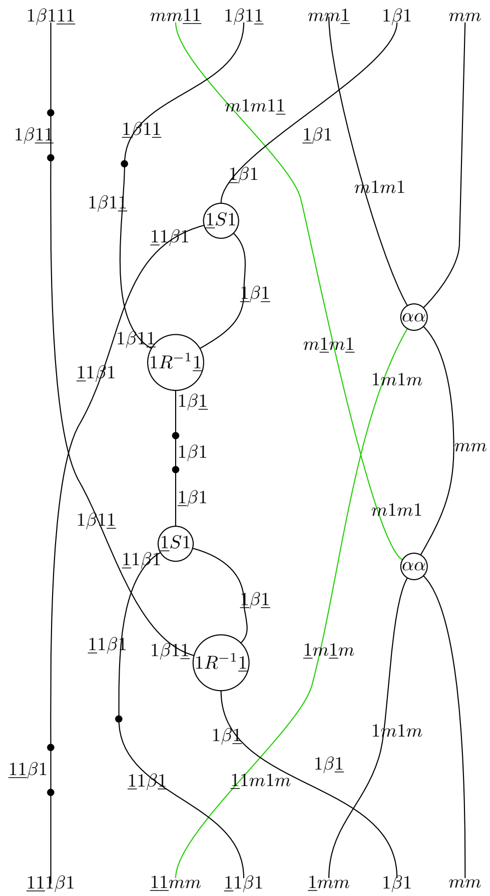
Now, by axiom (BA3), this turns out to be



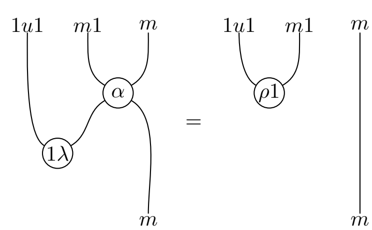
Then, if we replace this, we get



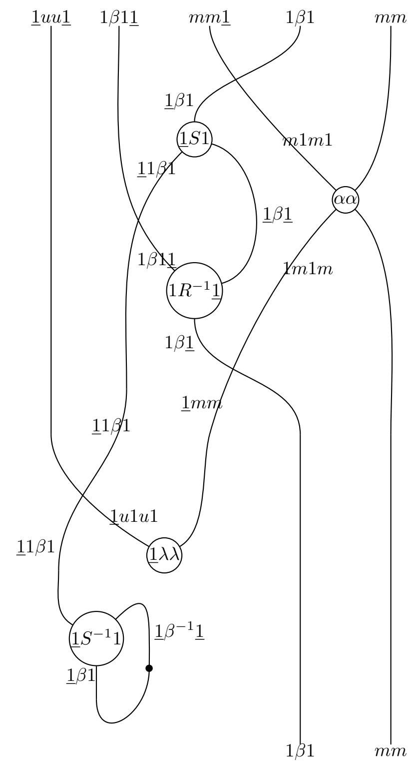
Then, we can again insert and push the unitors in order to find across the two 2-cells $1R^{-1}1$ and $1S1$, in order to get



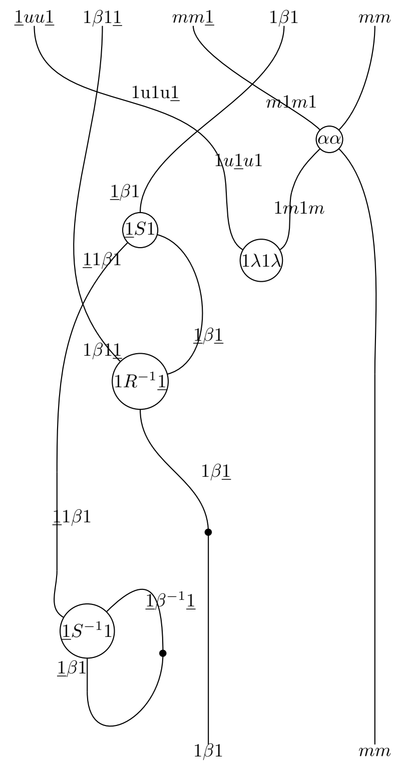
Eventually, sliding the two highlighted morphisms leads to the right hand side of axiom (AC) for $\mathcal{B} \otimes \mathcal{C}$, as desired. Let us now come to the identity coherence axiom, which under the assumption of working with semi-strict braided monoidal 2-categories, we recall, has the following form



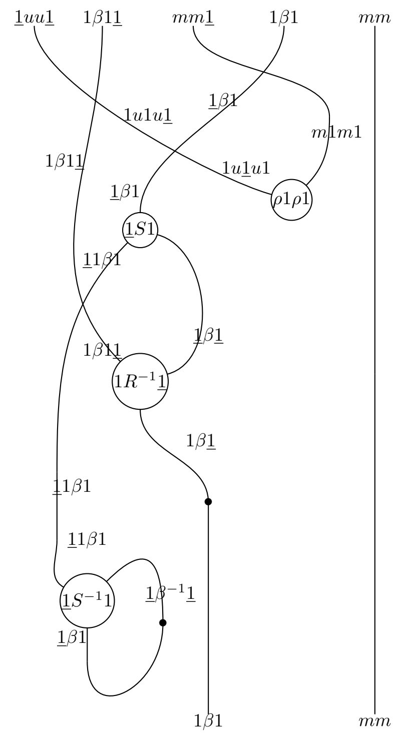
Then, our left hand side, by definition of $\underline{\lambda}$ and $\underline{\alpha}$ is



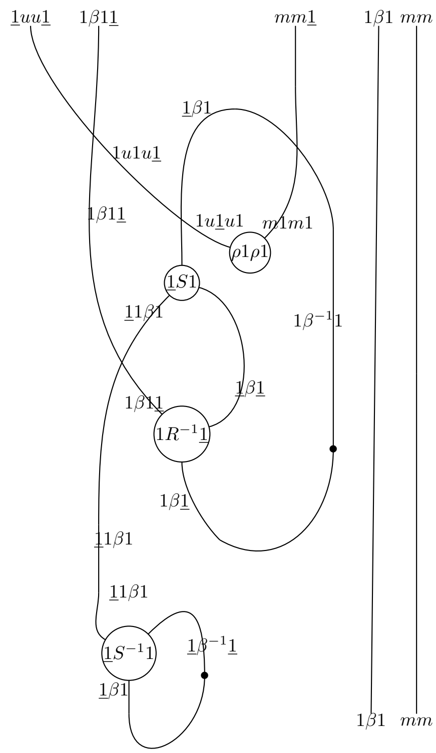
We can then slide $\underline{1}\lambda\lambda$ to the right of $1\beta 1$, by naturality, and it turns into $1\lambda 1\lambda$, and then move $1u\underline{1}u1$ on the left of the 2-cell formed by $1R^{-1}\underline{1}$ and $\underline{1}S1$. We find



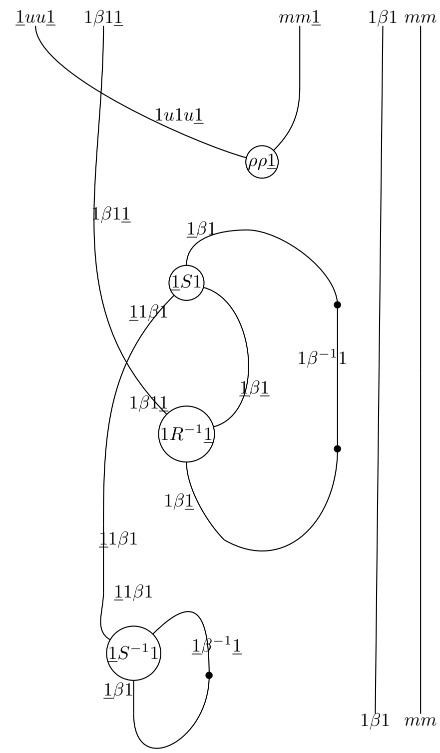
We recognize at this point the possibility to use (IC) for both $\mathcal{B}$ and $\mathcal{C}$ and get



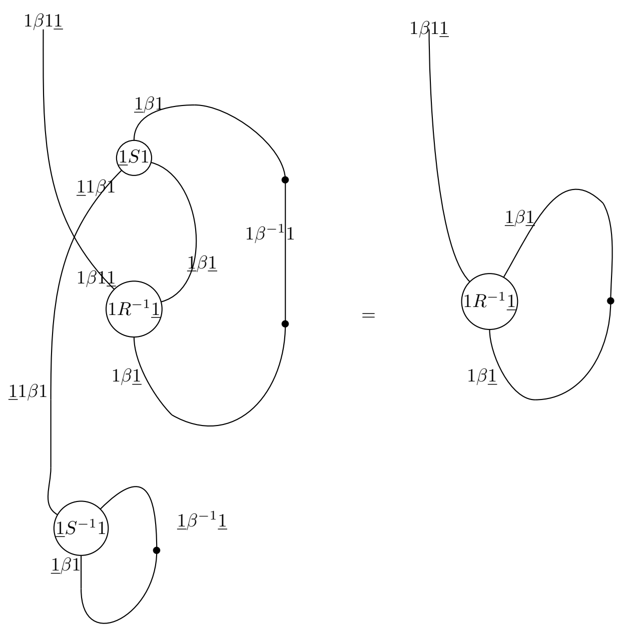
At this point we can link the two instances of $\underline{1}\beta 1$ as follows,



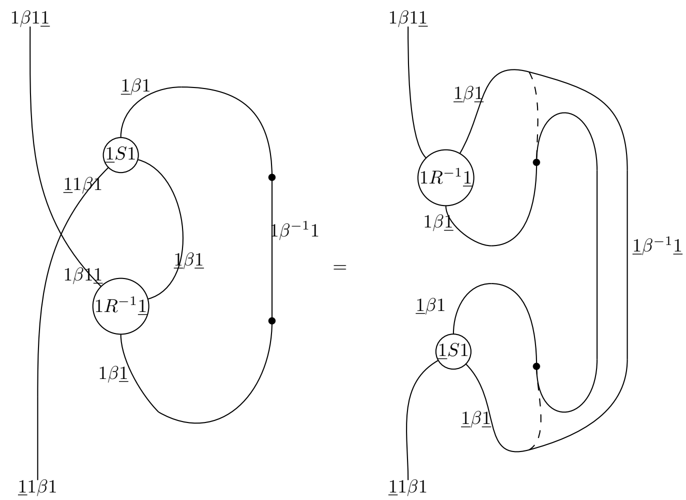
It suffices then to let $1\rho 1\rho$ to slide over $\underline{1}\beta 1$. This turns the 2-cell to



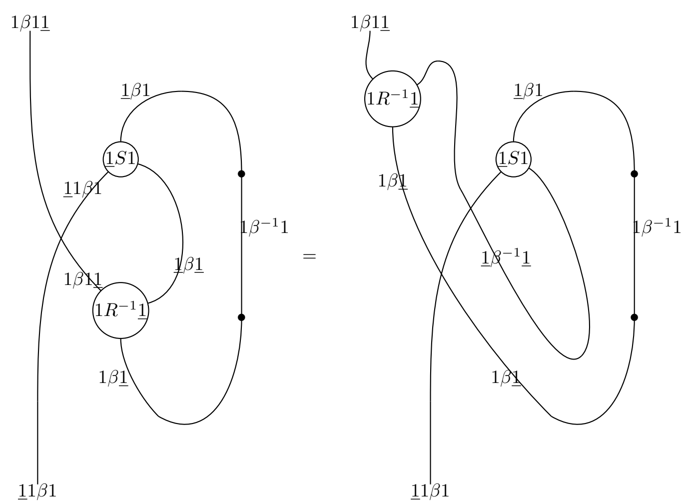
Eventually, we can argue that



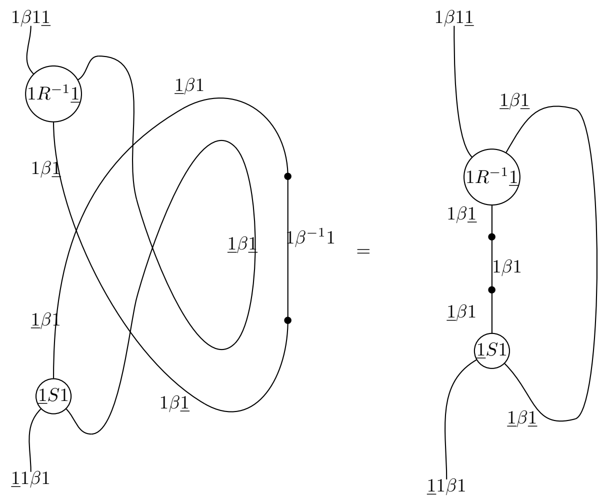
and this will conclude the proof, since the result is then $\underline{\rho}\underline{1}$ whiskered with $\underline{m}$, *i.e.* the right hand side of (IC). The last claim follows in fact by the following equality, where the dashed part has been loosen and the corresponding wires have been linked.



The former two 2-cells are then easily seen to be equal by applying the modification property and then simplifying as follows:



which eventually is



□

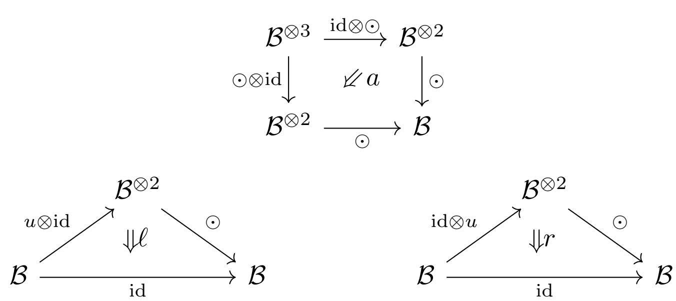
Remark 1.7.4. It is an interesting phenomenon the fact that for both the case of the opposite $\mathcal{V}$-bicategory and the case of the tensor product of $\mathcal{V}$-bicategories, we have the following. On one hand, the proof of the identity coherence does not require the use of the braiding axioms, but it is crucial for the braiding to be an equivalence. On the other hand, the associativity coherence make a heavy use of braiding axioms, but the argument would be perfectly fine even if β were not invertible. This could reflect a more general pattern that is probably worth studying in more detail.

1.8 Monoidal enriched bicategories

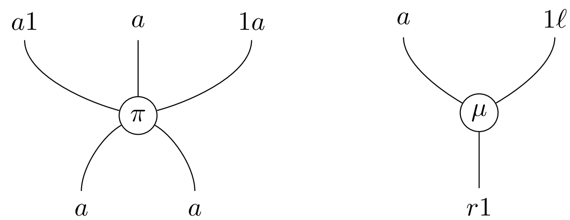
The following definition generalize in a straightforward manner the definition of monoidal bicategory (see Definition 1.1.8) in that it defines a monoidal $\mathcal{V}$-bicategory as a "generalized" monoid structure on a $\mathcal{V}$-bicategory. For that, it is fundamental the notion of tensor product of $\mathcal{V}$-bicategories explored in Section 1.7.

Definition 1.8.1. A *monoidal structure* on a $\mathcal{V}$-bicategory $\mathcal{B}$ is the data of

- $\odot\colon \mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$ and $u\colon \mathcal{J}\to\mathcal{B}$ tensor and unit $\mathcal{V}$-pseudofunctors
- a, ℓ, r adjoint equivalences in the bicategories $\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{B}^{\otimes 3},\mathcal{B})$ and $\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{B},\mathcal{B})$

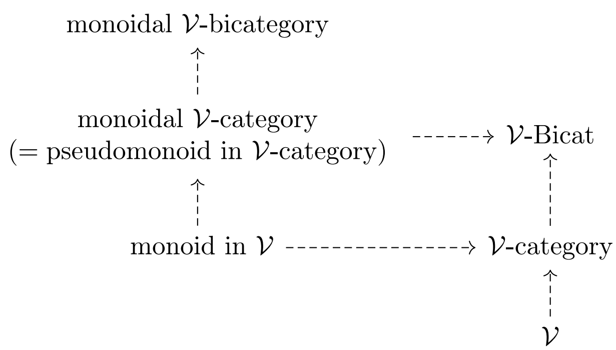


- Invertible $\mathcal{V}$-modifications π,μ



satisfying the non-abelian 4-cocycle condition 1.1.9.

Remark 1.8.2. The picture sketched in the following diagram (where the dashed arrows stand for "categorifies to", horizontally and vertically)



allows us to observe how the definition of monoidal structure on a ($\mathcal{V}$-)bicategory should directly generalize to the appropriate notion of *pseudomonoid* in whatever should be the correct definition of a monoidal tricategory. That is, the structure provided by tensor, unit, a, ℓ, r and π, μ, seen as cells of what a monoidal tricategory would be - together with the non-abelian 4-cocycle condition - define the second (or third, depending on where we start the counting) order categorification of the notion of monoid object.

Chapter 2

Enriched bi(co)ends

Ends and coends are universal objects associated to a functor of type $F\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{D}$. The most essential explanation for the importance of these objects is related to the ubiquity of functors of this sort.

The deeper motivation for the study of these objects, is that functors of this sort generalize modules (profunctors), and hence their study is at least as fruitful in understanding categories as the study of module theory is in understanding rings. From this viewpoint, ends and coends generalize homs and tensor products of modules, two essential notions of all linear algebra.

A first bicategorical treatment of the theory of ends and coends can be found in [Cor16]. The theory developed below reduces to its case if the base monoidal bicategory is Cat with its cartesian monoidal structure.

In this chapter, in particular, we will use all the time the opposite construction and the tensor product of enriched bicategories.

2.1 Extra-pseudonaturality

In this section we introduce the enriched version of the notion of extra-pseudonatural transformation between $\mathcal{V}$-pseudofunctors P, Q of sort

$$P\colon \mathcal{E} \otimes \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \longrightarrow \mathcal{D}$$
$$Q\colon \mathcal{E} \otimes \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C} \longrightarrow \mathcal{D}.$$

The non-enriched case was introduced in [Cor16]. Axioms that we are going to deal with are a lot, but still they are reduced in size by passing to the enriched context, just as it happens for usual pseudonatural transformations. Moreover, by keeping in mind that for every axiom (concerning $\mathcal{B}$) we have a symmetric one (concerning $\mathcal{C}$), everything reduces to keeping track of *unitality*, *functoriality* (as for pseudonatural transformations) and *compatibility* with parameters. What we get for free in the enriched context is *naturality*. We specify at the outset that because of the considerable space that writing the axioms for such structures requires, we will often use intuitive (and frequent in literature) abbreviations, such as $P_{ebb'}$ to mean $P(e, b, b')$ (and the same for Q), or omitting to explicitly write the parameter e if it happens to not being crucially involved in a specific diagram.

Definition 2.1.1. Let $P\colon \mathcal{E}\otimes\mathcal{B}^{\mathsf{op}}\otimes\mathcal{B}\to\mathcal{D}$ and $Q\colon \mathcal{E}\otimes\mathcal{C}^{\mathsf{op}}\otimes\mathcal{C}\to\mathcal{D}$ be $\mathcal{V}$-pseudofunctors. An (enriched) *extra-pseudonatural transformation* from P to Q, denoted $i\colon P \overset{..}{\Rightarrow} Q$, consists of, for every pair of objects b in $\mathcal{B}$, c in $\mathcal{C}$,

- An enriched pseudonatural transformation

$$i_{-,b,c}\colon P(-,b,b)\to Q(-,c,c)$$

- Two 2-isomorphisms in $\mathcal{V}$

$$\begin{tikzcd} \mathcal{B}(b,b') \arrow[r, "P(e,-,b)"] \arrow[d, "P(e,b',-)"'] & \mathcal{D}(P(e,b',b),P(e,b,b)) \arrow[d, "(i_{e,b,c})_*"] \arrow[dl, phantom, "\swarrow i_{e,bb',c}"] \\ \mathcal{D}(P(e,b',b),P(e,b',b')) \arrow[r, "(i_{e,b',c})_*"'] & \mathcal{D}(P(e,b',b),Q(e,c,c)) \end{tikzcd} \tag{2.1}$$

and

$$\begin{tikzcd} \mathcal{C}(c,c') \arrow[r, "Q(e,-,c')"] \arrow[d, "Q(e,c,-)"'] & \mathcal{D}(Q(e,c',c'),Q(e,c,c')) \arrow[d, "(i_{e,b,c'})^*"] \arrow[dl, phantom, "\swarrow i_{e,b,cc'}"] \\ \mathcal{D}(Q(e,c,c),Q(e,c,c')) \arrow[r, "(i_{e,b,c})^*"'] & \mathcal{D}(P(e,b,b),Q(e,c,c')) \end{tikzcd} \tag{2.2}$$

satisfying:

unitality, saying that for every triple of objects e,b,c it holds

$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "u"'] \arrow[dr, "u"] & \\ \mathcal{D}(P(e,b,b),P(e,b,b)) \arrow[dr, "i_*"'] & = & \mathcal{D}(P(e,b,b),P(e,b,b)) \arrow[dl, "i_*"] \\ & \mathcal{D}(P(e,b,b),Q(e,c,c)) & \end{tikzcd}$$

$$\overset{(EU1)}{=}$$

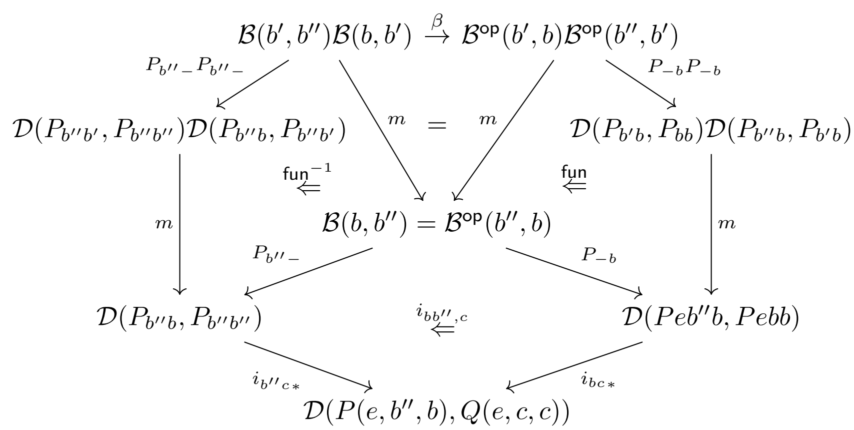
$$\begin{tikzcd} & \mathbb{1} \arrow[dl, "u"'] \arrow[d, "u"] \arrow[dr, "u"] & \\ \Uparrow \mathsf{un}^{-1} & \mathcal{B}(b,b) \arrow[dl, "P(e,b,-)"'] \arrow[dr, "P(e,-,b)"] & \Downarrow \mathsf{un} \\ \mathcal{D}(P(e,b,b),P(e,b,b)) \arrow[dr, "i_*"'] & \overset{i_{e,bb,c}}{\Leftarrow} & \mathcal{D}(P(e,b,b),P(e,b,b)) \arrow[dl, "i_*"] \\ & \mathcal{D}(P(e,b,b),Q(e,c,c)) & \end{tikzcd}$$

which can shortly be written, together with the analogue for $\mathcal{C}$, as

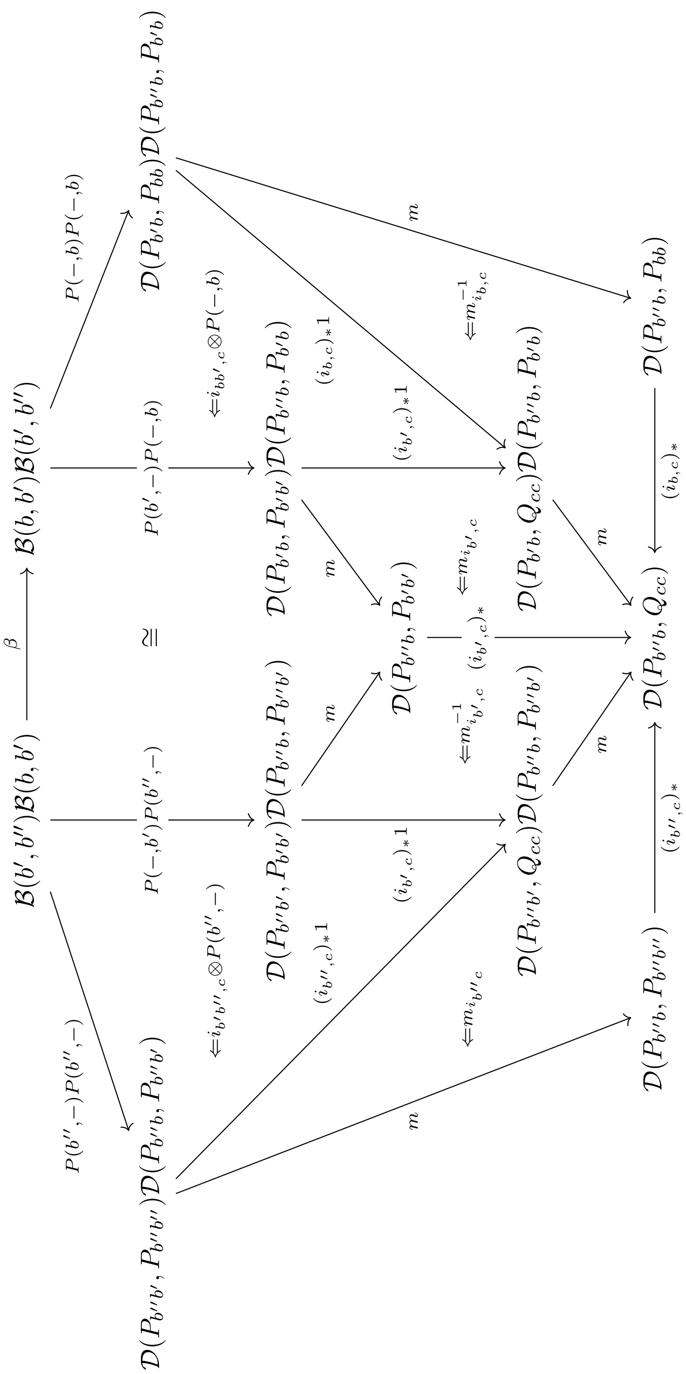
$$\mathrm{id}_{i_{e,b,c_*} u_{P(e,b,b)}} = \mathsf{un}^{-1} i_{e,bb,c} \mathsf{un}, \tag{EU1}$$

$$\mathrm{id}_{i_{e,b,c^*} u_{Q(e,c,c)}} = \mathsf{un}^{-1} i_{e,b,cc} \mathsf{un}. \tag{EU2}$$

functoriality, expressing how $i_{bb'',c}$ relates to $i_{bb',c}$ and $i_{b'b'',c}$ for every triple of objects b, b', b'' in $\mathcal{B}$ (and similarly for objects in $\mathcal{C}$) by saying that



is equal to the following 2-cell

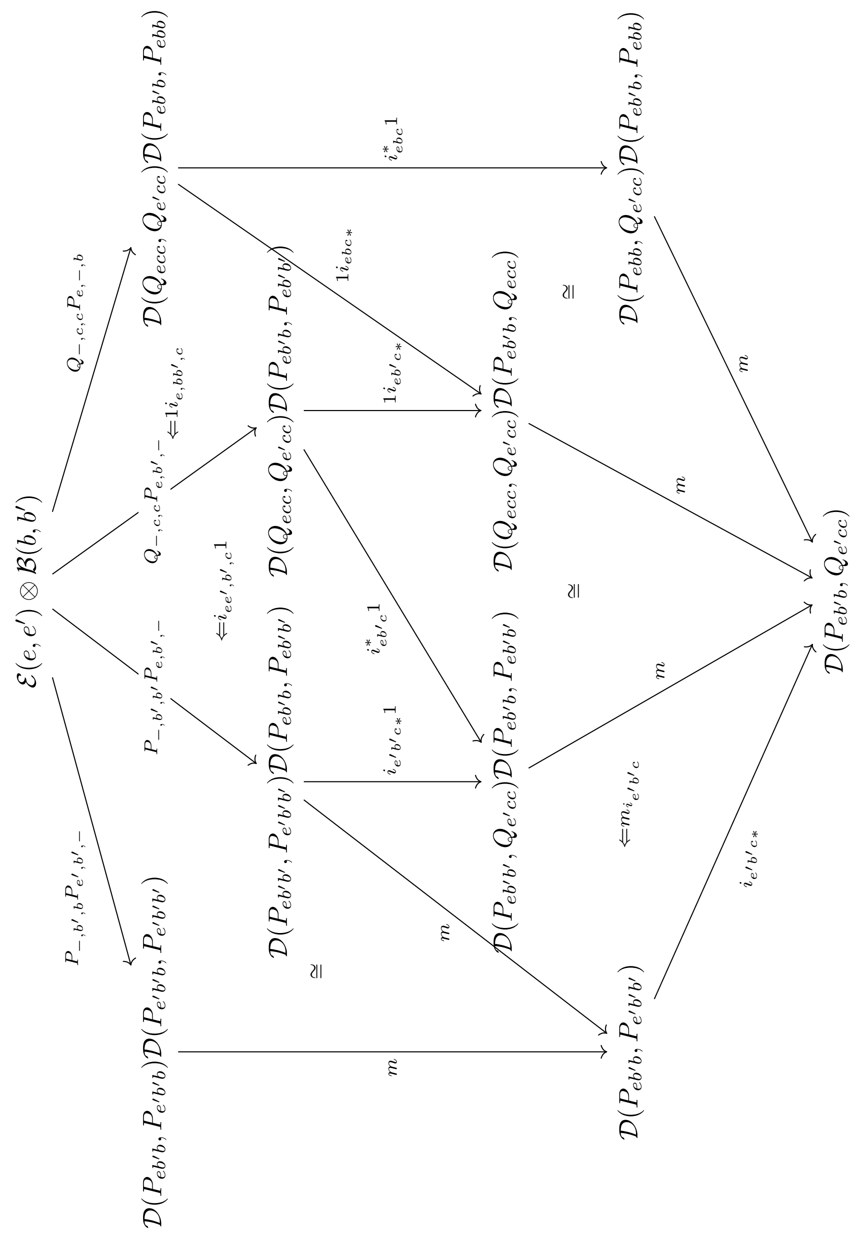


which can shortly be written, together with its symmetric analogue, as

$$\mathsf{fun}^{-1} \circ i_{bb'',c}\mathsf{fun} = i_{b'b'',c} \otimes P(b'',-) \circ i_{bb',c} \otimes P(-,b), \tag{EF1}$$

$$\mathsf{fun}^{-1} \circ i_{b,cc''}\mathsf{fun} = Q(c,-) \otimes i_{b,cc'} \circ Q(-,c'') \otimes i_{b,c'c''}. \tag{EF2}$$

And *compatibility*, between the parameters and the variables, expressed by the equality of diagrams



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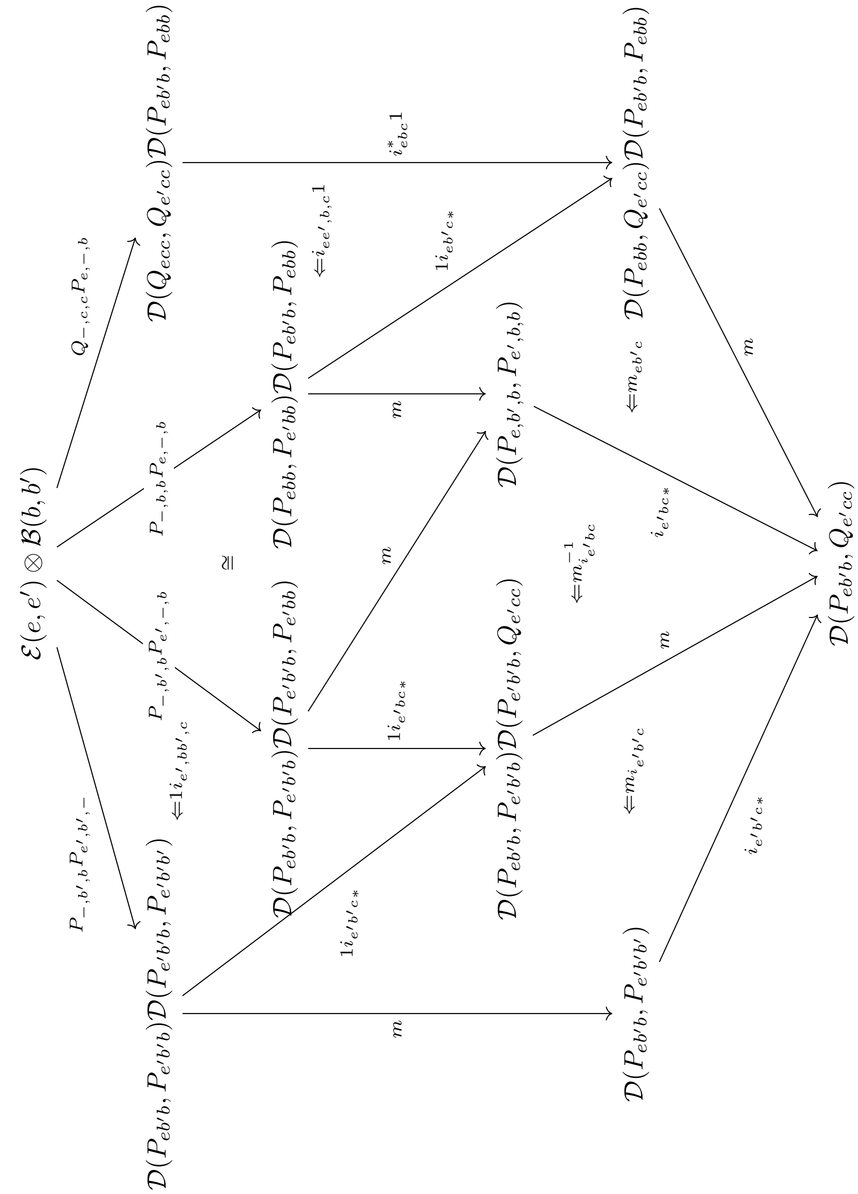


shortly written as

$$i_{ee',b',c} \otimes P_{e,b',-} \circ Q_{-c,c} \otimes i_{e,bb',c} = P_{-,b',b} \otimes i_{e',bb',c} \circ i_{ee',b,c} \otimes P_{e,-,b} \tag{EC1}$$

$$P_{-,b,b} \otimes i_{e',b,cc'} \circ i_{ee',b,c} \otimes Q_{e,c',-} = i_{ee',b,c'} \otimes Q_{e,-,c} \circ Q_{-,c,c'} \otimes i_{e,b,cc'}. \tag{EC2}$$

Remark 2.1.2. If $\mathcal{V} = \mathsf{Cat}$, by whiskering the two structural 2-cells (2.1) and (2.2), with an enriched morphisms $g\colon \mathbb{1} \to \mathcal{B}(b,b')$ and $h\colon \mathbb{1} \to \mathcal{C}(c,c')$ respectively, we get the non-enriched version of the structure, that is, the 2-isomorphisms $i_{g,c}$ and $i_{b,h}$

$$\begin{array}{ccccc} P(e,b',b) & \xrightarrow{P(e,g,b)} & P(e,b,b) & \xrightarrow{i_{e,b,c'}} & Q(e,c',c') \\ {\scriptstyle P(e,g,b')}\downarrow & i_{g,c}\swarrow & \downarrow{\scriptstyle i_{e,b,c}} & i_{b,h}\swarrow & \downarrow{\scriptstyle Q(e,h,c')} \\ P(e,b',b') & \xrightarrow[i_{e,b',c}]{} & Q(e,c,c) & \xrightarrow[Q(e,c,h)]{} & Q(e,c,c') \end{array}$$

and the corresponding axioms which can easily be checked to be those given in [Cor16].

There is a notion of morphism of extra-pseudonatural transformations, which will assemble them into a category.

Definition 2.1.3. Let $j, j' \colon P \Rrightarrow Q$ be extra-pseudonatural transformation of $\mathcal{V}$-pseudofunctors $P \colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{D}$ and $Q \colon \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C} \to \mathcal{D}$. A morphism $\Gamma \colon j \to j'$ is the data of an indexed family of 2-morphisms in $\mathcal{V}$

$$\Gamma_{b,c} \colon j_{b,c} \Longrightarrow j'_{b,c}$$

such that the following equalities hold true for all pairs (b, b') and (c, c') of objects of $\mathcal{B}$ and $\mathcal{C}$ respectively:

$$\begin{array}{ccc} \mathcal{B}(b,b') & \xrightarrow{P(-,b)} & \mathcal{D}(Pb'b,Pbb) \\ {\scriptstyle P(b',-)}\downarrow & \swarrow j_{bb',c} & \downarrow{\scriptstyle (j_{b,c})_*} \\ \mathcal{D}(Pb'b,Pb'b') & \underset{(j'_{b',c})_*}{\overset{(j_{b',c})_*}{\rightrightarrows}} {\scriptstyle \Downarrow(\Gamma_{b',c})_*} & \mathcal{D}(Pb'b,Qcc) \end{array} = \begin{array}{ccc} \mathcal{B}(b,b') & \xrightarrow{P(-,b)} & \mathcal{D}(Pb'b,Pbb) \\ {\scriptstyle P(b',-)}\downarrow & \swarrow j'_{bb',c} & {\scriptstyle (j'_{b,c})_*}\downarrow \overset{(\Gamma_{b,c})_*}{\Leftarrow} \downarrow{\scriptstyle (j_{b,c})_*} \\ \mathcal{D}(Pb'b,Pb'b') & \xrightarrow[j'_{b',c}]{} & \mathcal{D}(Pb'b,Qcc) \end{array} \tag{2.3}$$

$$\begin{array}{ccc} \mathcal{C}(c,c') & \xrightarrow{Q(-,c')} & \mathcal{D}(Qc'c',Qcc') \\ {\scriptstyle Q(c,-)}\downarrow & \swarrow j_{b,cc'} & \downarrow{\scriptstyle (j_{b,c'})^*} \\ \mathcal{D}(Qcc,Qcc') & \underset{(j'_{b,c})^*}{\overset{(j_{b,c})^*}{\rightrightarrows}} {\scriptstyle \Downarrow(\Gamma_{b,c})^*} & \mathcal{D}(Pbb,Qcc') \end{array} = \begin{array}{ccc} \mathcal{C}(c,c') & \xrightarrow{Q(-,c')} & \mathcal{D}(Qc'c',Qcc') \\ {\scriptstyle Q(c,-)}\downarrow & \swarrow j'_{b,cc'} & {\scriptstyle (j'_{b,c'})^*}\downarrow \overset{(\Gamma_{b,c'})^*}{\Leftarrow} \downarrow{\scriptstyle (j_{b,c'})^*} \\ \mathcal{D}(Qcc,Qcc') & \xrightarrow[(j'_{b,c})^*]{} & \mathcal{D}(Pbb,Qcc') \end{array} \tag{2.4}$$

This definition clearly gives rise to a category $\mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(P, Q)$.

Remark 2.1.4. A few observations of straightforward verification are useful:

(i) Extra-pseudonatural transformations are preserved by any pseudofunctor. If $F \colon \mathcal{D} \to \mathcal{H}$ is a $\mathcal{V}$-pseudofunctor between $\mathcal{V}$-bicategories, and $i \colon P \Rrightarrow Q$ is an extra-pseudonatural transformation between $\mathcal{D}$-valued pseudofunctors, then $F(i)$ defines an extra-pseudonatural transformation $FP \Rrightarrow FQ$.

(ii) Extra-pseudonatural transformations are preserved by adjunctions. Let $L \colon \mathcal{D} \to \mathcal{E}$ and $R \colon \mathcal{E} \to \mathcal{D}$ form a pseudoadjunction $L \dashv R$ between $\mathcal{V}$-bicategories, and P, Q be pseudofunctors

$$\begin{array}{ccc} L\colon \mathcal{D} & \underset{\longleftarrow}{\overset{\longrightarrow}{}} & \mathcal{E}\colon R \\ {\scriptstyle P}\nwarrow & & \nearrow{\scriptstyle Q} \\ & \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C} & \end{array}$$

Then, any extra-pseudonatural transformation $P \stackrel{..}{\Rightarrow} RQ$ defines an extra-pseudonatural transformation $LP \stackrel{..}{\Rightarrow} Q$, and vice versa. The proof is really straightforward: if $i\colon P \stackrel{..}{\Rightarrow} RQ$ is an extra-pseudonatural transformation, then each component of the transposed $j\colon LP \stackrel{..}{\Rightarrow} Q$ is clearly defined to be $j_{c,d} = \varepsilon_{Q(d,d)} \circ L(i_{c,d})$. The very same works at the level of 2-cells. The structural

$$\begin{array}{ccc} \mathcal{C}(c,c') & \xrightarrow{LP(-,c)} & \mathcal{D}(LP(c,c'),LP(c,c)) \\ {\scriptstyle LP(c',-)}\downarrow & \swarrow j_{cc',d} & \downarrow{\scriptstyle (j_{c,d})_*} \\ \mathcal{D}(LP(c',c),LP(c',c')) & \xrightarrow[(j_{c,d})_*]{} & \mathcal{D}(LP(c',c'),Q(d,d)) \end{array}$$

is defined to be the whiskering of

$$\begin{array}{ccc} \mathcal{C}(c,c') & \xrightarrow{LP(-,c)} & \mathcal{D}(LP(c,c'),LP(c,c)) \\ {\scriptstyle LP(c',-)}\downarrow & \swarrow L(i_{cc',d}) & \downarrow{\scriptstyle L(i_{c,d})_*} \\ \mathcal{D}(LP(c',c),LP(c',c')) & \xrightarrow[L(i_{c',d})_*]{} & \mathcal{D}(LP(c,c'),LRQ(d,d)) \end{array}$$

with $(\varepsilon_{D(d,d)})_*\colon \mathcal{D}(LP(c,c'),LRQ(d,d)) \to \mathcal{D}(LP(c',c'),Q(d,d))$. The extra-pseudonaturality axioms are then preserved by pseudofunctoriality.

(iii) With a bit of attention one can see that an enriched extra-pseudonatural transformation $i\colon P \stackrel{..}{\Rightarrow} Q$ between $\mathcal{V}$-pseudofunctors

$$P\colon \mathcal{E} \otimes \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \longrightarrow \mathcal{D}$$
$$Q\colon \mathcal{E} \otimes \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C} \longrightarrow \mathcal{D}$$

consists of the same data of an enriched extra-pseudonatural transformation $P^{\mathsf{op}} \stackrel{..}{\Rightarrow} Q^{\mathsf{op}}$ between the $\mathcal{V}$-pseudofunctors

$$P^{\mathsf{op}}\colon \mathcal{E}^{\mathsf{op}} \otimes (\mathcal{B}^{\mathsf{op}})^{\mathsf{op}} \otimes \mathcal{B}^{\mathsf{op}} \longrightarrow \mathcal{D}^{\mathsf{op}}$$
$$Q^{\mathsf{op}}\colon \mathcal{E}^{\mathsf{op}} \otimes (\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} \otimes \mathcal{C}^{\mathsf{op}} \longrightarrow \mathcal{D}^{\mathsf{op}}.$$

Remark 2.1.5. Observe that if the unit of a monoidal bicategory is not terminal, the constant pseudofunctor is not a concept that exists in general between enriched bicategories. However, there's a unit $\mathcal{V}$-bicategory $\mathcal{J}$ (Example 1.3.5), and for every $\mathcal{V}$-bicategory $\mathcal{D}$ and every object D in it, a constant object $\mathcal{V}$-pseudofunctor is defined from this $\mathcal{V}$-bicategory into $\mathcal{D}$:

$$\Delta D\colon \mathcal{J} \longrightarrow \mathcal{D}.$$

The object part is clearly defined by mapping the object $*$ to the object D, while the hom part is given by the unit arrow of D. Thanks to the canonical equivalence $\mathcal{J}^{\mathsf{op}} \otimes \mathcal{J} \simeq \mathcal{J}$, we can also consider constant $\mathcal{V}$-pseudofunctors of type $\mathcal{J}^{\mathsf{op}} \otimes \mathcal{J} \to \mathcal{D}$.

In the following we are going to work mainly with extra-pseudonatural transformations whose domain or codomain is a constant pseudofunctor in the sense of Remark 2.1.5. Therefore, it is customary and convenient to introduce the following notion.

Definition 2.1.6. A *biwedge* over a $\mathcal{V}$-pseudofunctor $P\colon \mathcal{E} \otimes \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{D}$ is an extra-pseudonatural transformation $D \ddot{\Rightarrow} P$ from a constant pseudofunctor over an object D in $\mathcal{D}$ to P. Dually, a *bicowedge* is an extra-pseudonatural transformation $P \ddot{\Rightarrow} D$.

A result similar to but fairly less trivial than Remark 2.1.4 (*ii*), is the following proposition, asserting that a closedness pseudoadjunction maps biwedges to bicowedges.

Proposition 2.1.7. *Let $\mathcal{V}$ be a closed monoidal bicategory, $\mathcal{M}$ a $\mathcal{V}$-bicategory, a, y objects in $\mathcal{V}$ and $P\colon \mathcal{M}^{\mathsf{op}} \otimes \mathcal{M} \to \mathcal{V}$ a $\mathcal{V}$-pseudofunctor. Then, biwedges of the form*

$$y \ddot{\Rightarrow} [P(-,-), a]$$

bijectively correspond to bicowedges of the form

$$y \otimes P(-,-) \ddot{\Rightarrow} a.$$

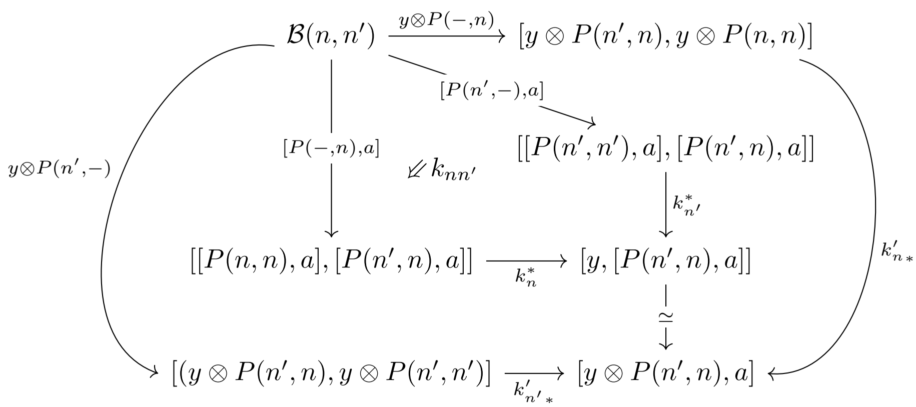
Proof. Suppose a bicowedge with components $k_n\colon y \to [P(n,n), a]$ is given. Then it is clear how to define the correspondent $k'_n\colon y \otimes P(n,n) \to a$ via the internal adjunction. The remaining part is the structural 2-cell

$$\begin{array}{ccc}
\mathcal{B}(n,n') & \xrightarrow{\;y\otimes P(-,n)\;} & [y \otimes P(n',n), y \otimes P(n,n)] \\
{\scriptstyle y\otimes P(n',-)}\big\downarrow & \swarrow k'_{nn'} & \big\downarrow{\scriptstyle (k'_n)_*} \\
[y \otimes P(n',n), y \otimes P(n',n')] & \xrightarrow[(k'_{n'})_*]{} & [y \otimes P(n',n), a]
\end{array}$$

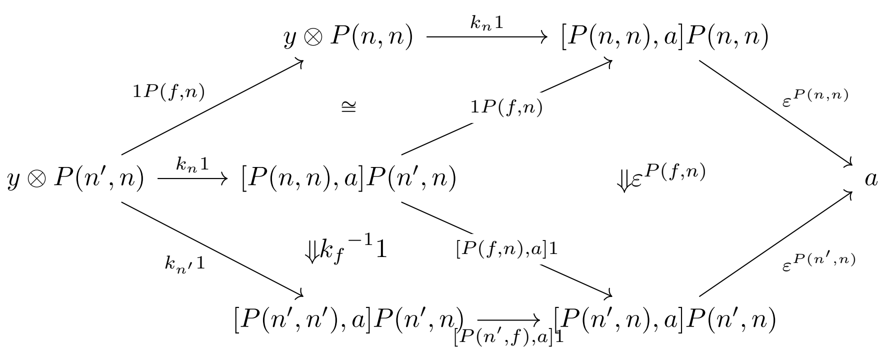
Now, we are going to prove that there is an isomorphism

$$\begin{array}{ccc}
\mathcal{B}(n,n') & \xrightarrow{\;y\otimes P(-,n)\;} & [y \otimes P(n',n), y \otimes P(n,n)] \\
{\scriptstyle [P(n',-),a]}\big\downarrow & & \\
[[P(n',n'), a], [P(n',n), a]] & \cong & \big\downarrow{\scriptstyle k'_{n*}} \\
{\scriptstyle k^*_{n'}}\big\downarrow & & \\
[y, [P(n',n), a]] & \xrightarrow{\;\simeq\;} & [y \otimes P(n',n), a]
\end{array} \tag{2.5}$$

and, similarly, there will be a symmetric one which together will allow us to fill the following diagram with isomorphisms whose composite will define $k'_{n,n'}$.



For the sake of space, we define the isomorphism (2.5) on each "object" f, but then it is not a conceptual challenge to translate it in enriched terms. By remembering how both k' and the adjunction isomorphism are explicitly given, we find out that one has to consider the extra-pseudonaturality structure of the counit together with the structural morphism k_f, in order to find the isomorphism:



Similarly, the symmetric one can also be constructed. Now, the so defined $k'_{nn'}$ is easily seen to satisfy bicowedges axioms, straightforwardly following for the correspondent axioms for k. □

The prototypical example of a $\mathcal{V}$-pseudofunctor of sort $\mathcal{C}^{\mathsf{op}}\otimes\mathcal{C}\to\mathcal{V}$ is the hom-pseudofunctor.

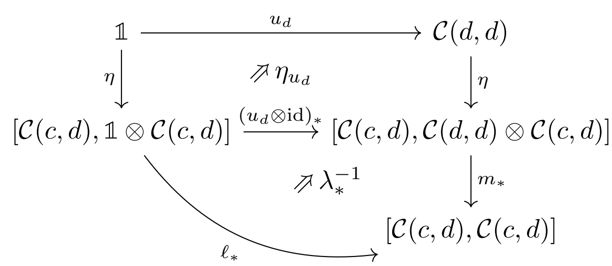
Definition 2.1.8 (Hom $\mathcal{V}$-pseudofunctor)**.** The $\mathcal{V}$-pseudofunctor $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathsf{op}}\otimes\mathcal{C}\to\mathcal{V}$ is defined on an object (c,c') to be $\mathcal{C}(c,c')$, while for every pair of objects $(c,c'),(d,d')$ the morphism

$$\mathcal{C}^{\mathsf{op}}\otimes\mathcal{C}((c,c'),(d,d'))\longrightarrow[\mathcal{C}(c,c'),\mathcal{C}(d,d')]$$

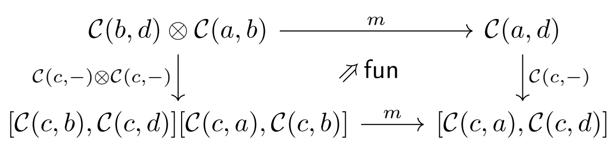
is defined as the transposition under the adjunction $-\otimes\mathcal{C}(c,c')\dashv[\mathcal{C}(c,c'),-]$ of the map

$$\mathcal{C}^{\mathsf{op}}(c,d)\otimes\mathcal{C}(c',d')\otimes\mathcal{C}(c,c')\xrightarrow{id\otimes m}\mathcal{C}^{\mathsf{op}}(c,d)\otimes\mathcal{C}^{\mathsf{op}}(d',c)\xrightarrow{m}\mathcal{C}(d,d'). \tag{2.6}$$

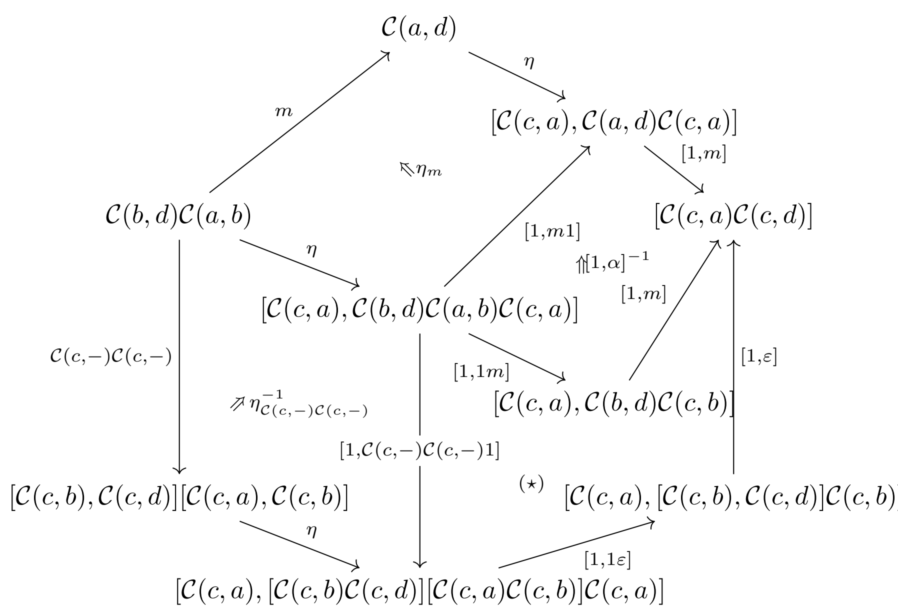
This also defines $\mathcal{V}$-pseudofunctors $\mathcal{C}(c,-)\colon\mathcal{C}\to\mathcal{V}$ and $\mathcal{C}(-,c')\colon\mathcal{C}^{\mathsf{op}}\to\mathcal{V}$. The further structure of $\mathcal{V}$-pseudofunctor is given below. For example, on $\mathcal{C}(c,-)$, we have un =



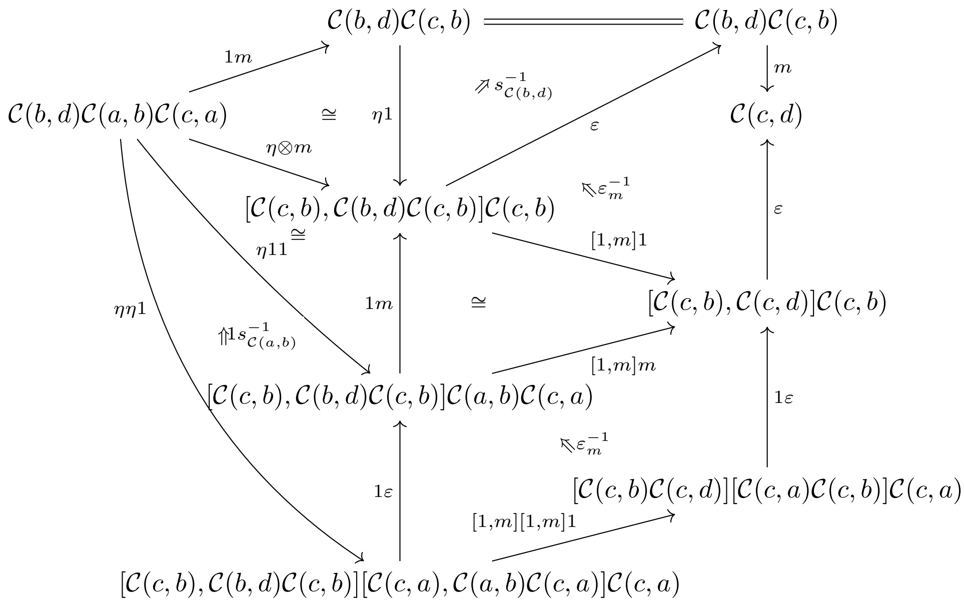
The definition of



is done using how the multiplication in $\mathcal{V}$ is defined (1.3.15) and considering the 2-cell



where $(\star)$ is the 2-cell consisting of the the functor $[\mathcal{C}(c,a), -]$ applied to



Similar constructions can be done for the pseudofunctors $\mathcal{C}(-,c')$ and $\mathcal{C}(-,-)$.

Now that we established how $\mathcal{C}(-,-)$ enjoys a structure of $\mathcal{V}$-pseudofunctor for any $\mathcal{V}$-bicategory $\mathcal{C}$, we can state the following.

Proposition 2.1.9. *Le $F\colon \mathcal{B}\to\mathcal{C}$ be a $\mathcal{V}$-pseudofunctor between $\mathcal{V}$-bicategories. Then, each component of F induces a $\mathcal{V}$-pseudonatural transformation*

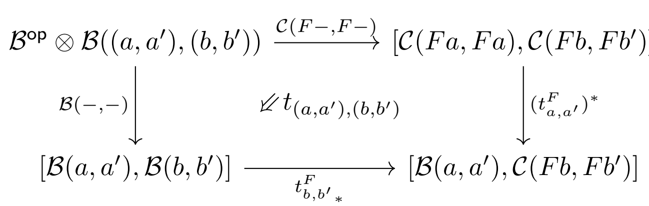
$$t^F\colon \mathcal{B}(-,-)\Longrightarrow \mathcal{C}(F-,F-)$$

of $\mathcal{V}$-pseudofunctors $\mathcal{B}^{\mathsf{op}}\otimes\mathcal{B}\to\mathcal{V}$.

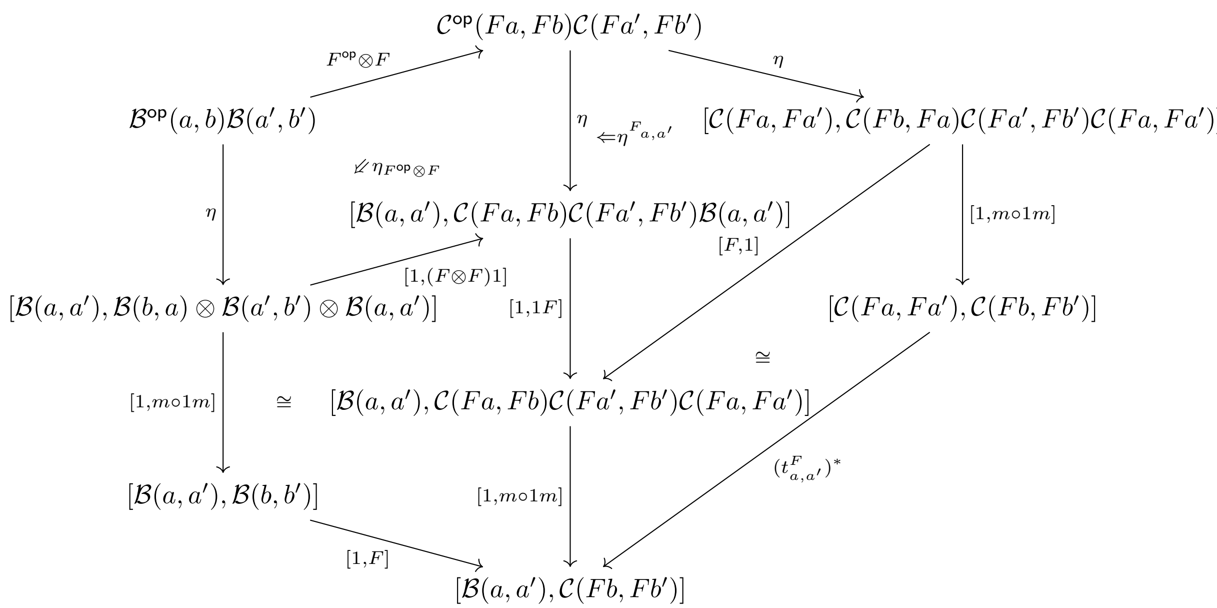
Proof. Each component of t^F is defined on the nose by the components of F

$$t^F_{b,b'} = F_{b,b'}\colon \mathcal{B}(b,b')\longrightarrow \mathcal{C}(Fb,Fb').$$

The structural 2-isomorphism is of the form



Since each component of $\mathcal{C}(-,-)$ is defined as the transpose of $m\circ 1\otimes m$ (as in 2.6), and since the $\mathcal{V}$-pseudofunctor $\mathcal{C}(F-,F-)$ is the composition $\mathcal{C}(-,-)\circ F^{\mathsf{op}}\otimes F$, the diagram that we look for is shaped, and the 2-cell $t_{(a,a'),(b,b')}$ defined, as the following using pseudonaturality and extra-pseudonaturality of the unit η for the adjunction.



The verification of the axioms of $\mathcal{V}$-pseudofunctor are left to the reader. □

2.1.1 Examples

A good motivation for introducing the notion of extra-(pseudo)natural transformation, or at least of (bi)wedge, is Proposition 2.1.10. Also, we can appreciate how extra-naturality arise very naturally in the context of enriched category theory. Later in this section we provide some other examples of extra-pseudonaturality, relating it to usual enriched pseudonaturality. All of these examples generalize well known cases treated in [Kel05], and the feeling is that these two notions of extra and plain (pseudo)naturality are so linked by canonical constructions bringing one to the other, that one could use the same term in order to mean any of the two, and the number of variables makes everything clear. Nonetheless, we prefer to keep different terminologies, since the amount of structure brought by the bicategorical setting could make the reading already complicated enough for the reader.

Proposition 2.1.10. *Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be $\mathcal{V}$-pseudofunctors. A $\mathcal{V}$-pseudonatural transformation $\alpha \colon F \Rightarrow G$ defines an extra-pseudonatural transformation*

$$\hat{\alpha} \colon \mathbb{1} \ddot{\Rightarrow} \mathcal{D}(F(-), G(-)),$$

and vice versa.

Proof. On the level of 1-cell, the data are of course the very same ($\hat{\alpha}_c = \alpha_c$), that is a morphism in $\mathcal{V}$

$$\mathbb{1} \longrightarrow \mathcal{D}(F(c), G(c))$$

for every c in $\mathcal{C}$. In order to make the higher structure correspond, we prove how 2-cells

$$\begin{array}{ccc} \mathcal{C}(c,c')\otimes 1 & \xrightarrow{G} & \mathcal{D}(Gc,Gc') \\ {\scriptstyle F}\big\downarrow & \Swarrow \alpha_{cc'} & \big\downarrow{\scriptstyle \alpha_c^*} \\ \mathcal{D}(Fc,Fc') & \xrightarrow[\alpha_{c'*}]{} & \mathcal{D}(Fc,Gc') \end{array} \tag{2.7}$$

and

$$\begin{array}{ccc} \mathcal{C}(c,c') & \xrightarrow{\mathcal{D}(F-,Gc')} & [\mathcal{D}(Fc',Gc'),\mathcal{D}(Fc,Gc')] \\ {\scriptstyle \mathcal{D}(Fc,G-)}\big\downarrow & \Swarrow \hat{\alpha}_{cc'} & \big\downarrow{\scriptstyle \alpha_{c'}^*} \\ [\mathcal{D}(Fc,Gc),\mathcal{D}(Fc,Gc')] & \xrightarrow[\alpha_c{}^*]{} & [\mathbb{1},\mathcal{D}(Fc,Gc')] \end{array} \tag{2.8}$$

respectively of the structure of pseudonatural transformation and of extra-pseudonatural transformation, are mapped, up to a 2-isomorphism, one to the other via the pseudoadjunction $-\otimes\mathbb{1}\dashv[\mathbb{1},-]$. It suffices thus to show how this pseudoadjunction maps each of the two sides of one square into a 1-cell in $\mathcal{V}$ isomorphic to the correspondent side of the other square.

First observe that the morphism $\alpha_c^*\circ G$ should properly be written as

$$\mathcal{C}(c,c')\otimes\mathbb{1}\xrightarrow{G}\mathcal{D}(Gc,Gc')\xrightarrow{\mathrm{id}\otimes\alpha_c}\mathcal{D}(Gc,Gc')\otimes\mathcal{D}(Fc,Gc)\xrightarrow{m}\mathcal{D}(Fc,Gc').$$

The transpose of (2.7) is then

$$\begin{array}{ccccc} \mathcal{C}(c,c') & \xrightarrow{\eta^{\mathbb{1}}} & [\mathbb{1},\mathcal{C}(c,c')] & \xrightarrow{[\mathbb{1},G]} & [\mathbb{1},\mathcal{D}(Gc,Gc')] \\ {\scriptstyle \eta^{\mathbb{1}}}\big\downarrow & & & & \big\downarrow{\scriptstyle [\mathbb{1},\mathrm{id}\otimes\alpha_c]} \\ [\mathbb{1},\mathcal{C}(c,c')] & & \Swarrow \bar{\alpha}_{cc'}=[\mathbb{1},\alpha_{cc}]*\eta^{\mathbb{1}} & & [\mathbb{1},\mathcal{D}(Gc,Gc')\otimes\mathcal{D}(Fc,Gc)] \\ {\scriptstyle [\mathbb{1},F]}\big\downarrow & & & & \big\downarrow{\scriptstyle [\mathbb{1},m]} \\ [\mathbb{1},\mathcal{D}(Fc,Fc')] & \xrightarrow[{[\mathbb{1},\alpha_{c'}\otimes\mathrm{id}]}]{} & [\mathbb{1},\mathcal{D}(Fc',Gc')\otimes\mathcal{D}(Fc,Fc')] & \xrightarrow[{[\mathbb{1},m]}]{} & [\mathbb{1},\mathcal{D}(Fc,Gc')] \end{array}$$

The top-right side of this square is then isomorphic to the bottom-left side of (2.8) via the 2-isomorphism

$$\begin{array}{ccccc} & & \mathcal{C}(c,c') & & \\ & \swarrow{\scriptstyle \eta^{\mathbb{1}}} & & \searrow{\scriptstyle G} & \\ [\mathbb{1},\mathcal{C}(c,c')\otimes\mathbb{1}] & & \Searrow\eta^{\mathbb{1}}_G & & \mathcal{D}(Gc,Gc') \\ {\scriptstyle \mathcal{A}(\mathbb{1},G\otimes\mathbb{1})}\big\downarrow & & \swarrow{\scriptstyle \eta^{\mathbb{1}}} & & \big\downarrow{\scriptstyle \eta^{\mathcal{D}(Fc,Gc)}} \\ [\mathbb{1},\mathcal{D}(Gc,Gc')\otimes\mathbb{1}] & & \Searrow(\eta^{\alpha_c})^{-1} & & [\mathcal{D}(Fc,Gc),\mathcal{D}(Gc,Gc')\otimes\mathcal{D}(Fc,Gc)] \\ {\scriptstyle [\mathbb{1},\mathrm{id}\otimes\alpha_c]}\big\downarrow & & \swarrow{\scriptstyle \alpha_c^*} & & \big\downarrow{\scriptstyle [\mathcal{D}(Fc,Gc),m]} \\ [\mathbb{1},\mathcal{D}(Gc,Gc')\otimes\mathcal{D}(Fc,Gc)] & & \Searrow[\alpha_c,m]^{-1} & & [\mathcal{D}(Fc,Gc),\mathcal{D}(Fc,Gc')] \\ & \searrow{\scriptstyle [\mathbb{1},m)]} & & \swarrow{\scriptstyle \alpha_c^*} & \\ & & [\mathbb{1},\mathcal{D}(Fc,Gc')] & & \end{array}$$

given by composing the structural isomorphism of the pseudonatural transformation

$$\eta^{\mathbb{1}} \colon \mathrm{id} \Rightarrow [\mathbb{1}, - \otimes \mathbb{1}],$$

of extra-pseudonatural transformation

$$\eta_{\mathcal{D}(Gc,Gc')} \colon \mathcal{D}(Gc, Gc') \ddot{\Rightarrow} [-, \mathcal{D}(Gc, Gc') \otimes -]$$

(see Theorem A.1.2), and of pseudonatural transformation

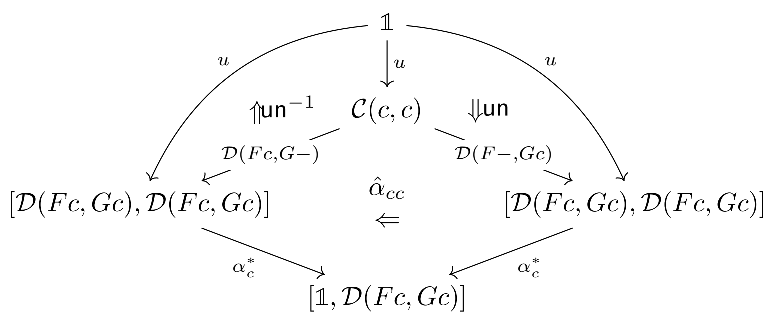
$$[-, m] \colon [-, \mathcal{D}(Gc, Gc') \otimes \mathcal{D}(Fc, Gc)] \Rightarrow [-, \mathcal{D}(Fc, Gc')].$$

A perfectly analogous 2-cell exhibit the isomorphism between the other two sides. That is, we can define the higher structure of $\hat{\alpha}$ to be the composition

$$\hat{\alpha}_{cc'} = [\alpha_c, m]^{-1} \circ (\eta^{\alpha_c})^{-1} \circ \eta^{\mathbb{1}}_G \circ [\mathbb{1}, \alpha_{c,c'}] \circ (\eta^{\mathbb{1}}_F)^{-1} \circ \eta^{\alpha_{c'}} \circ [\alpha_{c'}, m]$$

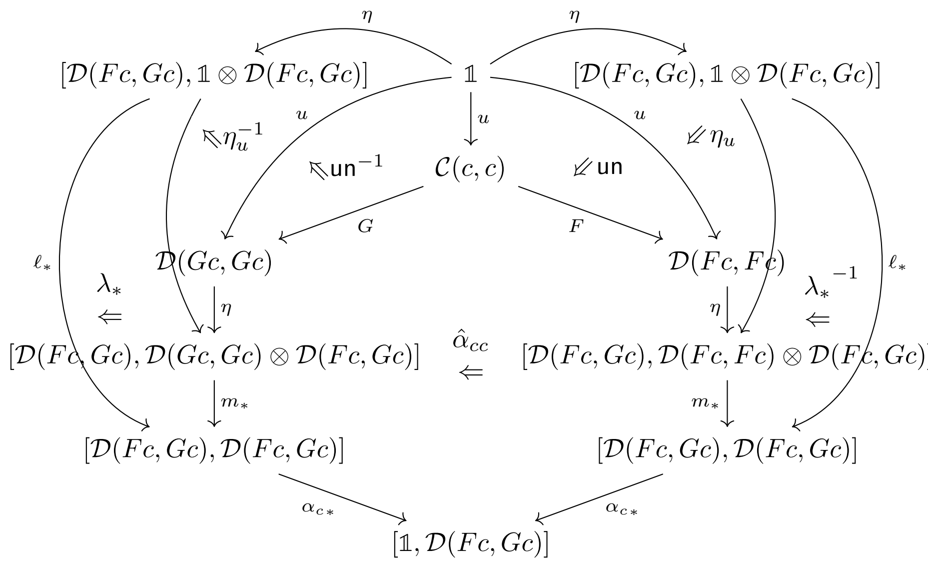
We are now going to see how axioms correspond. First, observe how axioms to be checked are here, for both structures, just the respective unitality and functoriality. The structure of the proof is analogous for both. Thus, since pretty huge diagrams are involved, we limit ourselves to prove unitality:

$\mathrm{id}_{\alpha_c^* \circ u} =$

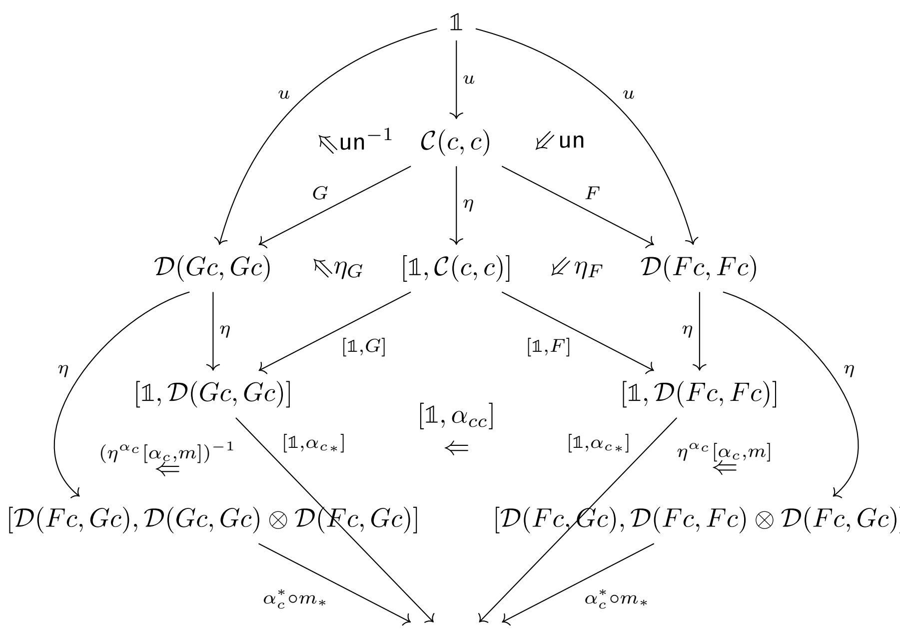


(2.9)

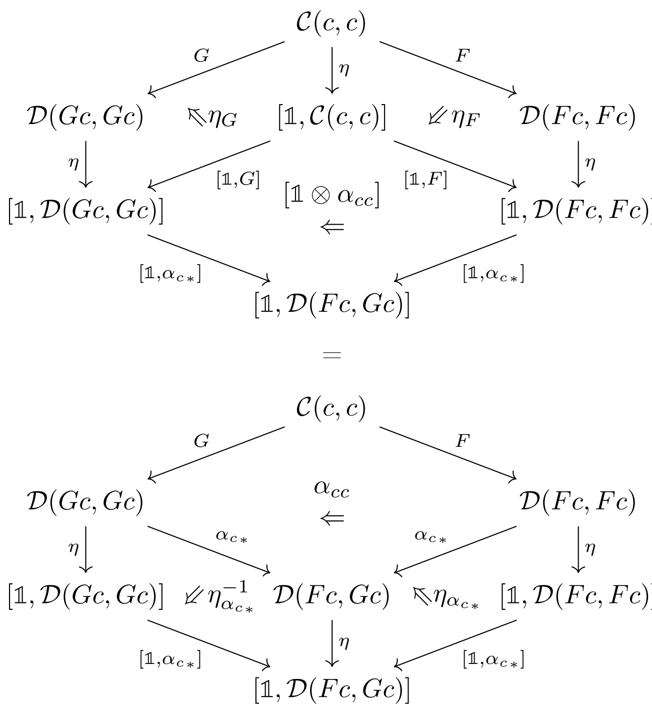
for the extra-pseudonatural $\hat{\alpha} \colon \mathbb{1} \ddot{\Rightarrow} \mathcal{D}(F-, G-)$ build from the pseudonatural $\alpha \colon F \Rightarrow G$. The right hand side can then be written, using the definition of unitors for the pseudofunctor $\mathcal{D}(F-, G-)$ (see Definition 2.1.8) and the fact that the unitor of the composition of pseudofunctor is the composition of the unitors, as



If we focus on $\mathsf{un}^{-1} \circ \hat{\alpha}_{cc} \circ \mathsf{un}$, we can expand



and observe that from the naturality and the functoriality of the pseudonatural transformation $\eta \colon \mathrm{id} \Rightarrow [\mathbb{1}, - \otimes \mathbb{1}]$ we have equality



Therefore, one can use unitality for the pseudonatural transformation α, which says

$$\mathsf{un}^{-1}\alpha_{cc}\mathsf{un} = \mathrm{id}.$$

This allows us to consequently simplify the rest of the structure on both sides of the 2-cell, showing how indeed (2.9) holds true. The proof for the functoriality axiom, though quite more space-consuming, follows a similar structure. □

Proposition 2.1.11. *Let $F\colon \mathcal{B}\otimes\mathcal{C}\to\mathcal{D}$ be a $\mathcal{V}$ pseudofunctor. Then, for every pair of objects b, b' in $\mathcal{B}$, the family of morphisms*

$$\left\{F(-,c)\colon \mathcal{B}(b,b') \longrightarrow \mathcal{D}(F(b,c),F(b',c))\right\}_c$$

defines an extra-pseudonatural transformation.

Proof. The argument is the same used in [Kel05] 1.8 (c). □

There is at least another property expressing an interplay between naturality and extra-naturality that worth to be generalized to this bicategorical context and which will be essential in proving Fubini theorem in Section 2.5. It is the following, and is a generalization of a fact which can be found in [Kel05].

Lemma 2.1.12. *Let $F, G\colon \mathcal{B}\to\mathcal{C}$, and $H\colon \mathcal{B}\otimes\mathcal{B}^{\mathsf{op}}\otimes\mathcal{B}\to\mathcal{C}$ be three $\mathcal{V}$-pseudofunctors. Suppose moreover to have families of morphisms*

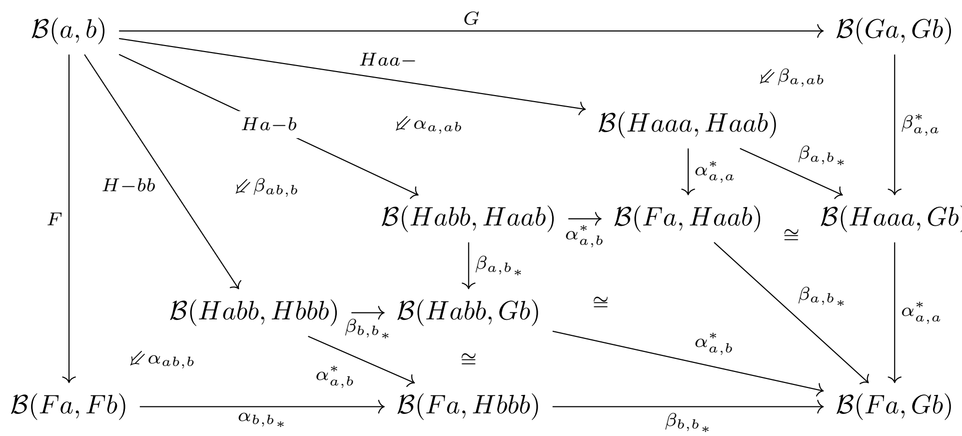
$$\alpha_{a,b}\colon Fa \longrightarrow H(a,b,b)$$
$$\beta_{a,b}\colon H(a,a,b) \longrightarrow Gb$$

carrying each at the same time a structure of natural transformation in the variable a and of extra-pseudonatural transformation in the variable b (α) and the converse for β. Then, the family

$$Fa \xrightarrow{\alpha_{a,a}} H(a,a,a) \xrightarrow{\beta_{a,a}} Ga$$

is part of an enriched pseudonatural transformation.

Proof. The proof is a matter of defining the higher structure for this pseudonatural transformation, which can be produced as



The reader can verify with no surprise how unitality and functoriality for both structures (extra and plain pseudonatural transformation) will provide unitality and functoriality for this pseudonatural transformation. □

2.2 Definition of $\mathcal{V}$-bi(co)ends

The definition of enriched bi(co)ends, as most of concepts in 2-category theory, generalize the 1-categorical definition of enriched (co)end by introducing as part of the structure a 2-isomorphism that weakens the universal property. Then, this 2-isomorphism is subject to two axioms (BC1 and BC2 below, for bicoends), which at first could sound arbitrary and complicated. However, it is convenient to keep in mind as of now that these two axioms correspond precisely to essential surjectivity and fully faithfulness of an equivalence exhibiting the bicoend of a pseudofunctor $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{D}$ as a representing object of the functor $\mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(P,-)$ (Proposition 2.2.6 and Remark 2.2.8).
As usual for universal properties in enriched category theory, the definition is first given for $\mathcal{V}$-valued pseudofunctors, and then the general case is provided representably. Otherwise, the resulting definition would be too weak for most of purposes.

2.2.1 $\mathcal{V}$-valued bi(co)ends

Definition 2.2.1. Let $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$ be a $\mathcal{V}$-pseudofunctor. The *(enriched) bicoend* of P is, if it exists, an object $\int^b P(b,b)$ of $\mathcal{V}$ together with an (enriched) extra-pseudonatural transformation $i\colon P \overset{..}{\Rightarrow} \displaystyle\int^b P(b,b)$ (to the constant pseudofunctor) such that the following two axioms hold true.

(BC1) For any object x in $\mathcal{V}$ and any extra-pseudonatural transformation $j\colon P \overset{..}{\Rightarrow} x$ there's a 1-cell $\tilde{j}\colon \int^b P(b,b) \to x$, and for every a in $\mathcal{B}$ a 2-isomorphism J_a

$$\begin{array}{ccccc} & & \int^b P(b,b) & & \\ & \nearrow^{i_a} & \Downarrow J_a & \searrow^{\tilde{j}} & \\ P(a,a) & & \xrightarrow[j_a]{\qquad\qquad} & & x \end{array}$$

giving the following identity of 2-cells:

$$\begin{array}{ccccc} \mathcal{B}(a,c) & \xrightarrow{P(-,a)} & [P(c,a),P(a,a)] & & \\ {\scriptstyle P(c,-)}\downarrow & \swarrow j_{ac} & \downarrow{\scriptstyle j_{a*}} & \overset{J_{a*}}{\Leftarrow} \quad \searrow^{i_{a*}} & \\ [P(c,a),P(c,c)] & & & & [P(c,a),\int^b P(b,b)] \\ & \searrow_{j_{c*}} & & \swarrow_{\tilde{j}_*} & \\ & & [P(c,a),x] & & \end{array}$$

$$=$$

$$\begin{array}{ccccc} \mathcal{B}(a,c) & \xrightarrow{P(-,a)} & [P(c,a),P(a,a)] & & \\ {\scriptstyle P(c,-)}\downarrow & \swarrow i_{ac} & & \searrow^{i_{a*}} & \\ [P(c,a),P(c,c)] & & \xrightarrow{\quad i_{c*} \quad} & & [P(c,a),\int^b P(b,b)] \\ & \searrow_{j_{c*}} & \swarrow J_{c*} & \swarrow_{\tilde{j}_*} & \\ & & [P(c,a),x] & & \end{array}$$

compactly stated as

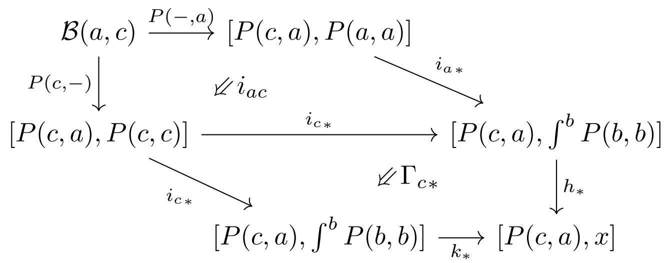
$$j_{ac} \circ J_{a*} = J_{c*} \circ i_{ac},$$

and

(BC2) For every pair of 1-cells $h, k\colon \int^b P(b,b) \to x$ and any family of 2-cells $\Gamma_a\colon hi_a \Rightarrow ki_a$ such that

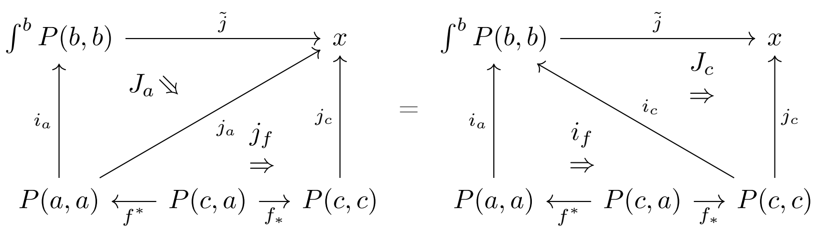
$$\begin{array}{ccccc} \mathcal{B}(a,c) & \xrightarrow{P(-,a)} & [P(c,a),P(a,a)] & & \\ {\scriptstyle P(c,-)}\downarrow & \swarrow i_{ac} & \downarrow{\scriptstyle i_{a*}} & \searrow^{i_{a*}} & \\ [P(c,a),P(c,c)] & & & & [P(c,a),\int^b P(b,b)] \\ & \searrow_{i_{c*}} & & \swarrow \Gamma_{a*} & \downarrow{\scriptstyle h_*} \\ & & [P(c,a),\int^b P(b,b)] & \xrightarrow[k_*]{} & [P(c,a),x] \end{array}$$

$$=$$

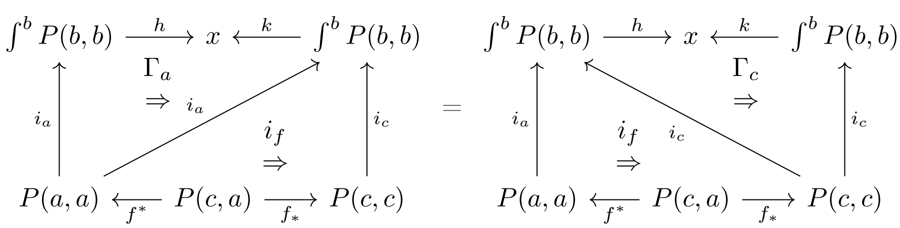


there is a unique 2-cell $\gamma\colon h \Rightarrow k$ such that $\Gamma_a = \gamma * i_a$.

Remark 2.2.2. Compare the above definition with the one in [Cor16], after whiskering the structure with a morphism $f\colon \mathbb{1} \to \mathcal{B}(a,c)$. Any extra-pseudonatural transformation $j\colon P \dddot{\Rightarrow} x$ induces a pair $(\tilde{j}, \{J_a\}_a)$ such that for every morphism $f\colon a \to c$



The second axiom also specialize to the non-enriched case, since whenever $(h, k, \{\Gamma_a\}_a)$ are as required, then it holds for every $f\colon a \to c$



And under this circumstances, the axiom in [Cor16] demand the existence of a unique 2-cell $\gamma\colon h \Rightarrow k$ such that $\Gamma_a = \gamma * i_a$.

Remark 2.2.3. Dually, one can define the opposite notion of *biend*. Given a $\mathcal{V}$-pseudofunctor $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$, an enriched extra-pseudonatural transformation $\int_b P(b,b) \dddot{\Rightarrow} P$ defines a biend whether the equivalent (in virtue of Remark 2.1.4 (*iii*)) data

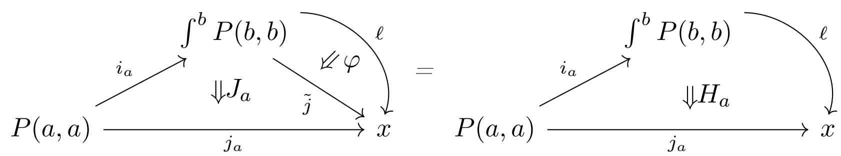
$$P^{\mathsf{op}} \dddot{\Rightarrow} \int_b P(b,b)$$

defines a bicoend.

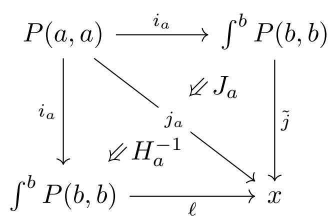
Proposition 2.2.4. *Let* $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$ *be a* $\mathcal{V}$*-pseudofunctor with a bicoend* $\int^b P(b,b)$*. If* $j\colon P \dddot{\Rightarrow} x$ *is an extra-pseudonatural transformation, then the induced pair* $(\tilde{j}, \{J_a\})$ *in BC1 is unique up to a unique isomorphism, meaning that if* $(\ell\colon \int^b P(b,b) \to x, \{H_a\colon \ell i_a \Rightarrow j_a\})$ *is another pair satisfying*

$$j_{ac} \circ H_a = H_c \circ i_{ac} \tag{2.10}$$

then, there's a unique isomorphism $\varphi\colon l \cong \tilde{j}$ *such that*



Proof. Consider the family of 2-cells $\Gamma_a \coloneqq H_a^{-1} \circ J_a$ in the 2-cell below



Now, the triple $(\tilde{j}, \ell, \{\Gamma_a\})$ is as required by axiom BC2, in the sense that we have equality of 2-cells

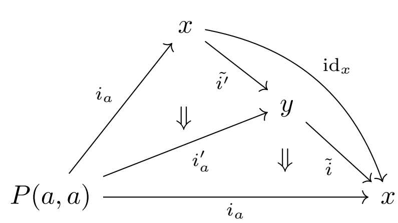
$$i_{ac} \circ \Gamma_a = i_{ac} \circ H_a^{-1} \circ J_a \overset{(2.10)}{=} H_c^{-1} \circ j_{ac} \circ J_a = H_c^{-1} \circ J_c \circ i_{ac} = \Gamma_c \circ i_{ac}.$$

Whence, the unique isomorphism $\gamma \colon \tilde{j} \cong \ell$ whose inverse is the desider one, making $\Gamma_a = \gamma * i_a$, and therefore $H_a = J_a \circ (\gamma^{-1}) * i_a$. ☐

As a straightforward consequence we have the essential uniqueness of the bicoend object.

Corollary 2.2.5. *The bicoend of a $\mathcal{V}$-pseudofunctor $P \colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$ is unique up to equivalence in $\mathcal{V}$.*

Proof. If $i \colon P \ddot{\Rightarrow} x, i' \colon P \ddot{\Rightarrow} y$ are two bicoends for P, then we have induced morphisms $\tilde{i'} \colon x \to y$ and $\tilde{i} \colon y \to x$ fitting, together with the respective 2-isomorphisms, the following diagram.



Therefore, if we consider the identity morphism for x, together with the identity 2-cell of i_a, we get, by Proposition 2.2.4, the desired isomorphism $\mathrm{id}_x \cong \tilde{i}\tilde{i'}$, and similarly $\mathrm{id}_y \cong \tilde{i'}\tilde{i}$. ☐

At this stage, the two bicoend axioms, which may look kind of obscure at first, deserve an explanation that will make them much clearer. The key point lies in the following proposition, which provides a higher point of view on bi(co)ends by defining them as representing objects.

Proposition 2.2.6. *Let $P \colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$ be a pseudofunctor admitting a bicoend. Then, there is for every object x in $\mathcal{V}$ an equivalence, pseudonatural in x, between the categories*

$$\mathcal{V}(\int^b P(b,b), x) \simeq \mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(P, x). \tag{2.11}$$

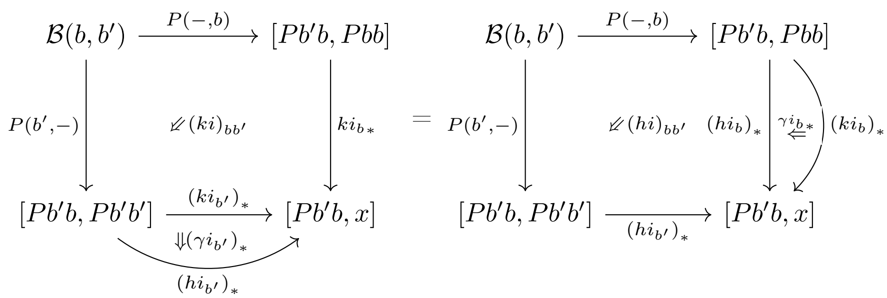
Dually, if P admits a biend, there is for every x an equivalence

$$\mathcal{V}(x, \int_b P(b,b)) \simeq \mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(x, P). \tag{2.12}$$

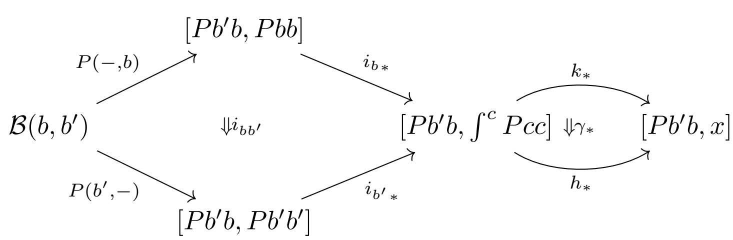
Proof. Let us us prove the first statement, and call $i \colon P \stackrel{..}{\Rightarrow} \int^b Pbb$ the bicoend of P. The dual statement will clearly follow a symmetric argument. The equivalence is here explicitly given on one side as the precomposition

$$\mathcal{V}(\int^b P(b,b), x) \longrightarrow \mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(P, x) \tag{2.13}$$

defined on a $k \colon \int^b P(b,b) \to x$ to be the extra-pseudonatural transformation $k \circ i$ with components $(k \circ i)_b = k \circ i_b$, and, on morphisms $\gamma \colon k \Rightarrow h$, the obvious whiskering $\gamma * i$, with components given by $\gamma * i_b \colon ki_b \Rightarrow hi_b$. One needs to check is that $\gamma * i$ actually defines a morphism of extra-pseudonatural transformation, which is the equality



This is true, since expanding the composition of both sides of the squares we find both squares to be equal to the horizontal composition $\gamma_* i_{bb'}$.



Fully faithfulness of (2.13) is now precisely axiom BC2, which states indeed that for all families $\Gamma_b \colon ki_b \Rightarrow hi_b$ defining a morphism of extra-pseudonatural transformations there is a unique morphism $h \Rightarrow k$ in the domain category which is mapped on it by precomposition with i. On the other hand, axiom BC1 says that for all objects $j \colon P \stackrel{..}{\Rightarrow} x$ in the codomain category, there exists a $\tilde{j} \colon \int^b Pbb \to x$ in the domain and an isomorphism of extra-pseudonatural transformations $J \colon \tilde{j} \circ i \cong j$, which is just essential surjectivity. □

2.2.2 Arbitrary-valued bi(co)ends

Now, the general definition of a bi(co)end valued in any $\mathcal{V}$-bicategory $\mathcal{D}$ can be given representably.

Definition 2.2.7. Let $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{D}$ be a $\mathcal{V}$-pseudofunctor, where $\mathcal{D}$ is any $\mathcal{V}$-bicategory. A *biend* for P is an object $\int_b P(b,b)$ in $\mathcal{D}$ together with an extra-pseudonatural transformation $i\colon \int_b P(b,b) \ddot{\Rightarrow} P$ such that for every object d in $\mathcal{D}$ the extra-pseudonatural transformation

$$\mathcal{D}(d,i)\colon \mathcal{D}(d, \int_b P(b,b)) \ddot{\Rightarrow} \mathcal{D}(d, P(-,-))$$

is a biend for $\mathcal{D}(d, P(-,-))\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$.

A *bicoend* for P is an object $\int^b P(b,b)$ in $\mathcal{D}$ together with an extra-pseudonatural transformation $i\colon P \ddot{\Rightarrow} \int^b P(b,b)$ such that for every object d in $\mathcal{D}$ the extra-pseudonatural transformation

$$\mathcal{D}(i,d)\colon \mathcal{D}(\int^b P(b,b), d) \ddot{\Rightarrow} \mathcal{D}(P(-,-), d)$$

is a biend for $\mathcal{D}(P(-,-), d)\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$.

It follows immediately the pair of equivalences in $\mathcal{V}$

$$\mathcal{D}(d, \int_b P(b,b)) \simeq \int_b \mathcal{D}(d, P(b,b)), \tag{2.14}$$

$$\mathcal{D}(\int^b P(b,b), d) \simeq \int_b \mathcal{D}(P(b,b), d). \tag{2.15}$$

Proposition 2.3.4 will witness that this definition of bi(co)end is coherent with the previous one for the case $\mathcal{D} = \mathcal{V}$.

Remark 2.2.8. We want to have the analogue result of Proposition 2.2.6 in the case of $\mathcal{D}$-valued bi(co)ends for an arbitrary $\mathcal{V}$-bicategory $\mathcal{D}$. In order to do so, we need to "gather" the category of bicowedges $\mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(P,d)$ into an object in $\mathcal{V}$. In other words, we want an object $\underline{\mathcal{V}\text{-}\mathsf{PsFun}^{\mathsf{e}}}(P,d)$ giving the desired equivalence

$$\mathcal{D}(\int^b P(b,b), d) \simeq \underline{\mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}}(P,d).$$

for every object (and constant pseudofunctor) d. The obvious choice is hence to define $\underline{\mathcal{V}\text{-}\mathsf{PsFun}^{\mathsf{e}}}(P,d) \coloneqq \int_b \mathcal{D}(P(b,b),d)$ (compare that with Definition 2.3.3), and then use (2.14).

2.3 The enriched pseudofunctor bicategory

In the non-enriched setting, we have the following result, expressing pseudonatural transformations as a biend:

Proposition 2.3.1 (Biend formula for PsNat)**.** *Let $L, S\colon \mathcal{C} \to \mathcal{D}$ be pseudofunctors between bicategories. Then, the bicoend of the pseudofunctor*

$$\mathcal{D}(L-, S-)\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Cat}$$

exists, and there's an equivalence of categories

$$\mathsf{PsNat}(L,S) \simeq \int_c \mathcal{V}(Lc, Sc).$$

Proof. We are going to show that $ev_d \colon \mathsf{PsNat}(L,S) \to \mathcal{D}(Ld, Sd)$ mapping $\gamma \mapsto \gamma_d$ is part of the data of a terminal extra-pseudonatural transformation, and then conclude by uniqueness of biends up to equivalence. We need the rest of the structure, the 2-isomorphisms, for every $g \colon d \to d'$

$$\begin{array}{ccc} \mathsf{PsNat}(L,S) & \xrightarrow{ev_d} & \mathcal{D}(Ld, Sd) \\ \downarrow{\scriptstyle ev_{d'}} & \swarrow ev_g & \downarrow{\scriptstyle g_*} \\ \mathcal{D}(Ld', Sd') & \xrightarrow[g^*]{} & \mathcal{D}(Ld, Sd') \end{array}$$

given on any $\gamma \colon L \Rightarrow S$ by

$$g_*\gamma_d = Sg \circ \gamma_d \longrightarrow \gamma_{d'} \circ Lg = g^*\gamma_{d'}$$

which is nothing but the isomorphism of pseudonaturality for γ.

Now, in order to prove terminality, let τ be another extra-pseudonatural transformation $Y \stackrel{..}{\Rightarrow} \mathcal{D}(L-, S-)$. We can just set $\tilde{\tau} \colon Y \to \mathsf{PsNat}(L,S)$ to be the functor mapping $y \mapsto \tilde{\tau}(y)$ defined by $\tilde{\tau}(y)_d = \tau_d(y)$, and therefore the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad\tilde{\tau}\quad} & \mathsf{PsNat}(L,S) \\ & {\scriptstyle \tau_d}\searrow \quad = \quad \swarrow{\scriptstyle ev_d} & \\ & \mathcal{D}(Ld, Sd) & \end{array}$$

is (even strictly) commutative. □

By virtue of the previous result, we are now ready to properly enrich the *enriched functor bicategory*. The following lemma will provide the structure induced for unit and composition for hom-objects.

Lemma 2.3.2. *Let $F, G, H \colon \mathcal{C} \to \mathcal{D}$ be $\mathcal{V}$-pseudofunctors, and suppose the relevant biends to exist. There are extra-pseudonatural transformations*

$$u_F \colon \mathbb{1} \stackrel{..}{\Rightarrow} \mathcal{D}(F-, F-)$$

and

$$\underline{m}_{F,G,H} \colon \int_c \mathcal{D}(Gc, Hc) \otimes \int_c \mathcal{D}(Fc, Gc) \stackrel{..}{\Rightarrow} \mathcal{D}(F-, H-).$$

Proof. For what concerns u_F, one can define it via Proposition 2.1.10 to be the same data of the $\mathcal{V}$-pseudonatural $\mathrm{id} \colon F \Rightarrow F$. One has hence each 1-component at d given (coherently with the notation) as $u_{Fd} \colon \mathbb{1} \to \mathcal{D}(Fd, Fd)$.

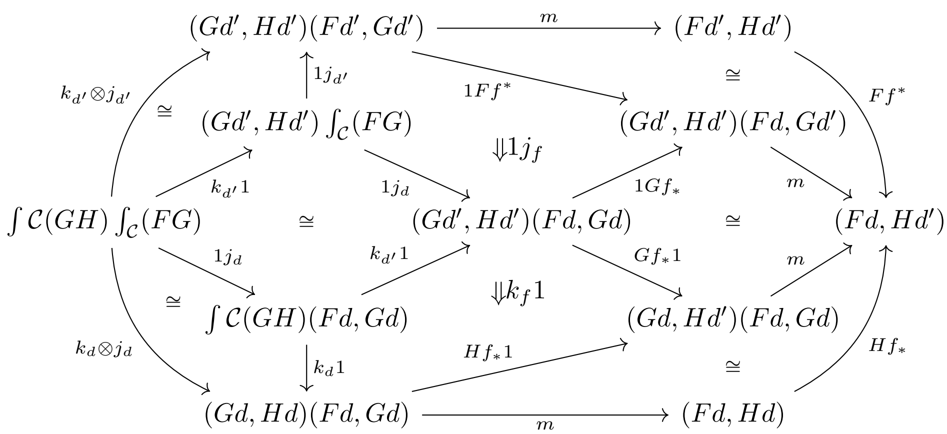
Each d-component of the second claimed to be extra-pseudonatural transformation is the composition

$$\begin{array}{ccc} \int_c \mathcal{D}(Gc, Hc) \otimes \int_c \mathcal{D}(Fc, Gc) & \xrightarrow{\qquad \underline{m}_d \qquad} & \mathcal{D}(Fd, Hd) \\ & {\scriptstyle k_d \otimes j_d}\searrow \qquad \nearrow{\scriptstyle m} & \\ & \mathcal{D}(Gd, Hd) \otimes \mathcal{D}(Fd, Gd) & \end{array}$$

and 2-isomorphisms are built in the following way. For reasons mostly of space and readability, we define the structure as if one could evaluate it at an object f in $\mathcal{C}(d,d')$. It is routine, then, to translate it in more general enriched terms. The idea, in order to construct m_f, which is

$$\begin{array}{ccc} \mathcal{C}(d,d') & \xrightarrow{\mathcal{D}(F-,Hd')} & [\mathcal{D}(Fd',Hd'),\mathcal{D}(Fd,Hd')] \\ {\scriptstyle \mathcal{D}(Fd,H-)}\big\downarrow & \swarrow m_{dd'} & \big\downarrow{\scriptstyle \underline{m}_*} \\ [\mathcal{D}(Fd,Hd),\mathcal{D}(Fd,Hd')] & \xrightarrow[\underline{m}_*]{} & [\int_c \mathcal{D}(Gc,Hc)\otimes\int_c \mathcal{D}(Fc,Gc),\mathcal{D}(Fd,Hd')] \end{array}$$

at an object f, is to separately use the structures of k and j. The notation in the following is that parenthesis stand for hom-objects in $\mathcal{D}$, and in the end we omit variables. As usual, juxtaposition stands for the tensor product. Then $\underline{m}_f$ is given by



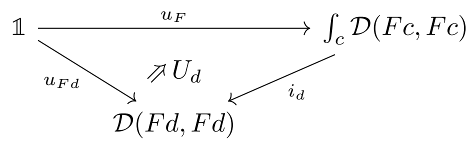
Extra-pseudonatural transformation axioms, then follow from the correspondent for j and k. □

Let us now enhance the structure of the bicategory $\mathcal{V}$-$\mathsf{PsFun}(\mathcal{C},\mathcal{D})$ to a $\mathcal{V}$-bicategory, which we will refer as $[\![\mathcal{C},\mathcal{D}]\!]$, in order to distinguish from the plain bicategory.

Definition 2.3.3 (The $\mathcal{V}$-bicategory of $\mathcal{V}$-pseudofunctors)**.** Let $\mathcal{V}$ be a monoidal bicategory and $\mathcal{C},\mathcal{D}$ be two $\mathcal{V}$-bicategories, then $[\![\mathcal{C},\mathcal{D}]\!]$ is the $\mathcal{V}$-bicategory:

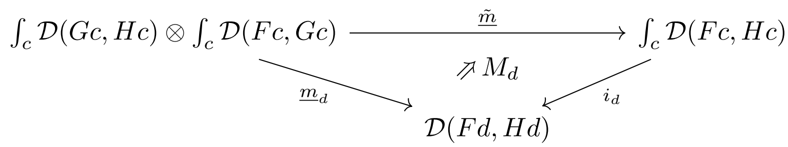
- objects are the $\mathcal{V}$-pseudofunctors $F\colon \mathcal{C}\to\mathcal{D}$
- each hom-object $[\![\mathcal{C},\mathcal{D}]\!](F,G)$ (also denoted as $\underline{\mathcal{V}\text{-}\mathsf{PsNat}}(F,G)$) is the object in $\mathcal{V}$ defined by the biend
$$\int_c \mathcal{D}(Fc,Gc) \xrightarrow{i_d} \mathcal{D}(Fd,Gd)$$
of the $\mathcal{V}$-pseudofunctor $\mathcal{D}(F-,G-)\colon \mathcal{C}^{\mathsf{op}}\otimes\mathcal{C}\to\mathcal{V}$.

- the unit is defined via the extra-pseudonatural transformation $u_F \colon \mathbb{1} \dddot{\Rightarrow} \mathcal{D}(F-, F-)$ of Lemma 2.3.2, which provides a unique pair $(u_F, \{U_d\})$



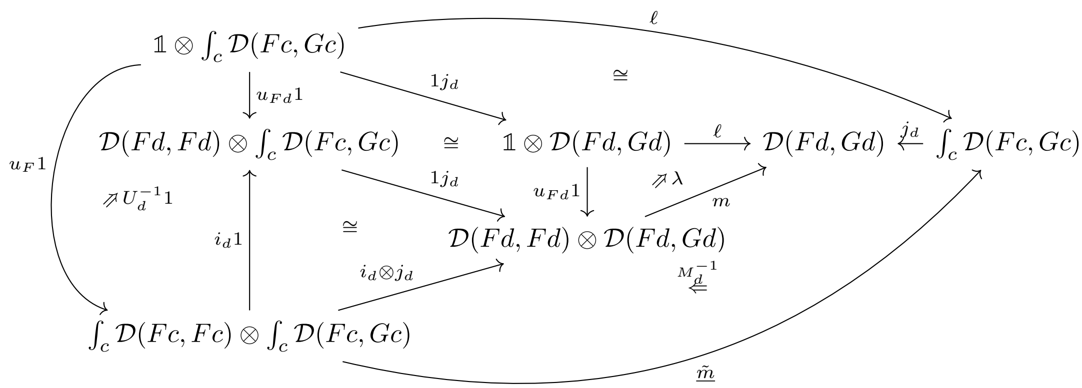
(2.16)

- The composition is defined analogously as induced by the other extra-pseudonatural transformation $\underline{m}$ of Lemma 2.3.2. That is,



(2.17)

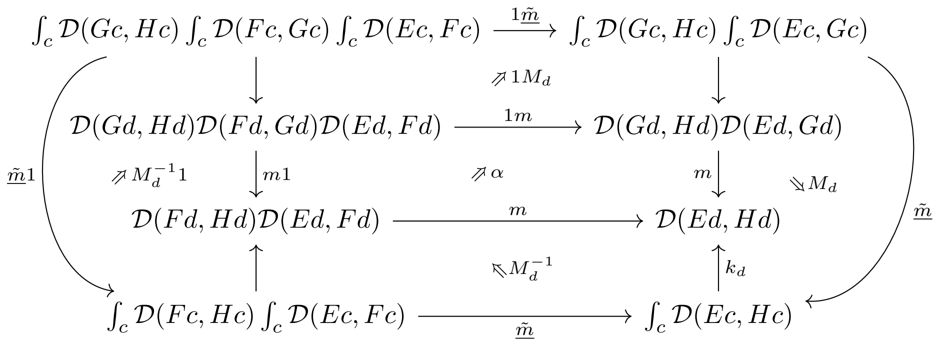
- The left unitor $\underline{\lambda}$ is defined via the equivalence (2.12) by defining (each component of) a morphism of extra-pseudonatural transformations (biwedges) by composing the inverses (*sic*!) of U and M, and the unitor λ for $\mathcal{D}$ as follows.



(2.18)

Similarly, the right unitor.

- The associator $\underline{\alpha}$ is analogously built as corresponding to a morphism of extra-pseudonatural transformations (biwedges). The latter having components as follows, where the arrows without a name clearly are just the tensor products of the structural biend morphisms.



(2.19)

Coherence identities (IC) and (AC) can then be proved with these definitions, and then be transposed again via (2.12). The proof is an easy check.

Proposition 2.3.4. *Let $\mathcal{M}$ be a $\mathcal{V}$-bicategory, for $\mathcal{V}$ a closed braided monoidal bicategory, and $P\colon \mathcal{M}^{\mathsf{op}} \otimes \mathcal{M} \to \mathcal{V}$ a $\mathcal{V}$-pseudofunctor for which the coend $i\colon P(-,-) \ddot{\Rightarrow} \int^m P(m,m)$ exists. There is, for every object a in $\mathcal{V}$, an equivalence in $\mathcal{V}$*

$$\left[\int^m P(m,m), a\right] \simeq \int_m [P(m,m), a].$$

Proof. The proof uses essential uniqueness of enriched biends. For, it suffices to prove that the extra-pseudonatural transformation

$$[i,a]\colon \left[\int^m P(m,m), a\right] \ddot{\Rightarrow} [P(-,-), a]$$

is terminal. Let us take another extra-pseudonatural $k\colon y \ddot{\Rightarrow} [P(-,-),a]$ and claim that each component of k provide, via a chain of equivalences of hom-categories for the bicategory $\mathcal{V}$,

$$\frac{\dfrac{\dfrac{k\colon y \longrightarrow [P(n,n), a]}{k'_n\colon y \otimes P(n,n) \longrightarrow a}}{k''_n\colon P(n,n) \otimes y \longrightarrow a}}{j_n\colon P(n,n) \longrightarrow [y,a],}$$

the component of another extra-pseudonatural transformation $j\colon P(-,-) \ddot{\Rightarrow} [y,a]$. The structure of extra-pseudonatural transformation of

$$y \otimes P(n,n) \longrightarrow a$$

can be deduced from the previous step by means of Proposition 2.1.7. The structures for k'' is then easily deduced from the braiding, and the fact that at the last step we also find an extra-pseudonatural transformation j is given by Remark 2.1.4 (*ii*). Thus, by the bicoend property we get $\tilde{j}\colon \int^m P(m,m) \to \mathcal{V}(y,a)$ and, for every n the 2-iso J_n

$$\begin{array}{ccc} & \int^m P(m,m) & \\ {}^{i_n}\nearrow & \Downarrow J_n & \searrow^{\tilde{j}} \\ P(n,n) & \xrightarrow[j_n]{} & [y,a] \end{array}$$

satisfying the bicoend axioms. Then, we can consider the equivalence of categogires

$$\Phi\colon \mathcal{V}(\int^m P(m,m),[y,a]) \longrightarrow \mathcal{V}(y,[\int^m P(m,m),a])$$

and define $\tilde{k}=\Phi(\tilde{j}), K_n=\Phi(J_n)$. Biend axioms are then automatic to be checked, being preserved by pseudofunctors. □

Proposition 2.3.5. *The enriched biend construction defines a $\mathcal{V}$-pseudofunctor*

$$\int_{\mathcal{B}}\colon [\![\mathcal{B}^{\mathsf{op}}\otimes\mathcal{B},\mathcal{D}]\!] \longrightarrow \mathcal{D} \tag{2.20}$$

$$P \longmapsto \int_b P(b,b) \tag{2.21}$$

Proof. On hom-objects the biend pseudofunctor is defined via the map

$$\underline{\mathcal{V}\text{-}\mathsf{PsNat}}(P,Q) \longrightarrow \mathcal{D}(\int_b P(b,b), \int_{b'} Q(b',b')),$$

which by definition of the enriched pseudonatural transformation object and the isomorphism (2.14) is the same thing as a map

$$\tilde{k}\colon \int_{b,b'} \mathcal{D}(P(b,b'),Q(b,b')) \longrightarrow \int_{b'} \mathcal{D}(\int_b P(b,b),Q(b',b')), \tag{2.22}$$

which we define as the induced by the extra-pseudonatural

$$k\colon \int_{b,b'} \mathcal{D}(P(b,b'),Q(b,b')) \stackrel{\cdot\cdot}{\Rightarrow} \mathcal{D}(\int_b P(b,b),Q(-,-))$$

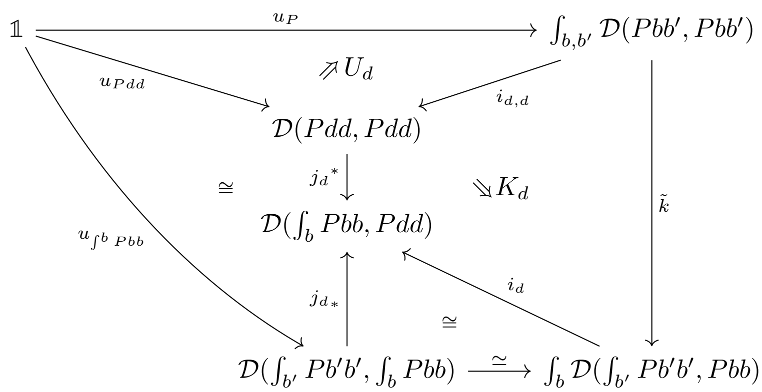
defined in turn on each component by the composition

$$k_d\colon \int_{b,b'} \mathcal{D}(P(b,b'),Q(b,b')) \xrightarrow{i_{d,d}} \mathcal{D}(P(d,d),Q(d,d)) \xrightarrow{j_d^*} \mathcal{D}(\int_b P(b,b),Q(d,d)).$$

Here, i and j are clearly the biend structure for the pseudonatural transformation objects and for $\int_b P(b,b)$, respectively. Analogously, it works for bicoends. The rest of the structure consists of un and fun. The first goes as

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{u_P} & \int_{b,b'} \mathcal{D}(Pbb',Pbb') \\ & \nearrow \mathsf{un} & \downarrow \tilde{k} \\ & \searrow_{u_{\int_b Pbb}} & \int_b \mathcal{D}(\int_{b'} Pb'b',Pbb) \end{array}$$

and is a 2-cell of morphisms $\mathbb{1} \to \int_{b'} \mathcal{D}(Pb'b', \int^b Pbb)$. It can therefore be defined via the equivalence (2.12) of Proposition 2.11 by establishing a morphism of extra-pseudonatural transformations, each of whose d-components is given by



Where U_d is the 2-iso of (2.16), K_d is induced together with (2.22) and the other two isomorphisms arise by a general fact for every closed bicategory and by definition of the structure of biend ℓ. It is no surprise that this composition defines a morphism of extra-pseudonatural transformations, as one can easily check since both K and U are, by definition. The construction of fun differs then more technically than conceptually from the case of un. $\square$

2.4 Yoneda lemma

In this section we prove a version of the Yoneda lemma for enriched bicategories. Our definition of the Yoneda pseudofunctor will be possible thanks to the following Proposition, stating that the constructions of the tensor product of $\mathcal{V}$-bicategories in Section 1.7 and of the enriched pseudofunctor enriched bicategory in Section 2.3 define a pseudoadjunction.

Proposition 2.4.1. *For $\mathcal{B}, \mathcal{C}, \mathcal{D}$ three $\mathcal{V}$-bicategories, there's a biequivalence of bicategories*

$$\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{B} \otimes \mathcal{C}, \mathcal{D}) \simeq \mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{B}, [\![\mathcal{C}, \mathcal{D}]\!]). \tag{2.23}$$

Proof. The argument is standard. If we start with a $\mathcal{V}$-pseudofunctor $F\colon \mathcal{B} \otimes \mathcal{C} \to \mathcal{D}$, we can define $\hat{F}\colon \mathcal{B} \to [\![\mathcal{C}, \mathcal{D}]\!]$ by letting

$$\begin{aligned} \hat{F}\colon \mathcal{B} &\longrightarrow [\![\mathcal{C}, \mathcal{D}]\!] \\ b &\longmapsto \quad \hat{F}b\colon c \longmapsto F(b,c) \\ &\qquad\qquad \mathcal{C}(c,c') \xrightarrow{u_b 1} \mathcal{B}(b,b)\mathcal{C}(c,c') \xrightarrow{F} \mathcal{D}(F(b,c), F(b,c')) \end{aligned}$$

For what concerns the hom-object part of the pseudofunctor $\hat{F}$, we define the morphism

$$\mathcal{B}(b,b') \longrightarrow [\![\mathcal{C}, \mathcal{D}]\!](\hat{F}b, \hat{F}b') = \int_c \mathcal{D}(F(b,c), F(b',c))$$

to be the map induced by biend property from the extra-pseudonatural transformation with components $i_c = F(-,c)\colon \mathcal{B}(b,b') \ddot{\Rightarrow} \mathcal{D}(F(b,c), F(b',c))$ (Proposition 2.1.11). Verification that this defines a biequivalence are a tedious though not conceptually challenging task, and we refer to the analogous argument explained at Proposition 4.3.4 in [Lor21]. $\square$

Definition 2.4.2. The *Yoneda pseudofunctor* $y\colon \mathcal{C}^{\mathsf{op}} \to [\![\mathcal{C}, \mathcal{V}]\!]$ is defined to be the image under the biequivalence (2.23) of the $\mathcal{V}$-pseudofunctor $\mathcal{C}(-,-)\colon \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C} \to \mathcal{V}$. Concretely, on objects, y is the $\mathcal{V}$-pseudofunctor mapping c to the $\mathcal{V}$-pseudofunctor $yc = \mathcal{C}(c,-)$.

Lemma 2.4.3 (Yoneda)**.** *Let $F\colon \mathcal{C} \to \mathcal{V}$ be a $\mathcal{V}$-pseudofunctor. There is a $\mathcal{V}$-equivalence of $\mathcal{V}$-pseudofunctors*

$$\int_d [\mathcal{C}(-,d), Fd] \simeq F.$$

Proof. The argument uses essential uniqueness for enriched biends. Therefore, each c-component of this equivalence of $\mathcal{V}$-pseudofunctors is going to be an equivalence in $\mathcal{V}$

$$Fc \xrightarrow{\simeq} \int_d [\mathcal{C}(c,d), Fd] \tag{2.24}$$

provided by a suitable extra-pseudonatural transformation

$$i\colon Fc \ddot{\Rightarrow} [\mathcal{C}(c,-), F(-)]. \tag{2.25}$$

The latter is defined by the following correspondence

$$\begin{array}{c} F_{c,-}\colon \mathcal{C}(c,-) \Rightarrow [Fc, F(-)] \\ \hline \mathbb{1} \ddot{\Rightarrow} [\mathcal{C}(c,-), [Fc, F(-)]] \\ \hline \mathbb{1} \ddot{\Rightarrow} [Fc, [\mathcal{C}(c,-), F(-)]] \\ \hline i\colon Fc \ddot{\Rightarrow} [\mathcal{C}(c,-), F(-)], \end{array}$$

where steps are respectively given by Proposition 2.1.10, by the equivalence $[\mathcal{C}(c,-), [Fc, F(-)]] \simeq [Fc, [\mathcal{C}(c,-), F(-)]]$, and by Remark 2.1.4 part (ii), for the adjunction $- \otimes Fc \dashv [Fc, -]$, which eventually give an extra-pseudonatural of sort (2.25) as desired. Now if $j\colon x \ddot{\Rightarrow} [\mathcal{C}(c,-), F(-)]$ is a biwedge, we claim that one can consider

$$\tilde{j} = \varepsilon_{Fc} \circ j_c \otimes u_c \colon x \longrightarrow [\mathcal{C}(c,c), Fc] \otimes \mathcal{C}(c,c) \longrightarrow Fc,$$

as terminal morphism. In other terms, we want to prove (2.24) by showing that there is an equivalence of categories

$$\mathcal{V}(x, Fc) \simeq \mathcal{V}\text{-}\mathsf{PsNat}^{\mathsf{e}}(x, [\mathcal{C}(c,-), F(-)]) \tag{2.26}$$

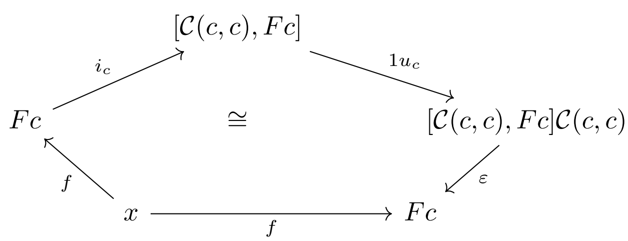
$$f \mapsto i \circ f \tag{2.27}$$

$$\varepsilon_{Fc} \circ j_c \otimes u_c \leftarrow\!\shortmid j \tag{2.28}$$

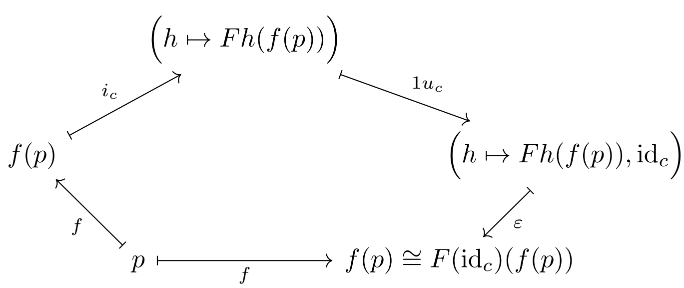
As one can see, the definition of the extra-pseudonatural transformation witnessing Fc as a biend is fairly involved and requires a few non-trivial steps. Therefore, let us make more clear what is happening by looking at "objects" of our objects in $\mathcal{V}$. Then, to reconstruct the proof in the properly enriched case will be a easy though meticulous exercise (left to the reader). Let us then understand the steps required to define i, whose each component i_d is defined by the following long composition

$$
i_d = \quad
\begin{array}{ccc}
Fc & & a \\
\downarrow {\scriptstyle \eta 1} & & \downarrow \\
[\mathcal{C}(c,d), \mathbb{1}\mathcal{C}(c,d)]Fc & & (\mathrm{id}, a) \\
\downarrow {\scriptstyle (F_{c,-})_* 1} & & \downarrow \\
[\mathcal{C}(c,d), [Fc, Fd]]Fc & & (f \mapsto Ff, a) \\
\downarrow {\scriptstyle (-\otimes Fc)1} & & \downarrow \\
[\mathcal{C}(c,d)Fc, [Fc, Fd]Fc]Fc & & ((f,b) \mapsto (Ff, b), a) \\
\downarrow {\scriptstyle \varepsilon_* 1} & & \downarrow \\
[\mathcal{C}(c,d)Fc, Fd]Fc & & ((f,b) \overset{ev}{\mapsto} Ff(b), a) \\
\downarrow {\scriptstyle \beta^* 1} & & \downarrow \\
[Fc\,\mathcal{C}(c,d), Fd]Fc & & ((b,f) \overset{ev}{\mapsto} Ff(b), a) \\
\downarrow {\scriptstyle [\mathcal{C}(c,d),-]1} & & \downarrow \\
[[\mathcal{C}(c,d), Fc\,\mathcal{C}(c,d)], [\mathcal{C}(c,d), Fd]]Fc & & (g \mapsto (h \mapsto ev(g(h))), a) \\
\downarrow {\scriptstyle \eta^* 1} & & \downarrow \\
[Fc, [\mathcal{C}(c,d), Fd]]Fc & & (p \mapsto (h \mapsto Fh(p)), a) \\
\downarrow {\scriptstyle \varepsilon} & & \downarrow \\
[\mathcal{C}(c,d), Fd] & & h \mapsto Fh(a)
\end{array}
$$

Then, to prove the equivalence (2.26) one can start on one hand with a map $f\colon x \to Fc$ and exhibit an isomorphism



This is given be the functoriality for F, since one has



On the other hand, if we start with an extra-pseudonatural $j\colon x \ddot{\Rightarrow} [\mathcal{C}(c,-), F(-)]$ we need an isomorphism

$$\begin{array}{ccccc} & & [\mathcal{C}(c,c), Fc]\mathcal{C}(c,c) & & \\ & \overset{1u_c}{\nearrow} & & \overset{\varepsilon}{\searrow} & \\ [\mathcal{C}(c,c), Fc] & & \cong & & Fc \\ & \underset{j_c}{\nwarrow} & & \underset{i_d}{\swarrow} & \\ & x & \xrightarrow[j_d]{} & [\mathcal{C}(c,d), Fd] & \end{array}$$

On objects, this is

$$p \mapsto j_c(p) \mapsto j_c(p)(\mathrm{id}_c) \mapsto (h \mapsto Fh(j_c(p)(\mathrm{id}_c))),$$

in one direction, while clearly $h \mapsto j_d(p)$ in the other. The structural 2-isomorphism of j

$$\begin{array}{ccc} x & \xrightarrow{j_d} & [\mathcal{C}(c,d), Fd] \\ {\scriptstyle j_c}\downarrow & \swarrow j_h & \downarrow{\scriptstyle \mathcal{C}(c,h)^*} \\ [\mathcal{C}(c,c), Fc] & \xrightarrow[Fh_*]{} & [\mathcal{C}(c,c), Fd] \end{array}$$

then concludes the proof. □

An analogous argument allows to prove the opposite version of the Yoneda lemma.

Proposition 2.4.4 (co-Yoneda)**.** *There is, for $F\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{V}$ a $\mathcal{V}$-pseudofunctor, a $\mathcal{V}$-equivalence of $\mathcal{V}$-pseudofunctors*

$$F \simeq \int^{d} \mathcal{C}(-,d) \otimes F(d)$$

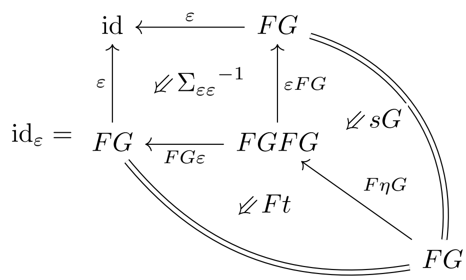
2.5 Fubini rule

A main result for (co)ends is the exchange rule, saying that the operation of taking (co)ends for several variable functors

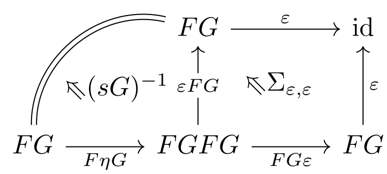
$$P\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{B}^{\mathsf{op}} \times \mathcal{C} \times \mathcal{B} \to \mathcal{D}$$

can be done all at once (on the category $\mathcal{C} \times \mathcal{B}$) or either one variable at a time, and the result is the same, no matter the order. This is called, for the similarity with the well known calculus rule, the Fubini theorem. There's a canonical argument, working also for bicategories, as shown for bicoends in [Cor16], consisting of building a canonical equivalence between the desired bicoends. Generalizing [Lor21], we can prove the result with a more elegant argument which realizes the operation of taking $\mathcal{V}$-valued bicoends (the $\mathcal{V}$-pseudofunctor (2.20)) as a right pseudoadjoint, and concludes by general uniqueness properties of pseudoadjunctions. Also, the following Remark will be used in order to avoid an explicit definition of one of the two triangulators, which is apparently quite non-trivial to find explicitly.

Remark 2.5.1. The strictification theorem for pseudoadjunctions (Theorem A.0.6) tells us that *the data* of a pseudoadjunction (pseudofunctors, unit, counit and the two triangulators) is sufficient, by altering one of the two triangulators, to also have a proper pseudoadjunction, meaning that the swallowtail equations are satisfied. This requires, anyway, to initially have both triangulators. However, if we do not have available of one of the two, we can sometimes (under the condition explained below) reconstruct it by imposing one of the swallowtail equations to be true. If for example we have s, it is sufficient to consider the equation



and define what is going to be $\varepsilon * Ft$ to be the 2-cell



For this to be sufficient to provide a definition of t, one clearly has to ensure that the functor $\varepsilon * F \colon \mathsf{PsNat}(G, G) \to \mathsf{PsNat}(FG, \mathrm{id})$ is fully faithful.

Theorem 2.5.2. *The $\mathcal{V}$-pseudofunctor*

$$\int_{\mathcal{B}} \colon [\![\mathcal{B}^{\mathsf{op}} \otimes \mathcal{B}, \mathcal{V}]\!] \longrightarrow \mathcal{V}$$

admits a left pseudoadjoint $H_{\mathcal{B}} \colon \mathcal{V} \to [\![\mathcal{B}^{\mathsf{op}} \otimes \mathcal{B}, \mathcal{V}]\!]$.

Proof. The $\mathcal{V}$-pseudofunctor $H_{\mathcal{B}}$ is defined on objects to be

$$H_{\mathcal{B}}(a) = \mathcal{B}(-,-) \otimes a,$$

and on hom-objects as the map

$$\mathcal{V}(a, a') \longrightarrow \int_{b,b'} [\mathcal{B}(b, b') \otimes a, \mathcal{B}(b, b') \otimes a'] = \underline{\mathcal{V}\text{-}\mathsf{PsNat}}(H_{\mathcal{B}}(a), H_{\mathcal{B}}(a'))$$

induced by the extra-pseudonatural family

$$\Big\{ \mathcal{B}(c, c') \otimes - \colon \mathcal{V}(a, a') \to [\mathcal{B}(c, c') \otimes a, \mathcal{B}(c, c') \otimes a'] \Big\}_{c,c'}.$$

The unit and the counit for the pseudoadjunction are then defined as follows. Each component of the unit

$$\eta_a \colon a \longrightarrow \int_b (\mathcal{B}(b,b) \otimes a)$$

is the map induced by biend property via the extra-pseudonatural transformation $u \otimes \mathrm{id} \colon \mathbb{1} \otimes a \Rrightarrow \mathcal{B}(-,-) \otimes a$. The P-component of the counit is provided by the pseudonatural transformation having components

$$(\varepsilon_P)_{b,b'} \colon \mathcal{B}(b,b') \otimes \int_c P(c,c) \xrightarrow{P(b,-)\otimes i_b} [P(b,b), P(b,b')] \otimes P(b,b) \xrightarrow{\epsilon} P(b,b'),$$

where $\epsilon = \epsilon_{P(b,b')}$ is the counit for the closedness pseudoadjunction. There is a non-evident fact, here, since *a priori* there is no evident reason for ε_P so defined to be a morphism of pseudofunctors. The crucial point is Lemma 2.1.12, which applies to this setting thanks to the fact that ϵ is both pseudonatural and extra-pseudonatural depending on which variable we focus on (Theorem A.1.2). Triangulators are then the remaining part of the structure. Let us start with s

$$\begin{array}{ccc} H_{\mathcal{B}} & \xRightarrow{H_{\mathcal{B}}\eta} & H_{\mathcal{B}} \circ \int_{\mathcal{B}} \circ H_{\mathcal{B}} \\ & \nearrow s & \Big\Downarrow \varepsilon H_{\mathcal{B}} \\ & = & H_{\mathcal{B}} \end{array}$$

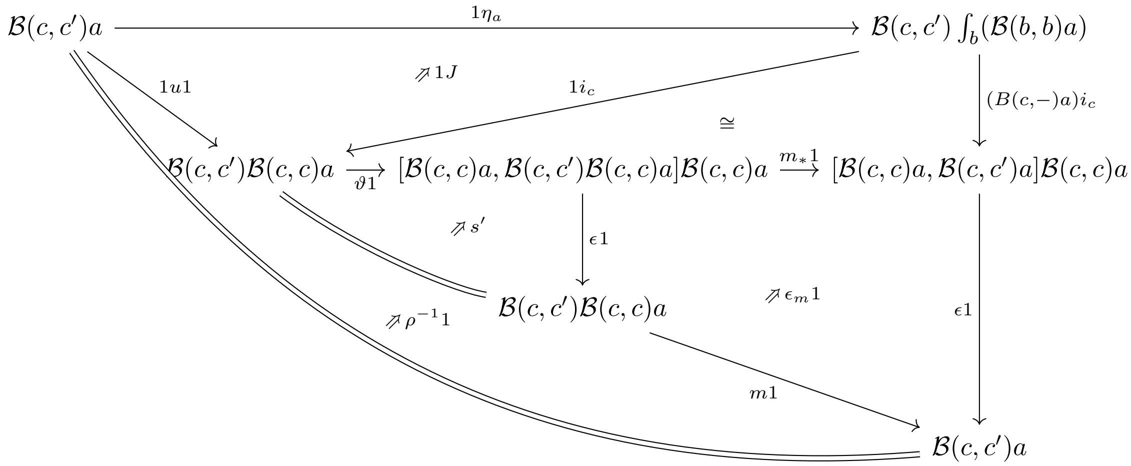
which is, for every object a in $\mathcal{V}$ and every pair of objects c, c' in $\mathcal{B}$, a 2-cell in $\mathcal{V}$

$$\begin{array}{ccc} \mathcal{B}(c,c') \otimes a & \xrightarrow{1\eta_a} & \mathcal{B}(c,c') \otimes \int_b (\mathcal{B}(b,b) \otimes a) \\ & \nearrow (s_a)_{c,c'} & \Big\downarrow (\varepsilon_{\mathcal{B}(-,-)\otimes a})_{c,c'} \\ & = & \mathcal{B}(c,c') \otimes a \end{array}$$

Remember how the pseudofunctor $\mathcal{B}(c,-)$ is defined, as the transpose of the multiplication via the adjunction $- \otimes \mathcal{B}(c,c) \vdash [\mathcal{B}(c,c), -]$. That means, explicitly

$$\mathcal{B}(c,-) \colon \mathcal{B}(c,c') \xrightarrow{\vartheta} [\mathcal{B}(c,c), \mathcal{B}(c,c')\mathcal{B}(c,c)] \xrightarrow{m_*} [\mathcal{B}(c,c), \mathcal{B}(c,c')].$$

for ϑ the unit for the closedness adjunction. The triangulator $(s_a)_{c,c'}$ is then defined to be the following composite.



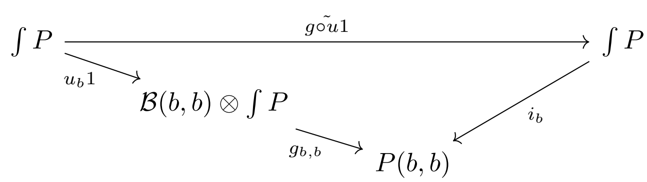
for J the morphism induced by biend property together with η, and s' the triangulator for the closedness pseudoadjunction. We are going to define the other triangulator by imposing one of the swallowtail equations to be true. The strictification theorem will then conclude the proof, since the two swallowtail equations will then automatically be satisfied. In order to do so, we can use Remark 2.5.1 and limit ourselves to directly prove the equivalence

$$\varepsilon * \mathcal{B}(=,=)\colon \mathcal{V}\text{-}\mathsf{PsNat}(\int, \int) \simeq \mathcal{V}\text{-}\mathsf{PsNat}(\mathcal{B}(=,=)\otimes -, -).$$

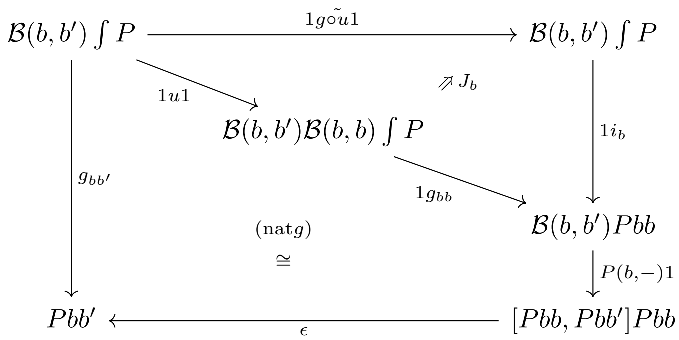
mapping each member f_P of a natural family of morphisms $f_P\colon \int P \to \int P$ to the pseudonatural transformation with components

$$\mathcal{B}(b,b')\otimes \int P \xrightarrow{1f} \mathcal{B}(b,b')\otimes \int P \xrightarrow{P(b,-)\otimes i_b} [Pbb, Pbb']Pbb \xrightarrow{\epsilon} Pbb'$$

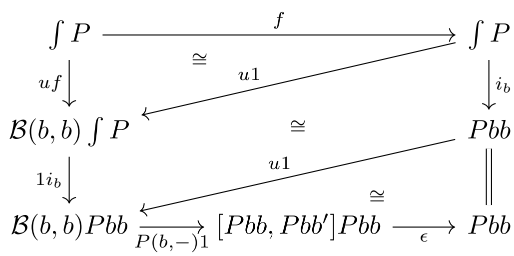
This functor has indeed an explicit inverse, given by mapping a pseudonatural transformation $g\colon \mathcal{B}(-,-)\otimes \int P \to P$ to the morphism $\int P \to \int P$ induced by



The fact that this defines an equivalence of categories is exhibited with the following isomorphisms



for one composition of the functors, and for the other one



□

Corollary 2.5.3 (Fubini)**.** *Let* $P\colon \mathcal{B}^{\mathsf{op}}\otimes\mathcal{B}\otimes\mathcal{C}^{\mathsf{op}}\otimes\mathcal{C}\to\mathcal{V}$ *be a* $\mathcal{V}$*-pseudofunctor. Then, there are equivalences of* $\mathcal{V}$*-pseudofunctors*

$$\int_{\mathcal{B}}\circ\int_{\mathcal{C}}\simeq\int_{\mathcal{B}\otimes\mathcal{C}}\simeq\int_{\mathcal{C}}\circ\int_{\mathcal{B}}$$

Proof. Since we ensured that the $\mathcal{V}$-pseudofunctor $\displaystyle\int_{\mathcal{B}}$ has a left pseudoadjoint $\mathcal{B}(-,-)\otimes =$, then, also

$$\int_{\mathcal{B}\otimes\mathcal{C}}\colon [\![(\mathcal{B}\otimes\mathcal{C})^{\mathsf{op}}\otimes(\mathcal{B}\otimes\mathcal{C}),\mathcal{V}]\!]\longrightarrow\mathcal{V}$$

has a left pseudoadjoint $(\mathcal{B}\otimes\mathcal{C})(-,-,-,-)\otimes =$, which is actually defined as $(\mathcal{B}(-,-)\otimes\mathcal{C}(-,-))\otimes =$. The monoidal associator then provides an equivalence between this $\mathcal{V}$-pseudofunctor and the composition

$$\mathcal{V}\xrightarrow{\mathcal{C}(-,-)\otimes =}[\![\mathcal{C}^{\mathsf{op}}\otimes\mathcal{C},\mathcal{V}]\!]\xrightarrow{\mathcal{B}(-,-)\otimes =}[\![(\mathcal{B}\otimes\mathcal{C})^{\mathsf{op}}\otimes(\mathcal{B}\otimes\mathcal{C}),\mathcal{V}]\!]$$

which is the composite of two left pseudoadjoints. Then, it is straightforward, generalizing the well known argument for usual adjunctions, that pseudoadjunctions can compose, as well as the fact that pseudoadjoints to equivalent $\mathcal{V}$-pseudofunctors are equivalent. Therefore, we get that the composition $\int_{\mathcal{C}}\circ\int_{\mathcal{B}}$ is equivalent to the $\mathcal{V}$-pseudofunctor $\int_{\mathcal{B}\otimes C}$. Analogously, by symmetry we can show that this is also equivalent to $\int_{\mathcal{B}}\circ\int_{\mathcal{C}}$. □

Remark 2.5.4. The Fubini theorem for arbitrary $\mathcal{D}$-valued $\mathcal{V}$-pseudofunctors $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{C}^{\mathsf{op}} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{D}$ is then given representably via the Yoneda lemma

$$\begin{aligned}\mathcal{D}(d, \int^b \int^c P(b,c,b,c)) &\simeq \int^b \int^c \mathcal{D}(d, P(b,c,b,c)) \\ &\simeq \int^{b,c} \mathcal{D}(d, P(b,c,b,c)) \simeq \mathcal{D}(d, \int^{b,c} P(b,c,b,c)).\end{aligned}$$

2.6 Kan extensions of pseudofunctors

In this section we introduce a notion of Kan extension for enriched pseudofunctors valued in the base $\mathcal{V}$ of the enrichment. To be more precise, we are going to define what based on the literature should be called *local* Kan extension. The work of Garner–Shulman [GS15] introduces a notion of *pointwise* Kan extension for enriched pseudofunctors, defined in terms of weighted bi(co)limits. In this setting, where we are going to treat extensions of pseudofunctors into the base monoidal bicategory $\mathcal{V}$, we are going to define pointwise Kan extension via certain bi(co)ends (see Remark 2.6.3), and we will show (Theorem 2.6.2) that they are in fact also local Kan extensions.

In particular, this result will make clear the universal property of the Day convolution in Section 2.7. Our discussion, as mentioned, will only cover the case of the extension along any $\mathcal{V}$-pseudofunctor of a $\mathcal{V}$-valued $\mathcal{V}$-pseudofunctors. The vast literature on the 1-categorical version of the subject allows to see left Kan extensions as coends whenever the target category is cotensored over the base of the enrichment. Dually, we also can see right Kan extension as ends whenever the target is tensored over the base of the enrichment. Our choice of $\mathcal{V}$ for the target is then a particularly specific context allowing both of these descriptions, but it will be sufficient for our purposes.

Definition 2.6.1. Let $\mathcal{V}$ be a monoidal bicategory, $\mathcal{M}, \mathcal{C}$ two $\mathcal{V}$-bicategories and $T\colon \mathcal{M} \to \mathcal{V}$, $K\colon \mathcal{M} \to \mathcal{C}$ two $\mathcal{V}$-pseudofunctors. A $\mathcal{V}$-pseudofunctor $L\colon \mathcal{C} \to \mathcal{V}$ together with an enriched pseudonatural transformation $\alpha\colon T \to LK$ (equivalently, a 1-cell $\alpha\colon \mathbb{1} \longrightarrow \int_m \mathcal{V}(Tm, LKm)$)

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad T \quad} & \mathcal{V} \\ & {\scriptstyle K}\searrow \;\Downarrow\alpha\; \nearrow{\scriptstyle L} & \\ & \mathcal{C} & \end{array}$$

is said to be a *(local) left Kan extension* of the $\mathcal{V}$-pseudofunctor T along K if the following composition is an equivalence $\underline{\mathcal{V}\text{-}\mathsf{PsNat}}(L,S) \simeq \underline{\mathcal{V}\text{-}\mathsf{PsNat}}(T,SK)$ in $\mathcal{V}$:

$$\int_c \mathcal{V}(Lc,Sc) \xrightarrow{\tilde{j_K}\otimes\alpha} \int_m \mathcal{V}(LKm,SKm) \otimes \int_m \mathcal{V}(Tm,LKm) \xrightarrow{\circ} \int_m \mathcal{V}(Tm,SKm).$$

The morphism $\tilde{j_K}$ is induced by the biend property as

$$\begin{array}{ccc} \int_c \mathcal{A}(Lc,Sc) & \overset{\tilde{j_K}}{\dashrightarrow} & \int_c \mathcal{A}(LKm,SKm) \\ {\scriptstyle j_{Kn}}\searrow & \swarrow J_{Kn} & \swarrow{\scriptstyle i_n} \\ & \mathcal{A}(LKn,SKn) & \end{array}$$

The $\mathcal{V}$-pseudofunctor L will also be denoted as $\mathsf{PsLan}_K T$.

Theorem 2.6.2. *Let $\mathcal{V}$ be a braided monoidal bicategory, $\mathcal{C}$ and $\mathcal{M}$ two $\mathcal{V}$-bicategories, and $T\colon \mathcal{M} \to \mathcal{V}$ as well as $K\colon \mathcal{M} \to \mathcal{C}$, $\mathcal{V}$-pseudofunctors. Suppose that for every $c \in \mathcal{C}$ the bicoend*

$$\eta^c\colon \mathcal{C}(K-, c) \otimes T(-) \stackrel{..}{\Rightarrow} \int^m \mathcal{C}(Km, c) \otimes Tm$$

exists. Then, so does the left Kan extension $L = \mathsf{PsLan}_K T$ computed as

$$\mathsf{PsLan}_K T(c) = \int^m \mathcal{C}(Km, c) \otimes Tm.$$

Proof. The argument proving the desired equivalence follows this chain of computations, generalizing the one which can be found in [Mac71].

$$\begin{aligned}
\underline{\mathcal{V}\text{-}\mathsf{PsNat}}(L, S) &\simeq \int_c [Lc, Sc] \\
&\simeq \int_c \left[\int^m C(Km, c) \otimes Tm, Sc\right] \\
&\stackrel{(2.14)}{\simeq} \int_c \int_m [C(Km, c) \otimes Tm, Sc] \\
&\stackrel{\text{Fubini}}{\simeq} \int_m \int_c [C(Km, c) \otimes Tm, Sc] \\
&\simeq \int_m \int_c [Tm, [C(Km, c), Sc]] \\
&\stackrel{(2.14)}{\simeq} \int_m \left[Tm, \int_c [C(Km, c), Sc]\right] \\
&\stackrel{\text{Yoneda}}{\simeq} \int_m [Tm, SKm] \\
&\simeq \underline{\mathcal{V}\text{-}\mathsf{PsNat}}(T, SK)
\end{aligned}$$

□

Remark 2.6.3. The bicoend $\int^m \mathcal{C}(Km, c) \otimes Tm$ is what should be called *pointwise* Kan extension of T along K, in the sense that we have it computed at a "point" c.

2.7 Day convolution

Let us start with a brief recollection of the 1-categorical version of the story. If $(\mathcal{C}, \odot, \mathbb{1})$ is a monoidal category enriched on a closed complete and cocomplete symmetric monoidal category $\mathcal{V}$, then we can equip the category of functors from $\mathcal{C}$ to $\mathcal{V}$ with a convolution product defined as the coend

$$(F \underset{\text{Day}}{\otimes} G)(X) = \int^{C,C'} \mathcal{C}(C \odot C', X) \otimes F(C) \otimes G(C').$$

An equivalent way to define this tensor product is as a left Kan extension of the exterior product $F \boxtimes G\colon \mathcal{C} \times \mathcal{C} \to \mathcal{V}$ along the monoidal product functor $\odot\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C} & \xrightarrow{F\boxtimes G} & \mathcal{V} \\ & {\scriptstyle\odot}\searrow \quad \Downarrow\alpha \quad \nearrow{\scriptstyle F \underset{\text{Day}}{\otimes} G} & \\ & \mathcal{C} & \end{array}$$

The universal property of the left Kan extension gives a natural isomorphism between *sets* of natural transformations

$$\mathsf{Nat}(F \underset{\text{Day}}{\otimes} G, H) \cong \mathsf{Nat}(F \boxtimes G, H(- \odot -)).$$

Definition 2.7.1. Let $(\mathcal{C}, \odot, \mathbb{1})$ be a monoidal $\mathcal{V}$-bicategory, and let $F, G \colon \mathcal{C} \to \mathcal{V}$ be two $\mathcal{V}$-pseudofunctors. The *Day convolution* product of the $\mathcal{V}$-pseudofunctors F and G is defined by the composition

$$F \underset{\text{Day}}{\otimes} G\colon\ \mathcal{C} \xrightarrow{\mathcal{C}(-\odot-,=)\otimes F(-)\otimes G(-)} \mathcal{V}\text{-}\mathsf{PsFun}((\mathcal{C}\otimes\mathcal{C})^{\mathsf{op}} \otimes \mathcal{C} \otimes \mathcal{C}) \xrightarrow{\int^{\mathcal{C}\otimes\mathcal{C}}} \mathcal{V}.$$

More explicitly, it is the $\mathcal{V}$-pseudofunctor whose evaluation at x in $\mathcal{C}$ as the enriched bicoend of the $\mathcal{V}$-pseudofunctor

$$\mathcal{C}(- \odot -, x) \otimes F(-) \otimes G(-) \colon (\mathcal{C}\otimes\mathcal{C})^{\mathsf{op}} \otimes \mathcal{C} \otimes \mathcal{C} \to \mathcal{V}. \tag{2.29}$$

Proposition 2.7.2. *If $F, G, H \colon \mathcal{C} \to \mathcal{V}$ are $\mathcal{V}$-pseudofunctors, and $(\mathcal{C}, \odot)$ is a monoidal $\mathcal{V}$-bicategory as above, then there is an equivalence in $\mathcal{V}$*

$$\underline{\mathcal{V}\mathsf{PsNat}}(F \underset{\text{Day}}{\otimes} G, H) \simeq \underline{\mathcal{V}\mathsf{PsNat}}(F \boxtimes G, H(- \odot -)).$$

Proof. It's an immediate consequence of Theorem 2.6.2 and the definition of Kan extension. □

Proposition 2.7.3. *The Day convolution product defines a $\mathcal{V}$-pseudofunctor*

$$- \underset{\text{Day}}{\otimes} - \colon [\![\mathcal{C}, \mathcal{V}]\!] \otimes [\![\mathcal{C}, \mathcal{V}]\!] \longrightarrow [\![\mathcal{C}, \mathcal{V}]\!].$$

Proof. The result uses pseudofunctoriality of the monoidal tensor pseudofunctor of $\mathcal{V}$ and of the operation of taking bicoends. More precisely, from the pseudofunctoriality of $\otimes_{\mathcal{V}}$ there is a $\mathcal{V}$-pseudofunctor

$$\begin{aligned} [\![\mathcal{C}, \mathcal{V}]\!] \otimes [\![\mathcal{C}, \mathcal{V}]\!] &\longrightarrow [\![(\mathcal{C}\otimes\mathcal{C})^{\mathsf{op}} \otimes (\mathcal{C}\otimes\mathcal{C}), [\![\mathcal{C}, \mathcal{V}]\!]]\!] \\ (F, G) &\longmapsto \mathcal{C}(- \odot -, =) \otimes F(-) \otimes G(-) \end{aligned}$$

Then, we can compose with the bicoend pseudofunctor $\int^{\mathcal{C}\otimes\mathcal{C}}$, and this composite clearly gives $- \underset{\text{Day}}{\otimes} -$. □

Remark 2.7.4. The following Lemma should be true for bicategories of pseudofunctors taking values in a more general $\mathcal{V}$-bicategory $\mathcal{D}$ than $\underline{\mathcal{V}\text{-}\mathsf{PsFun}}(\mathcal{C}, \mathcal{V})$. However, a notion of bicategory tensored over $\mathcal{V}$ should be introduced. Also, we state a weak form since it suffices for our goals. It should be clear at this point that most of biequivalences of bicategories which are also enriched can be enhanced to enriched biequivalences.

Lemma 2.7.5. *Let $\mathcal{C}$ be a $\mathcal{V}$-bicategory. There is a biequivalence induced by precomposition with the Yoneda pseudofunctor*

$$\mathcal{V}\text{-}\mathsf{PsFun}^{\mathsf{c}}([\![\mathcal{C},\mathcal{V}]\!],[\![\mathcal{C},\mathcal{V}]\!]) \xrightarrow{\simeq} \mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C}^{\mathsf{op}},[\![\mathcal{C},\mathcal{V}]\!]).$$

between the bicategory of weighted bicolimit-preserving $\mathcal{V}$-pseudofunctors from the pseudofunctor $\mathcal{V}$-bicategory and the one of all $\mathcal{V}$-pseudofunctors from $\mathcal{C}^{\mathsf{op}}$.

Proof. The correspondence, induced by $y\colon \mathcal{C}^{\mathsf{op}} \to [\![\mathcal{C},\mathcal{V}]\!]$, is given by

$$\begin{array}{rcl}
\mathcal{V}\text{-}\mathsf{PsFun}^{\mathsf{c}}([\![\mathcal{C},\mathcal{V}]\!],[\![\mathcal{C},\mathcal{V}]\!]) & \rightleftarrows & \mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C}^{\mathsf{op}},[\![\mathcal{C},\mathcal{V}]\!]) \\
F & \longmapsto & F\circ y \\
\int^c \mathcal{V}\text{-}\mathsf{PsNat}(y(c),-)\otimes L(c) & \longleftarrow\!\shortmid & L
\end{array}$$

On one hand, if we start with F, we go back and forth and apply the result to $G\colon \mathcal{C}\to\mathcal{V}$, we find, by iterating different form of Yoneda and by the preservation of bicoends,

$$\begin{aligned}
\int^c \underline{\mathcal{V}\text{-}\mathsf{PsNat}}(y(c),G)\otimes F(y(c)) &= \int^c \int_{c'} \mathcal{V}(y(c)(c'),G(c'))\otimes F(y(c)) \\
&\simeq \int^c G(c)\otimes F(y(c)) \\
&\simeq F\left(\int^c G(c)\otimes y(c)\right) \\
&\simeq F(G).
\end{aligned}$$

On the other hand, if we start by $L\colon \mathcal{C}^{\mathsf{op}} \to \mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C},\mathcal{V})$ and apply it to d, we get

$$\begin{aligned}
\int^c \underline{\mathcal{V}\text{-}\mathsf{PsNat}}(y(c),y(d))\otimes L(c) &= \int^c \int_{c'} \mathcal{V}(y(c)(c'),y(d)(c'))\otimes L(c) \\
&\simeq \int^c L(c)\otimes \int_{c'} \mathcal{V}(y(c)(c'),y(d)(c')) \\
&\simeq \int^c L(c)\otimes y(d)(c) \\
&\simeq L(d).
\end{aligned}$$

The fact that this correspondence is pseudofunctorial directly follows from the pseudofunctoriality of the construction involved. □

Proposition 2.7.6. *Let $(\mathcal{C},\odot)$ be a (braided) monoidal $\mathcal{V}$-bicategory. Then, the Day convolution defines a (braided) monoidal structure on the $\mathcal{V}$-bicategory $[\![\mathcal{C},\mathcal{V}]\!]$.*

Proof. One first needs to check that $-\underset{\text{Day}}{\otimes}-\colon [\![\mathcal{C},\mathcal{V}]\!]\otimes[\![\mathcal{C},\mathcal{V}]\!]\to[\![\mathcal{C},\mathcal{V}]\!]$ is a $\mathcal{V}$-pseudofunctor. The unit pseudofunctor is then defined to be

$$\underline{u} = y\circ u\colon \mathcal{J}\longrightarrow\mathcal{C}\longrightarrow[\![\mathcal{C},\mathcal{V}]\!],$$

for $\mathcal{J}$ the unit $\mathcal{V}$-bicategory, u the unit for the monoidal $\mathcal{V}$-bicategory $\mathcal{C}$, and y the Yoneda $\mathcal{V}$-pseudofunctor. Now observe that the Day convolution is actually a particular case of the Yoneda extension provided in the correspondence of Lemma 2.7.5. There is, in other terms, an equivalence $y(c) \underset{\text{Day}}{\otimes} y(c') \simeq y(c \odot c')$, provided by two iterations of the co-Yoneda lemma:

$$\begin{aligned} y(c) \underset{\text{Day}}{\otimes} y(c') &= \int^{c_1,c_2} \mathcal{C}(c_1 \odot c_2, -) \otimes \mathcal{C}(c, c_1) \otimes \mathcal{C}(c', c_2) \\ &\simeq \int^{c_1} \mathcal{C}(c, c_1) \otimes \int^{c_2} \mathcal{C}(c_1 \odot c_2, -) \otimes \mathcal{C}(c', c_2) \\ &\simeq \int^{c_1} \mathcal{C}(c, c_1) \otimes \mathcal{C}(c_1 \odot c', -) \\ &\simeq y(c \odot c'). \end{aligned}$$

Therefore, there is an equivalence of $\mathcal{V}$-pseudofunctors

$$\begin{array}{ccc} \mathcal{C}^{\mathsf{op}} \otimes \mathcal{C}^{\mathsf{op}} & \xrightarrow{\odot^{\mathsf{op}}} & \mathcal{C}^{\mathsf{op}} \\ {\scriptstyle y \otimes y}\downarrow & \simeq & \downarrow{\scriptstyle y} \\ [\![\mathcal{C}, \mathcal{V}]\!] \otimes [\![\mathcal{C}, \mathcal{V}]\!] & \xrightarrow[\underset{\text{Day}}{\otimes}]{} & [\![\mathcal{C}, \mathcal{V}]\!] \end{array} \tag{2.30}$$

and this will allow to lift the whole structure via the biequivalence of Lemma 2.7.5. Consider the monoidal structure for $\mathcal{C}$

$$\begin{array}{ccc} \mathcal{C}^{\otimes 3} & \xrightarrow{\mathrm{id} \otimes \odot} & \mathcal{C}^{\otimes 2} \\ {\scriptstyle \odot \otimes \mathrm{id}}\downarrow & \swarrow a & \downarrow{\scriptstyle \odot} \\ \mathcal{C}^{\otimes 2} & \xrightarrow[\odot]{} & \mathcal{C} \end{array}$$

$$\begin{array}{ccccc} & & \mathcal{C}^{\otimes 2} & & \\ & {\scriptstyle u \otimes \mathrm{id}}\nearrow & \Downarrow \ell & \searrow{\scriptstyle \odot} & \\ \mathcal{C} & & \xrightarrow[\mathrm{id}]{} & & \mathcal{C} \end{array} \qquad \begin{array}{ccccc} & & \mathcal{C}^{\otimes 2} & & \\ & {\scriptstyle \mathrm{id} \otimes u}\nearrow & \Downarrow r & \searrow{\scriptstyle \odot} & \\ \mathcal{C} & & \xrightarrow[\mathrm{id}]{} & & \mathcal{C} \end{array}$$

And let us call F_ℓ, F_r, F_a the $\mathcal{V}$-pseudofunctors domains of respectively ℓ, r, a, as well as G_ℓ, G_r, G_a the respective codomains. Consider then in general terms $F, G \colon \mathcal{D}^{\otimes n} \to \mathcal{E}$, and suppose to have extensions of F and G along the Yoneda pseudofunctor, that is pseudofunctors F', G' and equivalences

$$\begin{array}{ccc} \mathcal{D}^{\mathsf{op} \otimes n} & \xrightarrow{F^{\mathsf{op}}} & \mathcal{E}^{\mathsf{op}} \\ {\scriptstyle y^{\otimes n}}\downarrow & \simeq & \downarrow{\scriptstyle y} \\ [\![\mathcal{D}, \mathcal{V}]\!]^{\otimes n} & \xrightarrow[F']{} & [\![\mathcal{E}, \mathcal{V}]\!] \end{array} \qquad \begin{array}{ccc} \mathcal{D}^{\mathsf{op} \otimes n} & \xrightarrow{G^{\mathsf{op}}} & \mathcal{E}^{\mathsf{op}} \\ {\scriptstyle y^{\otimes n}}\downarrow & \simeq & \downarrow{\scriptstyle y} \\ [\![\mathcal{D}, \mathcal{V}]\!]^{\otimes n} & \xrightarrow[G']{} & [\![\mathcal{E}, \mathcal{V}]\!] \end{array} \tag{2.31}$$

Then, let us claim that there is an equivalence of categories

$$\mathcal{V}\text{-}\mathsf{PsNat}(F' \circ y^{\otimes n}, G' \circ y^{\otimes n}) \simeq \mathcal{V}\text{-}\mathsf{PsNat}(G, F). \tag{2.32}$$

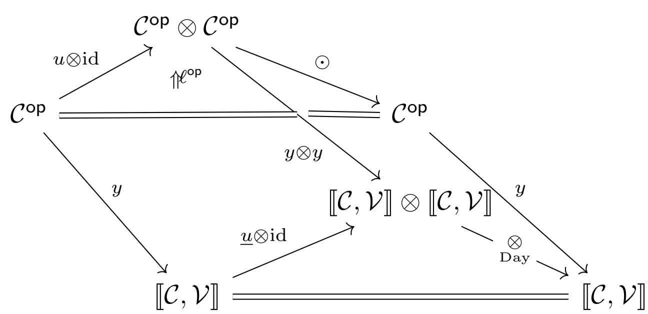
That is because a $\mathcal{V}$-pseudonatural transformation $\gamma \colon F' \circ y^{\otimes n} \Rightarrow G' \circ y^{\otimes n}$ consists of, for every object d in $\mathcal{D}^{\otimes n}$, a morphism in $\mathcal{V}$

$$\gamma_d \colon \mathbb{1} \longrightarrow [\![\mathcal{E}, \mathcal{V}]\!](F'(y^{\otimes n}(d)), G'(y^{\otimes n}(d))).$$

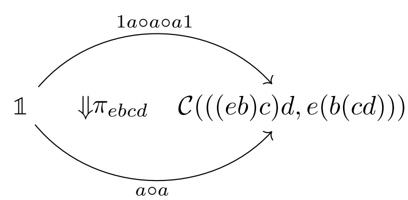
The latter is, by (2.31) and Yoneda lemma

$$\begin{aligned}\int_e [F'(y^{\otimes n}(d))(e), G'(y^{\otimes n}(d))(e)] &\simeq \int_e [y(F(d))(e), y(G(d))(e)] \\ &\simeq \int_e [\mathcal{C}(Fd, e), \mathcal{C}(Gd, e)] \\ &\simeq \mathcal{C}(Gd, Fd).\end{aligned}$$

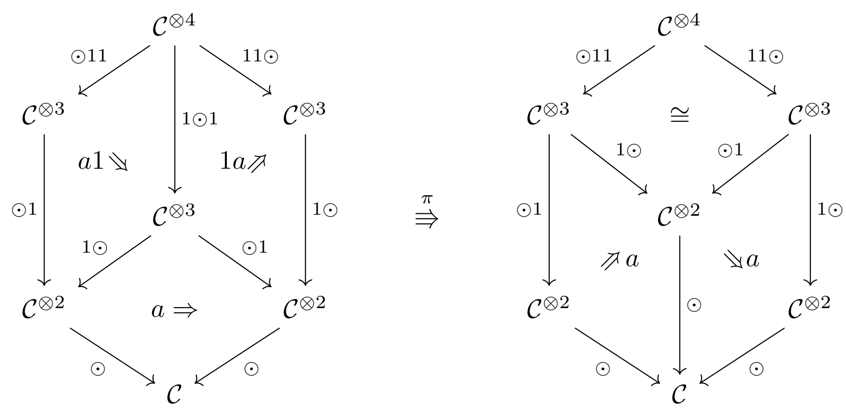
Hence, γ_d correspond to a map $\mathbb{1} \to \mathcal{C}(Gd, Fd)$. The same equivalences in $\mathcal{V}$ determine also the structural 2-cell corresponding to $\gamma_{dd'}$. This establishes the equivalence (2.32). Now observe that in virtue of the definition of the unit $\underline{u}$ for $[\![\mathcal{C}, \mathcal{V}]\!]$ and the equivalence (2.30), the situation depicted for these pseudonatural $F \Rightarrow G$ is the very same of $\ell^{\mathsf{op}} \colon G_\ell \Rightarrow F_\ell$, and similarly of r^{op} and a^{op}. The picture is that we are building parallelepipeds on each of the structures $\ell^{\mathsf{op}}, r^{\mathsf{op}}, a^{\mathsf{op}}$ with parallel Yoneda pseudofunctors



and looking for a top face to define the monoidal left unitor for $[\![\mathcal{C}, \mathcal{V}]\!]$. This is provided by computation (2.32) and Lemma 2.7.5. The same works for r and a. Now, let's come to the higher level structure. Consider π and μ for $(\mathcal{C}, \odot)$. These are 2-cells in the $\mathcal{V}$-bicategories $\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C}^{\otimes 4}, \mathcal{C})$ and $\mathcal{V}\text{-}\mathsf{PsFun}(\mathcal{C}^{\otimes 2}, \mathcal{C})$. Each component of the $\mathcal{V}$-modification π is a 2-cell in $\mathcal{V}$

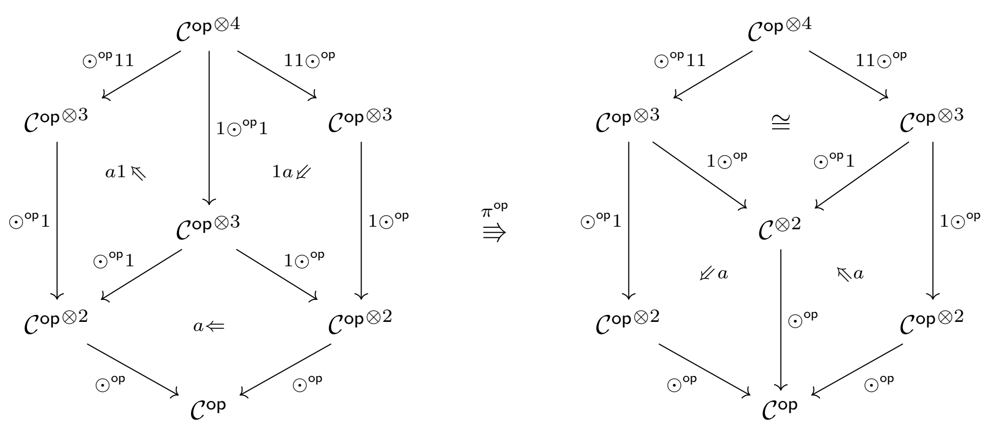


Therefore, π is a $\mathcal{V}$-modification defined between $\mathcal{V}$-pseudonatural transformations themselves going from $F = \odot \circ \odot 1 \circ \odot 11$ to $G = \odot \circ 1 \odot \circ 11 \odot$

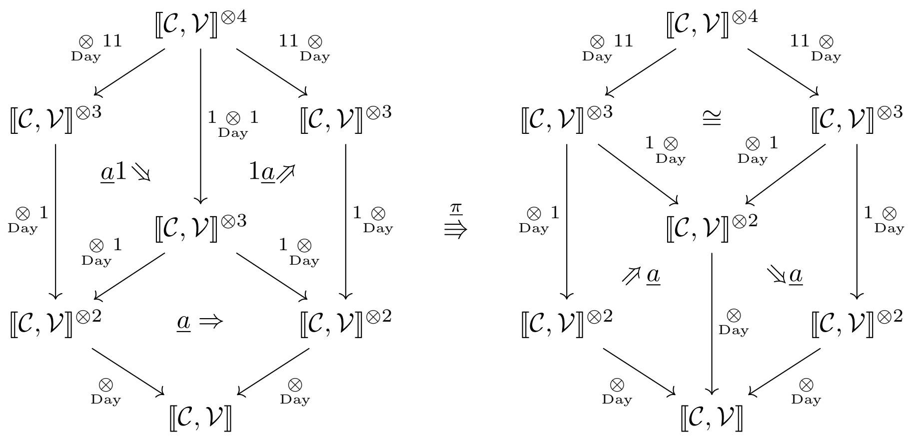


(2.33)

Thus, F and G so defined clearly are again such as in (2.31), with extensions F', G' obtained by simply replacing occurrences of $\odot$ with $\underset{\text{Day}}{\otimes}$. Let us now consider the opposite version of (2.33)

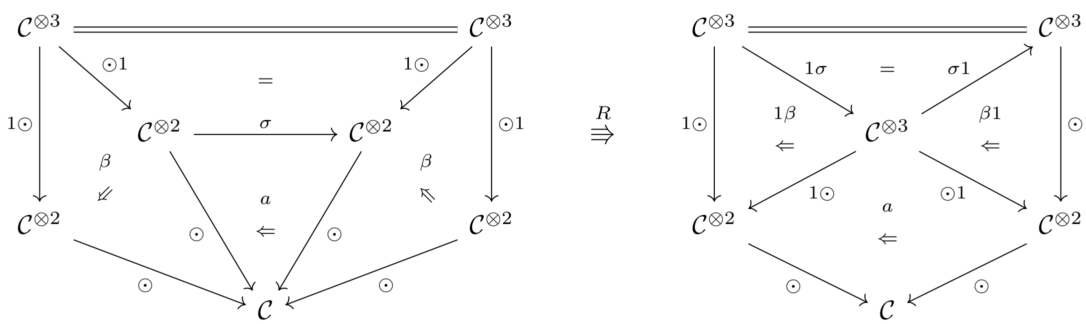


and build parallelepipeds along parallel Yoneda pseudofunctors. The same equivalences (2.32) and that of Lemma 2.7.5 (on 1- and 2-morphisms respectively, *i.e.* in both of cases on $\mathcal{V}$-modifications) will make π correspond to the desired modification



The non-abelian 4-cocycle condition on $\underline{\pi}$ then follows from the same condition on π. An analogous work can be done for μ, γ, δ, and the correspondent normalization axioms.

A similar argument also works for the braided structure and the braiding axioms. R is in fact a modification of the following sort



and it can be lifted via the correspondent equivalences to the same structure on the enriched pseudofunctor $\mathcal{V}$-bicategory. $\square$

Chapter 3

Mackey pseudofunctors

A starting point for the process of categorifying the notion of Mackey functor is the observation (which can be found in [SP07]) that the Mackey formula can be expressed in a natural fashion via categorical notions. Let $i\colon H \subseteq G$ and $j\colon K \subseteq G$ be subgroups of a finite group, and consider an equivariant object such as a representation over a fixed ring $\Bbbk$ of, let's say, H. That is precisely a $\Bbbk H$-module N, and the Mackey formula tells that one can compute the restriction to K of the induction of N to G without necessarily knowing the induction on G, via a choice of representatives in the set of double cosets $H\backslash G/K = \{[x] = HxK, x \in G\}$ as

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H N \cong \bigoplus_{x \in H\backslash G/K} \mathrm{Ind}^K_{H^x \cap K} c_x \mathrm{Res}^H_{H \cap {}^x K} N.$$

Here, we recall the notation for conjugate subgroups ${}^x H = \{x^{-1}hx,\ h \in H\}$, and similarly $K^x = \{xkx^{-1},\ k \in K\}$. Also, $c_x\colon \mathrm{Rep}_{\Bbbk}(H \cap {}^x K) \to \mathrm{Rep}_{\Bbbk}(H^x \cap K)$ is the functor induced by $H^x \cap K \to H \cap {}^x K$ mapping $g \mapsto xgx^{-1}$. If one allows oneself to deal with the more general notion of groupoid, one can observe that the very same double coset arise in the categorical construction of forming pseudopullbacks. Consider the groupoid $\coprod_{[x]\in H\backslash G/K} H \cap {}^x K$, whose set of objects can be evidently be indicated as $\{[x] = HxK\}$ (one for each connected component), and observe that there are two groupoid morphisms (functors)

$$\begin{array}{ccc} & \displaystyle\coprod_{[x]\in H\backslash G/K} H \cap {}^x K & \\ {}^{p}\swarrow & & \searrow^{q} \\ H & & K \end{array}$$

mapping evidently on objects, and defined on morphisms to be

$$p\colon g \mapsto g \text{ and } q\colon (g = xkx\colon [x] \to [x]) \mapsto k.$$

Also, there is an (invertible) natural transformation

$$\begin{array}{ccc} & \displaystyle\coprod_{[x]\in H\backslash G/K} H \cap {}^x K & \\ {}^{p}\swarrow & & \searrow^{q} \\ H & \overset{\sim}{\Rightarrow}\ \gamma & K \\ {}_{i}\searrow & & \swarrow_{j} \\ & G & \end{array} \tag{3.1}$$

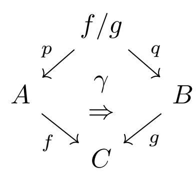
defined by $\gamma_{[x]} = x^{-1}$. Whenever $g = xkx^{-1} \colon [x] \to [x]$ the natural square

$$\begin{array}{ccc} * & \xrightarrow{\gamma_{[x]}} & * \\ {\scriptstyle ip(g)=g}\downarrow & & \downarrow{\scriptstyle k=jq(g)} \\ * & \xrightarrow[\gamma_{[x]}]{} & * \end{array}$$

commutes in G. This natural isomorphism happens to be a *pseudopullback* (see Definition 3.1.3), and this leads to consider another more natural way to express the Mackey formula (the one in the definition of a Mackey pseudofunctor 3.2.5). In this chapter we are going to recollect some definitions, constructions and results from the theory of Mackey pseudofunctors and Mackey motives. The main reference is [BD20], where the theory is introduced and developed.

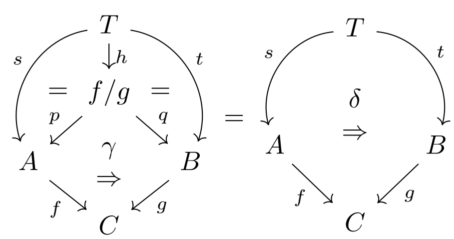
3.1 Pseudopullbacks

Definition 3.1.1. Let $\mathbb{G}$ be a 2-category. A *comma square* over a cospan $f \colon A \to C \leftarrow B : g$ of 1-morphisms in $\mathbb{G}$ is an object f/g together with a 2-cell



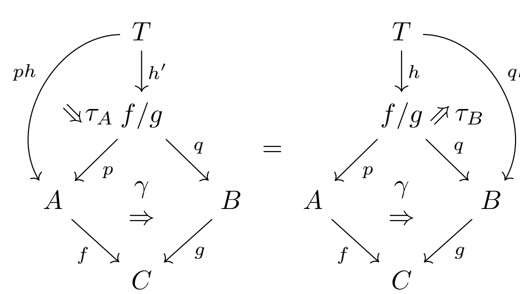
such that the two following properties hold true.

(a) For every triple $s \colon T \to A$, $t \colon T \to B$, $\delta \colon fs \Rightarrow gt$, there's a unique $h \colon T \to f/g$ such that



We will write $\langle s, t, \delta \rangle$ for such an induced h.

(b) For every pair of 1-cells $h, h' \colon T \to f/g$ and every pair of 2-cells τ_A, τ_B such that



There's a unique $\tau\colon h \to h'$ such that $p\tau = \tau_A$ and $q\tau = \tau_B$.

Example 3.1.2. Comma squares in Cat can be constructed explicitly as follows: for a cospan of functors $F\colon \mathcal{C} \to \mathcal{E} \leftarrow \mathcal{D} : G$ the category F/G has objects triples (C, D, γ) with C in $\mathcal{C}$, D in $\mathcal{D}$ and $\gamma\colon F(C) \to G(D)$ a morphism in $\mathcal{E}$, while a morphism $(C, D, \gamma) \to (C', D', \gamma')$ is a pair $(\alpha\colon C \to C', \beta\colon D \to D')$ such that the square

$$\begin{array}{ccc} FC & \xrightarrow{F\alpha} & FC' \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ GD & \xrightarrow[G\beta]{} & GD' \end{array}$$

commutes. Then, functors $\mathcal{C} \leftarrow F/G \to \mathcal{D}$ are the evident projections, so that each (C, D, γ)-component of the natural transformation in the comma square

$$\begin{array}{ccccc} & & F/G & & \\ & {\scriptstyle\pi_{\mathcal{C}}}\swarrow & & \searrow{\scriptstyle\pi_{\mathcal{D}}} & \\ \mathcal{C} & & \Rightarrow & & \mathcal{D} \\ & {\scriptstyle F}\searrow & & \swarrow{\scriptstyle G} & \\ & & \mathcal{E} & & \end{array}$$

is precisely $\gamma\colon FC \to GD$

On one side, the notion of comma square can be specialized to the notion of *iso-comma square* by demanding γ to be invertible and the property (a) to hold for every invertible δ. This last notion is what deserves the name of *2-pullback*. Clearly, in $(2, 1)$-categories $\mathbb{G}$ the two notions coincide. On the other hand, (iso-)comma squares happens to be too restrictive for our purposes, in the precise sense that they are not preserved by biequivalences. A more adequate notion is the one of *pseudo-comma*, and its subsequently defined "iso" version of *pseudopullback* (also called *Mackey squares* in [BD20]).

Definition 3.1.3. Let $\mathbb{G}$ be a (2,1)-category with comma squares. A 2-cell

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} & \\ A & & \overset{\alpha}{\Rightarrow} & & B \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} & \\ & & C & & \end{array}$$

is called a *pseudopullback* if the induced morphism

$$\langle p, q, \alpha\rangle\colon P \longrightarrow f/g$$

is an equivalence in $\mathbb{G}$.

Example 3.1.4. As explained in Remark 2.2.7 in [BD20], for a cospan $i\colon H \hookrightarrow G \hookleftarrow K : j$ of inclusion of subgroups, there's a non-canonical equivalence of categories

$$\coprod_{[x]\in H\backslash G/K} H \cap {}^{x}K \xrightarrow{\sim} i/j,$$

exhibiting the 2-cell (3.1) as a pseudopullback over the given cospan.

Proposition 3.1.5. *Let $\mathbb{G}$ be a (2,1)-category, $f\colon A \to C \leftarrow B : g$ be a cospan admitting a comma square and*

$$\begin{array}{ccccc} & & P & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ A & & \overset{\alpha}{\Rightarrow} & & B \\ & {}_{f}\searrow & & \swarrow_{g} & \\ & & C & & \end{array}$$

a 2-cell. Then α is a pseudopullback if and only if the functor

$$\mathcal{G}(T,P) \longrightarrow \mathcal{G}(T,f)/\mathcal{G}(T,g) \tag{3.2}$$
$$h \longmapsto (ph, qh, \alpha h) \tag{3.3}$$

is an equivalence of categories for all objects T.

Proof. This is Proposition 2.1.11 [BD20]. The proof goes by observing how conditions (a) and (b) for comma squares precisely give essential surjectivity and fully faithfulness for the functor (3.2). □

Remark 3.1.6. Proposition 3.1.5 suggests that we can (and we do, in this very moment) define pseudopullbacks by the equivalent condition, even in the case comma squares happen not to exist in $\mathbb{G}$. In fact, one could directly define pseudopullbacks as *pseudo limits* over the diagram $\begin{array}{cc} & \downarrow \\ \longrightarrow & \cdot \end{array}$

3.2 Mackey pseudofunctors

We consider now the hypotheses that are assumed on our base (2,1)-category $\mathbb{G}$ and its distinguished class of morphisms $\mathbb{J}$. Basically, these are an abstraction of the properties of the (2,1)-category of finite groupoids and its faithful morphisms, but to move from these to a more general setting will become necessary, especially in Section 3.4.

Definition 3.2.1. A 2-category $\mathbb{G}$ is said to be *extensive* if it admits finite coproducts and if the pseudofunctor between comma 2-categories

$$\mathbb{G}/H \times \mathbb{G}/K \longrightarrow \mathbb{G}/(H \sqcup K)$$
$$(f\colon P \to H, g\colon Q \to K) \longmapsto (f \sqcup g\colon P \sqcup Q \to H \sqcup K)$$

is a biequivalence.

Definition 3.2.2. An essentially small extensive (2,1)-category $\mathbb{G}$ together with a class of 1-cells $\mathbb{J}$ (2-full subcategory) is said to be a *spannable pair* if

- $\mathbb{J}$ contains all equivalences, is closed under composition, 2-isomorphism, and every map in $\mathbb{J}$ is faithful. Moreover, if $ij \in \mathbb{J}$, so is j.
- For every cospan of 1-cells $H \overset{i}{\to} G \overset{v}{\leftarrow} K$ with i in $\mathbb{J}$, the pseudopullback

$$\begin{array}{ccccc} & & P & & \\ & {}^{\tilde{v}}\swarrow & & \searrow^{\tilde{i}} & \\ H & & \overset{\gamma}{\Rightarrow} & & K \\ & {}_{i}\searrow & & \swarrow_{v} & \\ & & G & & \end{array}$$

exists in $\mathbb{G}$. Moreover, $\tilde{i}$ still belongs to $\mathbb{J}$.

- For every finite family $\{u_\ell : H_\ell \to G\}$ the coproduct maps $u: \coprod H_\ell \to G$ (exists and) is in $\mathbb{J}$ if and only if every u_ℓ is.

Moreover, such a pair $(\mathbb{G}, \mathbb{J})$ is said to be *cartesian* if $\mathbb{G}$ and $\mathbb{J}$ also admit finite products.

Example 3.2.3. The main examples of spannable and cartesian pairs to keep in mind are the (2,1)-category $\mathbb{G} = \mathsf{gpd}$ of finite groupoids with its class of all 1-cells $\mathbb{J}$, or with its class of faithful 1-cells gpd^f. Cartesian pairs are also the comma 2-categories $(\mathsf{gpd}/G, \mathsf{gpd}^f/G)$ for every object G. The pair $(\mathsf{gpd}^f, \mathsf{gpd}^f)$ is spannable, but not cartesian, since the product projections are not faithful.

Remark 3.2.4. If $(\mathbb{G}, \mathbb{J})$ is cartesian, so is $(\mathbb{G} \times \mathbb{G}, \mathbb{J} \times \mathbb{J})$. Every property is easily deduced componentwise.

Definition 3.2.5. Let $(\mathbb{G}, \mathbb{J})$ be a cartesian pair, and $\mathcal{C}$ be an additive bicategory (Definition B.0.11). A pseudofunctor $\mathcal{M}: \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ is a *$\mathcal{C}$-valued Mackey pseudofunctor* for $(\mathbb{G}, \mathbb{J})$ if the following conditions hold true.

(i) *Additivity:* For every finite family $\{G_\ell\}$ the canonincal map

$$\mathcal{M}(\coprod G_\ell) \to \bigoplus \mathcal{M}(G_\ell)$$

is an equivalence. In other words, $\mathcal{M}$ preserves finite products.

(ii) *Induction:* For every $i: H \to G$ in $\mathbb{J}$, the morphism $i^* = \mathcal{M}(i): \mathcal{M}(G) \to \mathcal{M}(H)$ in $\mathcal{C}$ has a left adjoint $i_!$ and a right adjoint i_*

(iii) *Mackey formulas:* For every pseudopullback

$$\begin{array}{ccccc} & & P & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ H & & \overset{\gamma}{\Rightarrow} & & K \\ & {}_{i}\searrow & & \swarrow_{v} & \\ & & G & & \end{array} \tag{3.4}$$

with i (and q) in $\mathbb{J}$, the left and right *mates* $\gamma_!$ and $(\gamma^{-1})_*$ defined as

$$\gamma_! = \begin{array}{ccccccc} \mathcal{M}(H) & \xrightarrow{i_!} & \mathcal{M}(G) & \xrightarrow{v^*} & \mathcal{M}(K) & = & \\ \| & \nearrow \eta & {\scriptstyle i^*}\downarrow & \nearrow \gamma^* & \downarrow{\scriptstyle q^*} & \nearrow \varepsilon & \| \\ & = & \mathcal{M}(H) & \xrightarrow[p^*]{} & \mathcal{M}(P) & \xrightarrow[q_!]{} & \mathcal{M}(K) \end{array} \tag{3.5}$$

$$(\gamma^{-1})_* = \begin{array}{ccccccc} \mathcal{M}(H) & \xrightarrow{i_*} & \mathcal{M}(G) & \xrightarrow{v^*} & \mathcal{M}(K) & = & \\ \| & \swarrow\varepsilon & {\scriptstyle i^*}\downarrow & \swarrow(\gamma^*)^{-1} & \downarrow{\scriptstyle q^*} & \swarrow\eta & \| \\ & = & \mathcal{M}(H) & \xrightarrow[p^*]{} & \mathcal{M}(P) & \xrightarrow[q_*]{} & \mathcal{M}(K) \end{array} \tag{3.6}$$

are isomorphisms.

(iv) *Ambidexterity:* For every i in $\mathbb{J}$ there's an isomorphism $i_! \cong i_*$.

It is useful for our next purposes to introduce a broader definition of a Mackey pseudofunctor, expressing the presence of just a left adjoint for every restriction, along which any pseudopullback has an invertible mate.

Definition 3.2.6. A *left Mackey pseudofunctor* is a pseudofunctor $\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ into an additive bicategory preserving products, *i.e.* $\mathcal{F}(\bigsqcup G_i) \simeq \bigoplus \mathcal{F}(G_i)$, and such that:

(a) For every 1-cell i in $\mathbb{J}$, the 1-cell $\mathcal{F}i$ admits a left adjoint $(\mathcal{F}i)_!$ in $\mathcal{C}$

(b) For any pseudopullback γ along an i in $\mathbb{J}$, its left mate $\gamma_!$ is invertible:

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} & \\ X & & \overset{\gamma}{\Rightarrow} & & Y \\ & {\scriptstyle i}\searrow & & \swarrow{\scriptstyle v} & \\ & & Z & & \end{array} \qquad \begin{array}{ccccc} & & \mathcal{F}P & & \\ & {\scriptstyle \mathcal{F}p}\nearrow & & \searrow{\scriptstyle (\mathcal{F}q)_!} & \\ \mathcal{F}X & & \Downarrow\gamma_! & & \mathcal{F}Y \\ & {\scriptstyle (\mathcal{F}i)_!}\searrow & & \nearrow{\scriptstyle \mathcal{F}v} & \\ & & \mathcal{F}Z & & \end{array}$$

Remark 3.2.7. Analogously, one defines a *right Mackey pseudofunctor* $\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ by demanding the existence of right adjoints $(Fi)_*$ to $\mathcal{F}i$ for every i in $\mathbb{J}$, as well as the correspondent right mates to be invertible. Often, moreover, it will be convenient to talk about pseudofunctors that enjoy all the properties of a (left, or right) Mackey one, except perhaps additivity, and not necessarily having target an additive bicategory, but any bicategory.

Definition 3.2.8. A *left quasi-Mackey pseudofunctor* $\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ into any bicategory $\mathcal{C}$ is a pseudofunctor such that conditions (a) and (b) of definition 3.2.6 hold true.

There is a natural notion of morphisms between all of these objects, regardless being left, right (see Remark 3.2.10 below) or quasi.

Definition 3.2.9. A *morphism of (left, quasi-)Mackey pseudofunctors* $t\colon \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ for $(\mathbb{G}, \mathbb{J})$ is a pseudonatural transformation such that for every i in $\mathbb{J}$, the mate $(t_i^{-1})_!$ of t_i^{-1}, defined as

$$\begin{array}{ccc} \mathcal{F}_1X & \xrightarrow{t_X} & \mathcal{F}_2X \\ {\scriptstyle (\mathcal{F}_1 i)_!}\downarrow & \swarrow(t_i^{-1})_! & \downarrow{\scriptstyle (\mathcal{F}_2 i)_!} \\ \mathcal{F}_1Y & \xrightarrow[t_Y]{} & \mathcal{F}_2Y \end{array} = \begin{array}{ccccccc} = & = & \mathcal{F}_1X & \xrightarrow{t_X} & \mathcal{F}_2X & \xrightarrow{(\mathcal{F}_2 i)_!} & \mathcal{F}_2Y \\ \| & \searrow\eta & {\scriptstyle \mathcal{F}_1 i}\uparrow & \searrow t_i^{-1} & \uparrow{\scriptstyle \mathcal{F}_2 i} & \searrow\varepsilon & \| \\ \mathcal{F}_1X & \xrightarrow[(\mathcal{F}_1 i)_!]{} & \mathcal{F}_1Y & \xrightarrow[t_Y]{} & \mathcal{F}_2Y & = & = \end{array}$$

is invertible.

Remark 3.2.10. For a morphism $t\colon \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ between Mackey pseudofunctors, the right mate

$$(t_i)_* = \begin{array}{ccccccc} \mathcal{F}_1X & \xrightarrow{i_*} & \mathcal{F}_1Y & \xrightarrow{t_Y} & \mathcal{F}_2Y & & \\ \| & \Swarrow\varepsilon & {\scriptstyle i^*}\downarrow & \Swarrow t_i & \downarrow{\scriptstyle i^*} & \Swarrow\eta & \| \\ & & \mathcal{F}_1X & \xrightarrow[t_X]{} & \mathcal{F}_2X & \xrightarrow{i_*} & \mathcal{F}_2Y \end{array}$$

is also invertible. For a morphisms between Add-valued Mackey pseudofunctors $t\colon \mathcal{F}_1 \Rightarrow \mathcal{F}_2$ this is proved at Proposition 6.3.1 in [BD20]. One can check that the proofs involved only require additivity of the target bicategory, but not specifically being Add.

Definition 3.2.11. $\mathcal{C}$-valued (left, quasi-)Mackey pseudofunctors for $(\mathbb{G}, \mathbb{J})$, together with morphisms of (left, quasi-)Mackey and all modifications between pseudofunctors form bicategories denoted $(\ell q)\mathsf{Mack}_{\mathcal{C}}(\mathbb{G}, \mathbb{J})$. The ℓ evidently standing for left, the q for *quasi*.

Clearly, one can independently consider either being left (or right) or being *quasi*-Mackey. We adopt intuitive and obvious variations in the name of the correspondent bicategory.

Remark 3.2.12. For the definition of (left, right) Mackey pseudofunctor $\mathcal{M}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ is not actually necessary to require the target bicategory $\mathcal{C}$ to be additive, but just to have biproducts.

However, we demand that since this is the case in many application, and since we have the feeling that it could be interesting to consider and develop more general enrichments in the target bicategory. For example, additive derivators (see [Gro13]).

3.3 The bicategory of Mackey 2-motives

In order to construct a universal category for the Mackey pseudofunctors, the idea realized in [BD20] is to start from $\mathbb{G}$ and freely add left adjoints to morphisms in $\mathbb{J}$. Then, this bicategory will be symmetrized at the level of 2-cells so that the left and right adjoints will coincide. In other words, as first step we are going to define a universal category for left Mackey pseudofunctors.

Definition 3.3.1. Let $(\mathbb{G}, \mathbb{J})$ be a spannable pair. The bicategory $\mathsf{Span} = \mathsf{Span}(\mathbb{G}, \mathbb{J})$ is defined by having

- 0-cells those of $\mathbb{G}$;
- 1-cells $H \to G$ are *spans* of 1-cells in $\mathbb{G}$;

  $$H \xleftarrow{u} P \xrightarrow{i} G$$

  with i in $\mathbb{J}$
- 2-cells from $H \overset{u}{\leftarrow} P \overset{i}{\rightarrow} G$ to $H \overset{u}{\leftarrow} Q \overset{i}{\rightarrow} G$ are equivalence classes of triples $[\alpha_1, a, \alpha_2]$

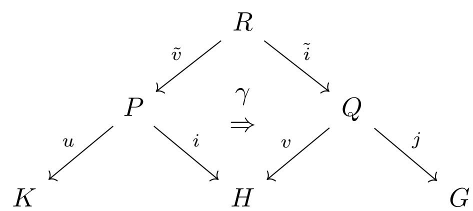
  $$\begin{array}{ccccc} H & \xleftarrow{u} & P & \xrightarrow{i} & G \\ \| & \Searrow\alpha_1 & \downarrow{\scriptstyle a} & \Nearrow\alpha_2 & \| \\ H & \xleftarrow[v]{} & Q & \xrightarrow[j]{} & G \end{array}$$

  under the equivalence relation $(\alpha_1, a, \alpha_2) \sim (\beta_1, b, \beta_2)$ if and only if there is a 2-cell $\varphi\colon a \Rightarrow b$ of $\mathbb{G}$ giving $\varphi\alpha_1 = \beta_1$ and $\beta_2\varphi = \alpha_2$.

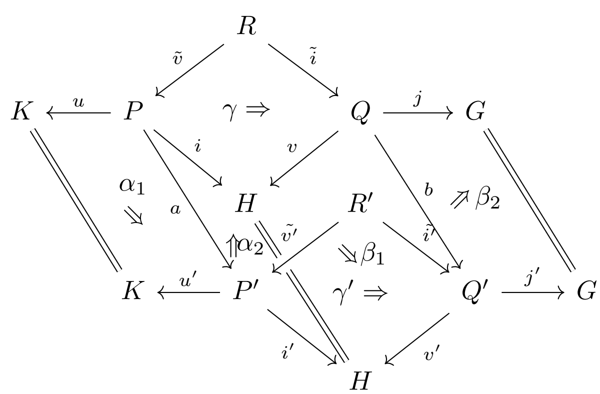
The composition functor

$$\circ_{K,H,G} \colon \mathsf{Span}(H,G) \times \mathsf{Span}(K,H) \longrightarrow \mathsf{Span}(K,G)$$

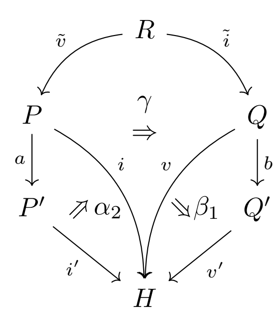
is provided by forming pseudopullbacks. On objects, the composition functor maps the pair $(H \xleftarrow{v} P \xrightarrow{j} G, K \xleftarrow{u} Q \xrightarrow{i} H)$ to $(K \xleftarrow{u\tilde{v}} R \xrightarrow{j\tilde{i}} G)$ defined by the pseudopullback



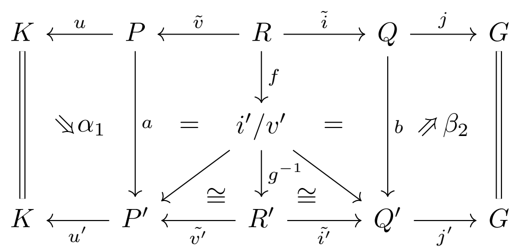
On morphisms, the composition functor is defined by inducing a map as follows. Consider two 2-cells $[\alpha_1, a, \alpha_2]$ and $[\beta_1, b, \beta_2]$, represented below together with the composition of 1-cells provided by pseudopullbacks:



The 2-cell γ' induces by the pseudopullback property an equivalence $g \colon R' \to (i'/v')$, while the composition



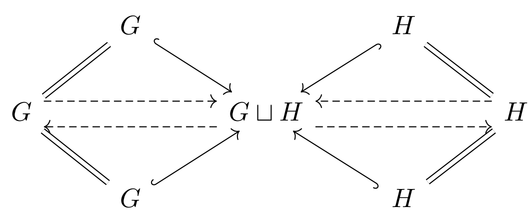
induces a morphism $f \colon R \to (i'/v')$. One can then define the composition



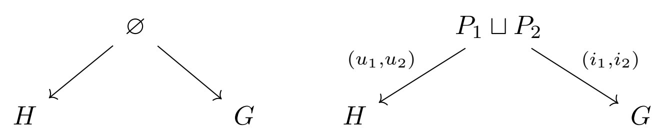
where the strictly commutative squares arise by definition of f, while the unlabeled isomorphism come from the definition of g and the isomorphism $gg^{-1} \cong \mathrm{id}$.

Remark 3.3.2. Observe how the construction in Definition 3.3.1 does not require extensivity on $\mathbb{G}$. However, this requirement is presented in order to induce from coproducts in $\mathbb{G}$ biproduct in the span bicategory, on objects and locally on morphisms.

Proposition 3.3.3. *Let* $(\mathbb{G}, \mathbb{J})$ *be a spannable pair. Then* $\mathsf{Span}(\mathbb{G}, \mathbb{J})$ *has biproducts, induced by coproducts in* $\mathbb{G}$*, and is locally semi-additive. More precisely,* $\varnothing$ *is the zero object, and the diagram*



is a biproduct diagram for G, H*. The diagrams*



provide the zero morphism and the direct sum of two morphisms $H \leftarrow P_k \to G$*, for* $k = 1, 2$.

Proof. This is Proposition 3.15 [Del19]. □

This last Proposition allows us to consider Span^+ (Definition B.0.8), the bicategory with the same objects and with the additive completion of $\mathsf{Span}(H, G)$ on every hom-category. The bicategory Span^+ is then an additive bicategory (see Definition B.0.11), and we can consider (left) Mackey pseudofunctors valued in it. The first example of a pseudofunctor valued in Span^+ that we deal with is the canonical morphism

$$\iota\colon \mathbb{G}^{\mathsf{op}} \hookrightarrow \mathsf{Span} \hookrightarrow \mathsf{Span}^+.$$

The two following properties show that such a canonical map is a left Mackey pseudofunctor, and though they are proved in [BD20] for Span, they clearly hold true for Span^+, too.

Proposition 3.3.4. *For every* $i\colon H \to G$ *in* $\mathbb{J}$ *there is in* $\mathsf{Span}(\mathbb{G}, \mathbb{J})$ *an adjunction*

$$(H \xleftarrow{\mathrm{id}} H \xrightarrow{i} G) \dashv (G \xleftarrow{i} H \xrightarrow{\mathrm{id}} H)$$

for every $i\colon H \to G$ *in* $\mathbb{J}$.

Proof. This is Proposition 5.1.21 in [BD20]. ∎

Proposition 3.3.5. *For every pseudopullback* $\begin{array}{ccccc} & & P & & \\ & {}^{p}\swarrow & & \searrow^{q} & \\ H & & \overset{\gamma}{\Rightarrow} & & K \\ & {}_{i}\searrow & & \swarrow_{u} & \\ & & G & & \end{array}$ *in* $\mathbb{G}$ *with two parallel sides* i, q *in* $\mathbb{J}$, *the mate* $\gamma_!$ *of the 2-cell* γ^* *in* $\mathsf{Span}(\mathbb{G}, \mathbb{J})$ *is an isomorphism.*

Proof. This is Lemma 5.1.25 in [BD20]. ∎

Corollary 3.3.6. *The canonical pseudofunctors* $\iota\colon \mathbb{G}^{\mathsf{op}} \to \mathsf{Span}$ *is a left Mackey pseudofunctor. The composition with the additive completion*

$$\mathbb{G}^{\mathsf{op}} \xrightarrow{\iota} \mathsf{Span} \hookrightarrow \mathsf{Span}^{+}$$

is a Mackey pseudofunctor.

Proof. This is a plain combination of Proposition 3.3.3, Proposition 3.3.4 and Proposition 3.3.5. ∎

We are now ready to state and appreciate a fundamental result in the theory, exhibiting the bicategory of spans as a universal object for left Mackey pseudofunctors.

Theorem 3.3.7. *Let* $(\mathbb{G}, \mathbb{J})$ *be a spannable pair. Let* $\mathcal{C}$ *be any additive bicategory and* $\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ *be a left Mackey pseudofunctor. Then there is an additive pseudofunctor* $\mathcal{F}_\star\colon \mathsf{Span}(\mathbb{G}, \mathbb{J}) \to \mathcal{C}$ *such that*

$$\begin{array}{ccc} \mathbb{G}^{\mathsf{op}} & \xrightarrow{\mathcal{F}} & \mathcal{C} \\ {\scriptstyle\iota}\downarrow & \nearrow_{\mathcal{F}_\star} & \\ \mathsf{Span}(\mathbb{G}, \mathbb{J}) & & \end{array}$$

is commutative. Moreover, such a $\mathcal{F}_\star$ *is unique up to a unique isomorphism. Conversely, any additive pseudofunctor* $\mathsf{Span}(\mathbb{G}, \mathbb{J}) \to \mathcal{C}$ *precomposed with* ι *enjoy the properties of a left Mackey pseudofunctor.*

Proof. This is Theorem 5.2.1 in [BD20]. Actually, the proof there is for the non-additive case, that is for a left *quasi*-Mackey pseudofunctors $\mathcal{F}$ into any bicategory $\mathcal{C}$. However, under the hypothesis of $\mathcal{F}$ being left Mackey, together with $\mathcal{C}$ being additive, the result makes $\mathcal{F}$ correspond to an additive pseudofunctors $\mathsf{Span} \to \mathcal{C}$. The fact that $\mathcal{F}_\star$ preserves biproducts is fairly obvious:

$$\mathcal{F}_\star(\bigoplus G_i) = \mathcal{F}_\star(\iota(\bigsqcup G_i)) = \mathcal{F}(\bigsqcup G_i) \simeq \bigoplus \mathcal{F}(G_i) = \bigoplus \mathcal{F}_\star(\iota G_i) = \bigoplus \mathcal{F}_\star(G_i).$$

It then follows from the general theory of additive bicategories that $\mathcal{F}_\star$ preserves then direct sums of morphisms, too (Proposition A.7.14 [BD20]). ∎

Remark 3.3.8. A symmetric result holds for a right Mackey pseudofunctor $\mathcal{G}$ by replacing the bicategory $\mathsf{Span}(\mathbb{G}, \mathbb{J})$ by its 2-cell dual $\mathsf{Span}(\mathbb{G}, \mathbb{J})^{\mathsf{co}}$, and hence giving rise to an extension

$$\mathcal{G}^\star\colon \mathsf{Span}(\mathbb{G}, \mathbb{J})^{\mathsf{co}} \to \mathcal{C}.$$

along the morphism $\mathbb{G}^{\mathsf{op}} \to \mathsf{Span}(\mathbb{G}, \mathbb{J})^{\mathsf{co}}$. The latter is the pseudofunctor mapping $u\colon H \to G$ to the 1-cell $[G \overset{u}{\leftarrow} H = H]$ and $\beta\colon u \Rightarrow v$ to $[\mathrm{id}, \beta^{-1}, \mathrm{id}]$.

Now, Theorem 5.3.7 in [BD20] makes also morphisms of left Mackey pseudofunctors and pseudonatural transformation of pseudofunctors $\mathsf{Span} \to \mathcal{C}$ correspond. Whence the biequivalence between the bicategory of $\mathcal{C}$-valued left Mackey pseudofunctors and the bicategory of additive pseudofunctors

$$\ell\mathsf{Mack}_{\mathcal{C}}(\mathbb{G}, \mathbb{J}) \simeq \mathsf{PsFun}_{\oplus}(\mathsf{Span}(\mathbb{G}, \mathbb{J}), \mathcal{C}).$$

Observe that, by virtue of Remark B.0.9, the two bicategories above are also biequivalent to $\mathsf{PsFun}_{\oplus}(\mathsf{Span}^{+}(\mathbb{G}, \mathbb{J}), \mathcal{C})$. The additivity may not necessarily hold on both sides and the equivalence would still hold true. More precisely, we can consider left *quasi*-Mackey pseudofunctors on the left hand side and the whole family of pseudofunctor from the (non-additive completed!) bicategory of spans to $\mathcal{C}$ on the right hand side, that is

$$\ell q\mathsf{Mack}_{\mathcal{C}}(\mathbb{G}, \mathbb{J}) \simeq \mathsf{PsFun}(\mathsf{Span}(\mathbb{G}, \mathbb{J}), \mathcal{C}).$$

At this point, the theory developed in [BD20] also defines the universal object for Mackey pseudofunctors, by doubling 2-cells in Span. The idea behind this is that once we add a left adjoint to the image of every morphism in $\mathbb{J}$ through the canonical inclusion $\mathbb{G}^{\mathsf{op}} \to \mathsf{Span}$, we can turn adjunctions in two sided ones with a very general construction. Precisely, we have

Lemma 3.3.9. *Let $(\mathbb{G}, \mathbb{J})$ be a spannable pair. Each hom category $\mathsf{Span}(H, G)$ admits pullbacks. Moreover, the composition functor preserves them in each variable.*

Proof. This is Proposition 5.4.1, together with Lemma 5.4.14 in [BD20] □

Definition 3.3.10. Let $(\mathbb{G}, \mathbb{J})$ be a spannable pair. The bicategory $\mathsf{Mot} = \mathsf{Mot}(\mathbb{G}, \mathbb{J})$ is defined to be the local span bicategory $\widehat{\mathsf{Span}}$ (Definition C.0.4, possible thanks to Lemma 3.3.9). It has then the same objects as those of Span, while on each hom-category we consider the ordinary category of spans (Definition C.0.1)

$$\mathsf{Mot}(H, G) = \widehat{\mathsf{Span}(H, G)}.$$

Remark 3.3.11. As for spans, it turns out that coproducts in $\mathbb{G}$ induce biproducts on $\mathsf{Mot}(\mathbb{G}, \mathbb{J})$, as well as biproduct on its hom-categories. This is proved for $\mathbb{G}$ a subcategory of gpd at Proposition 7.1.3 [BD20], but thanks to the above result for Span (Proposition 3.3.3), we can conclude by using the following lemma.

Lemma 3.3.12. *Let $\mathcal{B}$ be a bicategory for which the local span bicategory $\hat{\mathcal{B}}$ is possible. If $\mathcal{B}$ has biproducts and is locally semi-additive, then $\hat{\mathcal{B}}$ inherits biproducts and local semi-additivity.*

Proof. This is an easy consequence of the definition, since for $X \oplus Y$ to be a biproduct in $\mathcal{B}$ it means to have equivalences

$$\mathcal{B}(Z, X \oplus Y) \xrightarrow{\sim} \mathcal{B}(Z, X) \times \mathcal{B}(Z, Y)$$

$$\mathcal{B}(X \oplus Y, Z) \xrightarrow{\sim} \mathcal{B}(X, Z) \times \mathcal{B}(Y, Z)$$

For, it suffices to take the usual span of these equivalent categories, and find the canonical equivalences

$$\hat{\mathcal{B}}(Z, X \oplus Y) \xrightarrow{\sim} \widehat{\mathcal{B}(Z,X) \times \mathcal{B}(Z,Y)} \cong \widehat{\mathcal{B}(Z,X)} \times \widehat{\mathcal{B}(Z,Y)} = \hat{\mathcal{B}}(Z,X) \times \hat{\mathcal{B}}(Z,Y)$$

and similarly $\hat{\mathcal{B}}(X \oplus Y, Z) \xrightarrow{\sim} \hat{\mathcal{B}}(X,Z) \times \hat{\mathcal{B}}(Y,Z)$ The same argument exhibits the isomorphism required for $\hat{\mathcal{B}}(X,Y) = \widehat{\mathcal{B}(X,Y)}$ to have biproducts. □

Corollary 3.3.13. *Let* $(\mathbb{G}, \mathbb{J})$ *be a spannable pair. Then,* $\mathsf{Mot}(\mathbb{G}, \mathbb{J})$ *inherits biproducts of objects and biproducts on hom-categories from the inclusion*

$$\mathsf{Span}(\mathbb{G}, \mathbb{J}) \hookrightarrow \mathsf{Mot}(\mathbb{G}, \mathbb{J}).$$

Therefore, we can consider the additive completion of the bicategory of motives.

Definition 3.3.14. For a spannable pair $(\mathbb{G}, \mathbb{J})$ we define the bicategory of *additive motives* to be the additive completion $\mathsf{Mot}^+(\mathbb{G}, \mathbb{J})$ of the (locally additive) bicategory of motives.

Mot^+ is then additive, and we can evaluate Mackey pseudofunctors in it. A first example of such a Mackey pseudofunctor will be the canonical map

$$\mathsf{mot}\colon \mathbb{G}^{\mathsf{op}} \longrightarrow \mathsf{Mot} \hookrightarrow \mathsf{Mot}^+.$$

This follows form Proposition 6.1.11 of [BD20] and Lemma 3.3.12. From now on, we understand bicategories of both spans and motives to be their additive completion, even if not specified.

For what concerns the universal property of the bicategory of motives, it is expressed in terms of a slightly more restrictive notion, since the operation of doubling 2-cells doesn't just produces isomorphic, but properly *equal* two sided adjoints. Therefore, it's convenient to set the following definition

Definition 3.3.15. A *rectified* Mackey pseudofunctor $\mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ for $(\mathbb{G}, \mathbb{J})$ is one such that for every i in $\mathbb{J}$ left and right adjoints to i^* coincide, $i_! = i_*$, and such that the isomorphisms $(\gamma^{-1})_*$ and $\gamma_!$ of (3.5) are each others' inverses for every pseudopullback γ.

Proposition 3.3.16. *Let* $(\mathbb{G}, \mathbb{J})$ *be a spannable pair,* $\mathcal{C}$ *be an additive bicategory and* $\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ *a* rectified *Mackey pseudofunctor. Then, there's a unique - up to unique isomorphism - additive pseudofunctor* $\hat{\mathcal{F}}\colon \mathsf{Mot}(\mathbb{G}, \mathbb{J}) \to \mathcal{C}$ *extending* $\mathcal{F}$ *along* mot.

Proof. This is Theorem 6.1.13 in [BD20]. □

Proposition 3.3.17. *Let* $t\colon \mathcal{M} \Rightarrow \mathcal{N}$ *be any pseudonatural transformation between Mackey pseudofunctors* $\mathbb{G}^{\mathsf{op}} \to \mathcal{C}$ *into an additive bicategory* $\mathcal{C}$*. Then,* t *extends to a pseudonatural transformation* $\hat{\mathcal{M}} \Rightarrow \hat{\mathcal{N}}$ *if* t *is a morphism of Mackey pseudofunctors. Conversely, if* $s\colon \hat{\mathcal{M}} \Rightarrow \hat{\mathcal{N}}$ *is any pseudonatural transformation, then* $s \circ \mathsf{mot}\colon \mathcal{M} \Rightarrow \mathcal{N}$ *is a morphism of Mackey pseudofunctors.*

Proof. This follows from Theorem 6.3.1 at [BD20]. Even if there it is proved only for $\mathcal{C} = \mathrm{Add}$, the proof is essentially the same. □

The bicategorical version of the universal property follows immediately.

Corollary 3.3.18. *The precomposition with* $\mathsf{mot}\colon \mathbb{G}^{\mathsf{op}} \to \mathsf{Mot}(\mathbb{G}, \mathbb{J})$ *provides a biequivalence of bicategories*

$$\mathsf{Mack}_{\mathcal{C}}(\mathbb{G}, \mathbb{J}) \simeq \mathsf{PsFun}_{\oplus}(\mathsf{Mot}(\mathbb{G}, \mathbb{J}), \mathcal{C}).$$

3.4 Monoidal structure on motives

The goal of this section is to consider the cartesian structure of the cartesian pair $(\mathbb{G}, \mathbb{J})$, and use it in order to induce a braided monoidal structure on the bicategory of additive motives.

Lemma 3.4.1. *Let $\mathbb{G}$ be a 2-category with products. Then, the product 2-functor $\times\colon \mathbb{G}\times\mathbb{G} \to \mathbb{G}$ preserves comma squares and pseudopullbacks which exist in $\mathbb{G}$.*

Proof. This is just the fact that limits commute with limits, plus that products of equivalences are equivalences. □

Proposition 3.4.2. *The pseudofunctor*

$$\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} \xrightarrow{\times^{\mathsf{op}}} \mathbb{G}^{\mathsf{op}} \xrightarrow{\iota} \mathsf{Span}$$

admits an extension

$$\begin{array}{ccccc} \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} & \xrightarrow{\times^{\mathsf{op}}} & \mathbb{G}^{\mathsf{op}} & \xhookrightarrow{\iota} & \mathsf{Span} \\ {\scriptstyle \iota\times\iota}\big\downarrow & & & \nearrow_{\otimes} & \\ \mathsf{Span}\times\mathsf{Span} & & & & \end{array}$$

making the diagram strictly commute. Moreover, this extension is unique up to a unique isomorphism restricting to the identity on $\mathcal{F}$.

Proof. The proof is an application of Theorem 5.2.1 [BD20] using $\mathbb{G} \times \mathbb{G}$ instead of $\mathbb{G}$, together with the fact that $\mathsf{Span}(\mathbb{G} \times \mathbb{G}, \mathbb{J} \times \mathbb{J}) \simeq \mathsf{Span}(\mathbb{G}, \mathbb{J}) \times \mathsf{Span}(\mathbb{G}, \mathbb{J})$. The hypothesis of Theorem 5.2.1 asks for $\mathcal{F}$ to be left quasi-Mackey. Hence, we need to check that for every 1-cell $(i_1, i_2) \in \mathbb{J} \times \mathbb{J}$, the arrow $\mathcal{F}(i_1, i_2)$ admits an adjoint in Span, and this is true thanks to the fact that $\mathbb{J}$ is closed under products of 1-cells and that the property holds true for the left quasi-Mackey pseudofunctor $\iota\colon \mathbb{G} \to \mathsf{Span}$. Similarly, for any pseudopullback γ in $\mathbb{G} \times \mathbb{G}$, its image under the product 2-functor is again a pseudopullback (Lemma 3.4.1), and hence $\mathcal{F}\gamma$ is the image of a pseudopullback via the left Mackey pseudofunctor ι, so it has an invertible mate. □

The goal of this section is to prove that the pseudofunctor defined in Proposition 3.4.2 defines a braided monoidal structure on the bicategory of spans, and also on that of motives. The following analogous result also hold true.

Proposition 3.4.3. *The pseudofunctor*

$$\mathcal{H}\colon \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} \xrightarrow{\times^{\mathsf{op}}} \mathbb{G}^{\mathsf{op}} \longrightarrow \mathsf{Mot}$$

admits a unique up to unique isomorphism extension

$$\begin{array}{ccccc} \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} & \xrightarrow{\times^{\mathsf{op}}} & \mathbb{G}^{\mathsf{op}} & \hookrightarrow & \mathsf{Mot} \\ {\scriptstyle \mathsf{mot}\times\mathsf{mot}}\big\downarrow & & & \nearrow_{\otimes} & \\ \mathsf{Mot}\times\mathsf{Mot} & & & & \end{array}$$

Proof. The proof is perfectly analogous to the one of Proposition 3.4.2, and is based on Theorem 6.1.13 in [BD20]. In order to apply this result one need to ensure $\mathcal{H}$ to be such that:

(a) There are two equal left and right adjoints $i_! \dashv \mathcal{F}(i) \dashv i^*$ for every 1-cell i in $\mathbb{J}$. And this is true being true for $\mathbb{G}^{\mathsf{op}} \hookrightarrow \mathsf{Mot}$, and $\mathbb{J}$ being closed under products.

(b) Both mates of any comma square are isomorphisms. For that we can use the fact that comma squares are preserved by the product pseudofunctor (Lemma 3.4.1), and again by the fact that mot sends comma squares with two parallel sides in $\mathbb{J}$ to 2-cells with invertible mates.

(c) For every comma square γ it holds $(\gamma^{-1})_* = (\gamma_!)^{-1}$. Again, we have that comma squares are preserved by the product pseudofunctor and that the condition holds true for mot by Proposition 6.1.11 in [BD20].

□

Propositions 3.4.2 and 3.4.3 define then pseudofunctors underlying the monoidal structure on the bicategory of spans and motives respectively. From Theorem 5.3.7 and Theorem 6.3.6 in [BD20] we have that these monoidal structures are the images of the two pseudofunctors $\mathcal{F}$ and $\mathcal{H}$ towards the biequivalences

$$\ell\mathsf{Mack}_{\mathsf{Span}}(\mathbb{G}\times\mathbb{G},\mathbb{J}\times\mathbb{J}) \simeq \mathsf{PsFun}_{\oplus}(\mathsf{Span}\times\mathsf{Span},\mathsf{Span})$$

and

$$\mathsf{Mack}_{\mathsf{Mot}}(\mathbb{G}\times\mathbb{G},\mathbb{J}\times\mathbb{J}) \simeq \mathsf{PsFun}_{\oplus}(\mathsf{Mot}\times\mathsf{Mot},\mathsf{Mot}).$$

Analogously, for every natural n, we have biequivalences

$$\ell\mathsf{Mack}_{\mathsf{Span}}(\mathbb{G}^n,\mathbb{J}^n) \simeq \mathsf{PsFun}_{\oplus}(\mathsf{Span}^n,\mathsf{Span}). \tag{3.7}$$

$$\mathsf{Mack}_{\mathsf{Mot}}(\mathbb{G}^n,\mathbb{J}^n) \simeq \mathsf{PsFun}_{\oplus}(\mathsf{Mot}^n,\mathsf{Mot}) \tag{3.8}$$

and clearly we can dop the additivity on both sides, finding, for the non-additive completed bicategory of spans and motives,

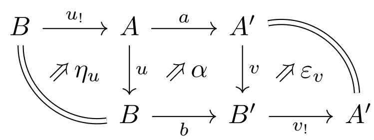
$$\ell q\mathsf{Mack}_{\mathsf{Span}}(\mathbb{G}^n,\mathbb{J}^n) \simeq \mathsf{PsFun}(\mathsf{Span}^n,\mathsf{Span}). \tag{3.9}$$

$$q\mathsf{Mack}_{\mathsf{Mot}}(\mathbb{G}^n,\mathbb{J}^n) \simeq \mathsf{PsFun}(\mathsf{Mot}^n,\mathsf{Mot}) \tag{3.10}$$

We want now to use these biequivalences to lift *all* the structure of the monoidal bicategory $(\mathbb{G}^{\mathsf{op}}, \times^{\mathsf{op}})$. That means, tensor and the identity pseudofunctors (Propositions 3.4.2 and 3.4.3), but also associator, unitors, pentagonator and 2-unitors. In order to do so, we needed first to ensure that tensor and unit pseudofunctors are (once composed with the canonical $\mathbb{G}^{\mathsf{op}} \to \mathsf{Mot}$), objects in the bicategory $q\mathsf{Mack}_{\mathsf{Mot}}(\mathbb{G},\mathbb{J})$.
Then, it is time to show that a, ℓ, r and π, μ provide in the same way pseudonatural transformations and modifications in the bicategory $q\mathsf{Mack}_{\mathsf{Mot}}(\mathbb{G}^n,\mathbb{J}^n)$ for the correct arity n. For the modifications the lift will be automatic, since all possible modifications are considered in this bicategory. Then, we will see that we can also lift the relations that they satisfy.

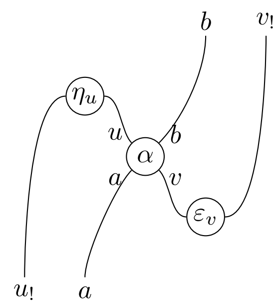
In order to do so, we need first to prove this key lemma, suited for the definition of morphism of Mackey pseudofunctors. We suspect this result to be known yet, because of the vast literature on mate theory, but we prove it here since we were not able to find it anywhere, as well as for the reader's convenience, who may use it as a toy model for a string diagrammatic proof.

Lemma 3.4.4. *Let $\mathcal{B}$ be a bicategory, $v_! \dashv v$ and $u_! \dashv u$ be two adjunctions between 1-morphisms, a, b two adjoint equivalences and α a 2-isomorphism, fitting into the following diagram.*



Then, the above composite 2-cell is an isomorphism $v_! b \overset{\sim}{\Rightarrow} a u_!$.

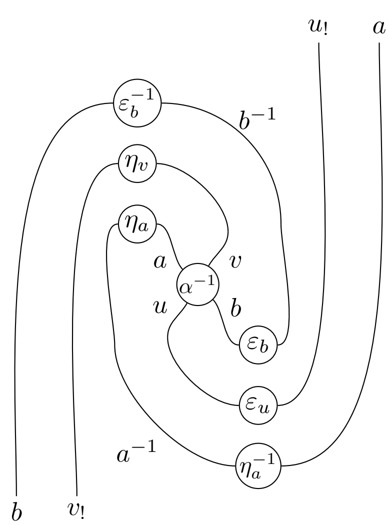
Proof. The 2-cell in question, $\alpha_!$ for short, is depicted as a string diagram as



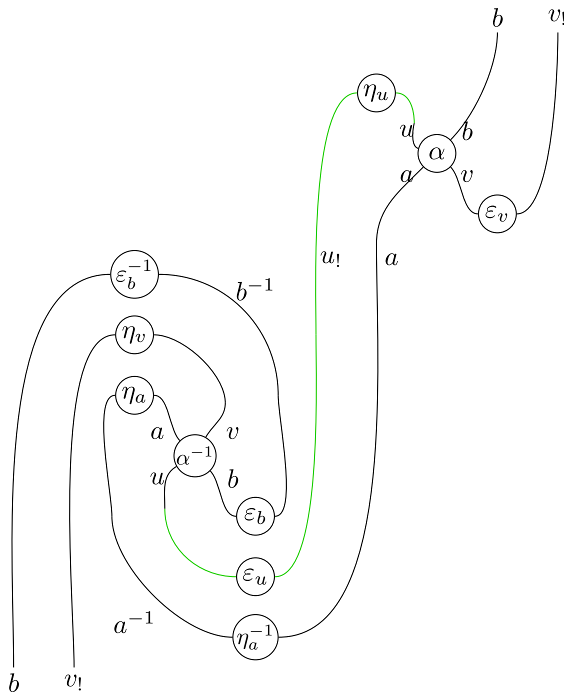
where notationally we consider 1-cells going from left to right and 2-cells going from top to bottom.

By assumption, besides the left adjoints of u and v and the inverse of α, we have invertible units and counits for the two adjoint equivalences $b^{-1} \dashv b$ and $a^{-1} \dashv a$. With these ingredients, the string diagram notation forces us to define the inverse of $\alpha_!$ as the 2-cell

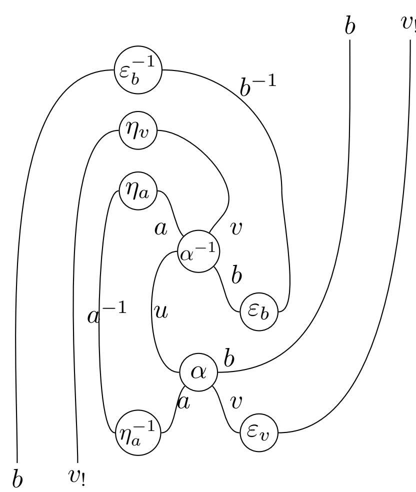
$\beta =$



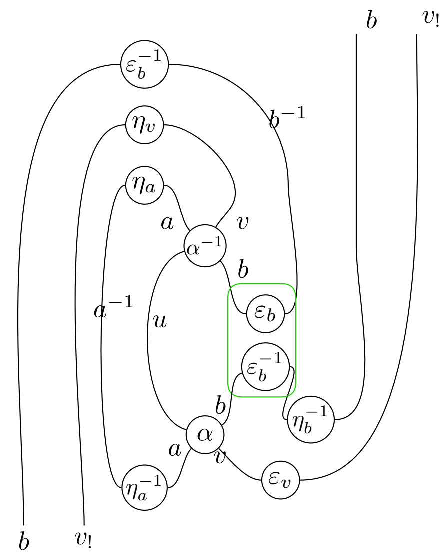
The following steps show how composition $\beta \circ \alpha_!$ gives the identity.



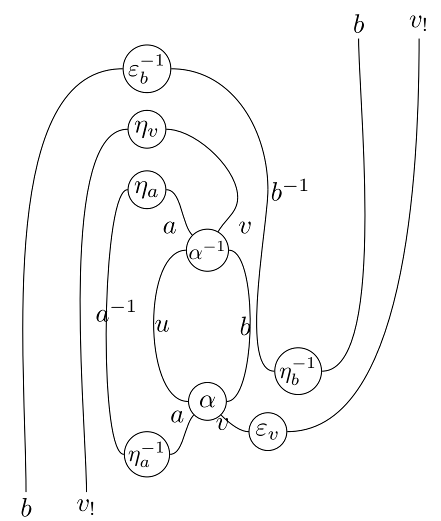
and the highlighted edge can be contracted by a triangular identity for the adjunction $u_! \dashv u$, giving



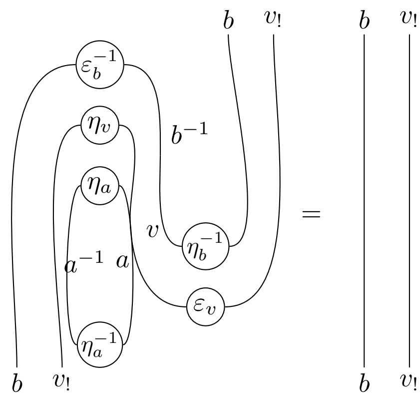
Now, by the a triangular identity for b, this is equal to



and the highlighted part now clearly compose to the identity of $b^{-1}b$, providing



Now, α and α^{-1} simplify, and everything that remains is again the identity by the triangular identity for the composition of adjoints.



The proof of the fact that the other composite $\alpha_! \circ \beta = \mathrm{id}$ follows the same structure. ☐

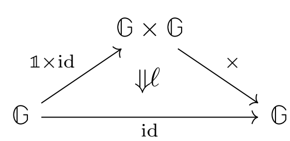
We are now ready to prove the following result.

Theorem 3.4.5. *Let* $(\mathbb{G}, \mathbb{J})$ *be cartesian. Then the monoidal structure given by the categorical product on* $\mathbb{G}$ *(*i.e. *the coproduct in* $\mathbb{G}^{\mathsf{op}}$*) extends to a monoidal structure on the bicategory* Span *via the inclusion* $\iota\colon \mathbb{G}^{\mathsf{op}} \to \mathsf{Span}$.

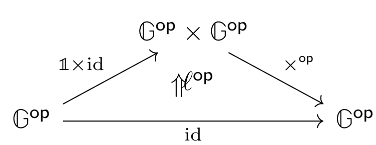
Proof. In order to prove the statement we need first of all to make the monoidal structure explicit. The unit is clearly given by the composition of pseudofunctors

$$1 \xrightarrow{\mathbb{1}} \mathbb{G}^{\mathsf{op}} \hookrightarrow \mathsf{Span},$$

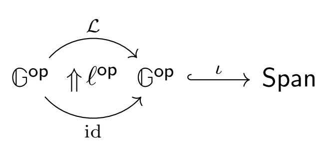
while the tensor pseudofunctor is the one defined in Proposition 3.4.2. The two are the correspondent under the biequivalence (3.9) of the pseudofunctors $\iota\mathbb{1}$ and $\iota\times^{\mathsf{op}}$, which are left quasi-Mackey (hence in the left hand side of (3.9)), for n respectively equal to 0 and 2. Let's now consider the left unitor ℓ for the monoidal structure on $(\mathbb{G}, \times)$



and let's look at its image by the 2-functor $(-)^{\mathsf{op}}$ on bicategories (this is actually a 3-functor), which goes as $(-)^{\mathsf{op}}\colon \mathsf{BiCat}^{\mathsf{co}} \to \mathsf{BiCat}$ (1-covariant and 2-contravariant), giving



Let us call $\mathcal{L}$ the upper composite pseudofunctor of two sides of this triangle. Then, let's define $\underline{\ell}$ to be the whiskering $\iota\ell^{\mathsf{op}}$



and unpack this definition: $\underline{\ell}$ is the data of a family of 1-equivalences $\underline{\ell}_G\colon G \to \mathbb{1}\times G$ in Span coming from $\mathbb{G}^{\mathsf{op}}$, along with, for every morphism $u\colon H \to G$ in $\mathbb{G}$, an invertible 2-cell $\underline{\ell}_u$ in Span (coming, again, from one of $\mathbb{G}^{\mathsf{op}}$)

$$\begin{array}{ccc} G & \xrightarrow{\underline{\ell}_G} & \mathbb{1}\times G \\ {\scriptstyle u^*}\big\downarrow & \swarrow \underline{\ell}_u & \big\downarrow{\scriptstyle \mathbb{1}\times u^*} \\ H & \xrightarrow[\underline{\ell}_H]{} & \mathbb{1}\times H \end{array}$$

subject to the usual requirement of unitality, functoriality and naturality (Definition 1.1.5).

What we want to prove is that $\underline{\ell}\colon \iota \Rightarrow \iota\mathcal{L}$ is a morphism of left quasi-Mackey pseudofunctors, so that we can look at its correspondent via the biequivalence (3.9) for $n = 1$. Hence, we first need to ensure $\iota\mathcal{L}$ to be left quasi-Mackey. This is a straightforward consequence of Proposition 3.3.4 and Proposition 3.3.5, since $\mathcal{L}$ preserves $\mathbb{J}$ and pseudopullbacks: if $j\colon H \to G$ is in $\mathbb{J}$, then $\mathcal{L}(j) = \mathrm{id}_{\mathbb{1}} \times j\colon 1\times H \to 1 \times G$, and being $\mathbb{J}$ closed by products and containing isomorphisms this also is in $\mathbb{J}$. Also, quite clearly, if γ is a pseudopullback, then so is $\mathbb{1}\times\gamma$ (since pseudopullbacks are preserved by the product pseudofunctor).

Now, by definition of morphism of (left, quasi-)Mackey pseudofunctors, we need to check that the mate of $\underline{\ell}_u^{-1}$ is an isomorphism. Such a mate $(\underline{\ell}_u^{-1})_!$ is

$$\begin{array}{ccccccc} & & \mathbb{1}\times G & \xleftarrow{\underline{\ell}_G} & G & \xleftarrow{u_!} & H \\ \Vert & \nwarrow\varepsilon & {\scriptstyle (\mathbb{1}\times u)^*}\big\downarrow & \nwarrow\underline{\ell}_u^{-1} & \big\downarrow{\scriptstyle u^*} & \nwarrow\eta & \Vert \\ \mathbb{1}\times G & \xleftarrow[(\mathbb{1}\times u)_!]{} & \mathbb{1}\times H & \xleftarrow[\underline{\ell}_H]{} & H & & \end{array}$$

Hence, we immediately conclude from Lemma 3.4.4. A completely analogous construction leads us to define the right unitor $\underline{r}$ and the associator $\underline{a}$, as well as to observe that these are morphisms of left quasi-Mackey pseudofunctors. This allows us to look at their image in the bicategories $\mathsf{PsFun}(\mathsf{Span}^n, \mathsf{Span})$, for $n = 3$ in the case of $\underline{a}$, and $n = 1$ for $\underline{\ell}$ and $\underline{r}$:

$$\begin{array}{ccc} \mathsf{Span}^3 & \xrightarrow{\mathrm{id}\times\otimes} & \mathsf{Span}^2 \\ {\scriptstyle \otimes\times\mathrm{id}}\big\downarrow & \swarrow \underline{a} & \big\downarrow{\scriptstyle \otimes} \\ \mathsf{Span}^2 & \xrightarrow[\otimes]{} & \mathsf{Span} \end{array}$$

$$\begin{array}{ccc} & \mathsf{Span}^2 & \\ {\scriptstyle \mathbb{1}\times\mathrm{id}}\nearrow & \Downarrow\underline{\ell} & \searrow{\scriptstyle \otimes} \\ \mathsf{Span} & \xrightarrow[\mathrm{id}]{} & \mathsf{Span} \end{array} \qquad \begin{array}{ccc} & \mathsf{Span}^2 & \\ {\scriptstyle \mathrm{id}\times\mathbb{1}}\nearrow & \Downarrow\underline{r} & \searrow{\scriptstyle \otimes} \\ \mathsf{Span} & \xrightarrow[\mathrm{id}]{} & \mathsf{Span} \end{array}$$

For what concerns the higher structure, we can start again from (π, μ, γ, ρ), 2-cells in $\mathsf{PsFun}(\mathbb{G}^n, \mathbb{G})$ for a suitable n (equal to 4 for π and 2 for the 2-unitors), part of the monoidal structure of $(\mathbb{G}, \times)$. These are all modifications of the form

$$\mathbb{G}^n \underset{\mathcal{F}_2}{\overset{\mathcal{F}_1}{\rightrightarrows}} \mathbb{G} \qquad \Downarrow\alpha_1 \overset{M}{\Rrightarrow} \Downarrow\alpha_2$$

If we look at what happens after applying the 3-functor $(-)^{\mathsf{op}}$, one can check that it is 3-covariant, and hence we end up with a modification

$$(\mathbb{G}^n)^{\mathsf{op}} \underset{\mathcal{F}_2^{\mathsf{op}}}{\overset{\mathcal{F}_1^{\mathsf{op}}}{\rightrightarrows}} \mathbb{G}^{\mathsf{op}} \qquad \Uparrow\alpha_1^{\mathsf{op}} \overset{M^{\mathsf{op}}}{\Rrightarrow} \Uparrow\alpha_2^{\mathsf{op}}$$

This is to say that we can just define $\underline{\pi}, \underline{\mu}$ (as well as $\underline{\gamma}, \underline{\rho}$) as a whiskering of π^{op} and μ^{op} (and eventually $\gamma^{\mathsf{op}}, \rho^{\mathsf{op}}$) with ι. This explicitly means that each component, *e.g.* $\underline{\pi}_G$, will be defined as the 2-cell in Span given by $\iota(\pi_G)$. Then, we can consider their image in $\mathsf{PsFun}(\mathsf{Span}^n, \mathsf{Span})$ via the usual biequivalence, since the 2-cells of $\ell q\mathsf{Mack}_{\mathsf{Span}}(\mathbb{G}^n, \mathbb{J}^n)$ consist of all possible modifications. Axioms defining a monoidal bicategory are then satisfied by pseudofunctoriality of the biequivalence. □

Now, the very same proof will prove part of the next result, saying that we can actually transfer the monoidal structure of the categorical product to a monoidal structure also on the bicategory of motives:

Theorem 3.4.6. *The cartesian monoidal structure on* $\mathbb{G}^{\mathsf{op}}$ *extends to a monoidal structure on the bicategory* Mot.

Proof. The proof is analogous to the previous one and uses an analogous result to that used in the previous one, namely the biequivalence (3.10). It's clear how tensor and unit are defined: one has the composite

$$\mathsf{mot} \circ \mathbb{1} \colon 1 \longrightarrow \mathbb{G}^{\mathsf{op}} \longrightarrow \mathsf{Mot}$$

and the extension (from Proposition 3.4.3):

$$\begin{array}{ccc} \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} & \xrightarrow{\times^{\mathsf{op}}} \mathbb{G}^{\mathsf{op}} \xhookrightarrow{\mathsf{mot}} & \mathsf{Mot} \\ {\scriptstyle \mathsf{mot}\times\mathsf{mot}} \big\downarrow & \nearrow_{\odot} & \\ \mathsf{Mot} \times \mathsf{Mot} & & \end{array}$$

It has to be shown that associator and unitors defined by whiskerings $\underline{\ell} = \mathsf{mot}\ell^{\mathsf{op}}$, $\underline{r} = \mathsf{mot}r^{\mathsf{op}}$ $\underline{a} = \mathsf{mot}a^{\mathsf{op}}$, for ℓ, r, a the monoidal structure for $\mathbb{G}$, are 1-cells in $q\mathsf{Mack}_{\mathsf{Mot}}(\mathbb{G}^n, \mathbb{J}^n)$. But our new structure is clearly the image of the structure built in Theorem 3.4.5 via the pseudofunctor

$$\mathsf{PsFun}((\mathbb{G}^{\mathsf{op}})^n, \mathsf{Span}) \longrightarrow \mathsf{PsFun}((\mathbb{G}^{\mathsf{op}})^n, \mathsf{Mot})$$

induced by the inclusion $\mathsf{Span} \hookrightarrow \mathsf{Mot}$. Hence, the isomorphic mates in Span will still remain isomorphisms in Mot, and as before this allows the lifting. □

3.5 The braided monoidal bicategory Add

In this section we start to deal with the case where $\mathcal{V}$ is the braided monoidal bicategory Add. As a bicategory, Add has additive categories as objects, additive functors as 1-cells, and natural transformations between them as 2-cells. The monoidal product of two additive categories $\mathcal{C}$ and $\mathcal{D}$ is subsequently defined in terms of the usual monoidal product of Ab-categories. Moreover Add is in fact a 2-category, since like for Cat composition is the usual composition of functors, and hence is strict.

Remark 3.5.1. As for any enrichment, Ab-Cat enjoys a monoidal structure induced by the one in Ab. The unit is $\mathbb{Z}$, seen like any ring as an Ab-category with one object. The tensor product (denoted $\otimes_{\mathbb{Z}}$ as the monoidal product of Ab, for sake of readability) of two Ab-categories $\mathcal{C}, \mathcal{D}$ is the Ab-category $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}$ having objects pairs of objects (C, D) in $\mathcal{C}_0 \times \mathcal{D}_0$, and hom-objects $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D}((C, D), (C', D')) = \mathcal{C}(C, C') \otimes_{\mathbb{Z}} \mathcal{D}(D, D')$. There's no reason for this structure to pass to the 2-subcategory Add. However, there is a universal constructions allowing to avoid this problem.

Definition 3.5.2. If $\mathcal{C}$ is an Ab-category, one can consider its *additive hull* $\mathcal{C}^{\oplus}$, which is the additive category defined by finite lists of objects in $\mathcal{C}$, and the evident matrices of maps in $\mathcal{C}$ as morphisms. More precisely, a morphism $[c_1, \ldots, c_n] \to [d_1, \ldots d_m]$ in $\mathcal{C}^{\oplus}$ is an $m \times n$ matrix $\left(f_{jk}\right)_{j,k}$ with $f_{jk} \colon c_k \to d_j$ in $\mathcal{C}$

The additive hull construction (see for example [DT14] for more details) defines a 2-functor

$$\text{Ab-Cat} \longrightarrow \text{Add}$$

which is left 2-adjoint to the inclusion Add $\hookrightarrow$ Ab-Cat, and is hence in particular such that for all additive categories $\mathcal{D}$ there is a natural equivalence of functor categories

$$\text{Add}(\mathcal{C}^{\oplus}, \mathcal{D}) \xrightarrow{\simeq} \text{Ab-Cat}(\mathcal{C}, \mathcal{D}). \tag{3.11}$$

This allows us to define a monoidal product of additive categories.

Definition 3.5.3. The monoidal structure on the 2-category Add of additive categories is defined by the unit $\mathbb{1} = \mathbb{Z}^{\oplus}$ and the tensor product $\mathcal{C} \otimes \mathcal{D} = (\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D})^{\oplus}$. The composition of two elementary tensor morphisms $(f \otimes g) \circ (f' \otimes g')$ is defined to be the elementary tensor $(ff' \otimes gg')$.

Remark 3.5.4. It is convenient to fix notation for addressing general elements of an additive completion of a linear category. Objects of $\mathcal{C}^{\oplus}$ are lists of objects in $\mathcal{C}$, and morphisms are matrices. A suggestive way to denote the list $[c_1, \ldots, c_n]$ is using the direct sum symbol $\bigoplus_{k=1}^{n} c_k$. Therefore, in particular, an object in a tensor product of additive categories $\mathcal{C} \otimes \mathcal{D} = (\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{D})^{\oplus}$ will be a list of pairs $\bigoplus_{k=1}^{n} (c_k, d_k)$.

Remark 3.5.5. Part of the structure of monoidal 2-category can be defined via the Yoneda lemma. Each component $\ell_{\mathcal{C}}$ of, let's say, the monoidal left unitor ℓ, is given from the chain of equivalences of enriched functor categories, for every additive category $\mathcal{C}'$,

$$\begin{aligned}
\text{Hom}(\mathbb{1} \otimes \mathcal{C}, \mathcal{C}') &\simeq \text{Hom}((\mathbb{1} \otimes_{\mathbb{Z}} \mathcal{C})^{\oplus}, \mathcal{C}') \overset{(3.11)}{\simeq} \text{Hom}(\mathbb{1} \otimes_{\mathbb{Z}} \mathcal{C}, \mathcal{C}') \\
&\simeq \text{Hom}(\mathbb{1}, \text{Hom}(\mathcal{C}, \mathcal{C}')) \overset{(3.11)}{\simeq} \text{Hom}(\mathbb{Z}, \text{Hom}(\mathcal{C}, \mathcal{C}')) \\
&\simeq \text{Hom}(\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{C}, \mathcal{C}') \simeq \text{Hom}(\mathcal{C}, \mathcal{C}').
\end{aligned}$$

Analogous arguments show how to construct the right unitor, but this does not work for the associator[1] The crucial point, even in 1-dimensional category theory, is that given a full reflective subcategory of a monoidal category one can always define a tensor product by tensoring in the largest category and then reflecting into the subcategory. However, this need not be associative. The Day reflection theorem ([Day72]) gives conditions under which this works (for the 1-categorical setting). In our case, we can define the associator explicitly as follows.

Definition 3.5.6. Given a general object in $(\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}$, which we denote

$$\bigoplus_{i=1}^{n}(\bigoplus_{k=1}^{\ell(i)}(a_{i_k},b_{i_k}),c_i),$$

we let the associator to be the functor defined (on objects)

$$\begin{gathered} a_{\mathcal{A},\mathcal{B},\mathcal{C}}\colon (\mathcal{A}\otimes\mathcal{B})\otimes\mathcal{C}\longrightarrow \mathcal{A}\otimes(\mathcal{B}\otimes\mathcal{C}) \\ \bigoplus_{i=1}^{n}\Big(\bigoplus_{k=1}^{\ell(i)}(a_{i_k},b_{i_k}),c_i\Big)\longmapsto \bigoplus_{i=1}^{n}\bigoplus_{k=1}^{\ell(i)}\big(a_{k_i},(b_{k_i},c_i)\big). \end{gathered}$$

The definition on morphisms follows very naturally: if

$$f\colon \bigoplus_{i=1}^{n}\Big(\bigoplus_{r=1}^{o(i)}(a_{i_r},b_{j_r}),c_i\Big)\longrightarrow \bigoplus_{j=1}^{m}\Big(\bigoplus_{k=1}^{p(j)}(a_{j_k},b_{j_k}),c_j\Big),$$

then f is a matrix

$$f=\begin{pmatrix} f_{11} & \dots & f_{1n}\\ \vdots & & \vdots\\ f_{m1} & \dots & f_{mn}\end{pmatrix}$$

where each f_{ji} is a morphism

$$\bigoplus_{r=1}^{o(i)}((a_{i_r},b_{i_r}),c_i)\longrightarrow\bigoplus_{k=1}^{p(j)}((a_{j_k},b_{j_k}),c_j).$$

That means, each f_{ij} is itself a matrix of size $p(j)\times o(i)$. That means,

$$f=\begin{pmatrix} \Big({f_{11}}^{kr}\Big)_{k,r} & \dots & \Big({f_{1n}}^{kr}\Big)_{k,r}\\ \vdots & & \vdots\\ \Big({f_{m1}}^{kr}\Big)_{k,r} & \dots & \Big({f_{mn}}^{kr}\Big)_{k,r}\end{pmatrix}.$$

The morphism $a(f)$ is then the matrix that we obtain by removing the internal parenthesis in the matrix defining f. Its size is then $\sum_{j=1}^{m}p(j)\times\sum_{i=1}^{n}o(i)$, and each entry is then seen as a one-entry matrix.

[1] Thanks to Steve Lack for pointing this out to me.

Remark 3.5.7. One can then carefully but easily verify that this structure makes the pentagon strictly commutative:

$$\begin{array}{ccc}
\bigoplus_{j=1}^{m}\Big(\bigoplus_{i=1}^{n(i)}\bigoplus_{k=1}^{p(k)}(a_{j_{i_k}},(b_{j_{i_k}},c_{j_i})),d_j\Big) & \overset{a}{\longmapsto} & \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n(i)}\bigoplus_{k=1}^{p(k)}(a_{j_{i_k}},((b_{j_{i_k}},c_{j_i}),d_j)) \\
\Big\uparrow{\scriptstyle a1} & & \Big\downarrow{\scriptstyle 1a} \\
\bigoplus_{j=1}^{m}\Big(\bigoplus_{i=1}^{n(i)}\Big(\bigoplus_{k=1}^{p(k)}(a_{j_{i_k}},b_{j_{i_k}}),c_{j_i}\Big),d_j\Big) & & \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n(i)}\bigoplus_{k=1}^{p(k)}(a_{j_{i_k}},(b_{j_{i_k}},(c_{j_i},d_j))) \\
\searrow{\scriptstyle a} & & \nearrow{\scriptstyle a} \\
& \bigoplus_{j=1}^{m}\bigoplus_{i=1}^{n(i)}\Big(\bigoplus_{k=1}^{p(k)}(a_{j_{i_k}},b_{j_{i_k}}),(c_{j_i},d_j)\Big) &
\end{array}$$

Therefore, one can choose π to be precisely the identity, and clearly the non-abelian 4-cocycle condition 1.1.9 is satisfied. On the other hand, the monoidal unitor μ is not, unfortunately, the identity: the triangle

$$\begin{array}{ccc}
(A\otimes\mathbb{1})\otimes B & \xrightarrow{\;a\;} & A\otimes(\mathbb{1}\otimes B) \\
& {\scriptstyle r1}\searrow \quad \swarrow{\scriptstyle 1\ell} & \\
& A\otimes B &
\end{array}$$

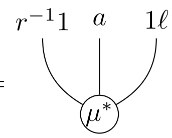
is not strictly commutative in Add, since the diagram

$$\begin{array}{ccc}
\bigoplus_{j=1}^{m}\Big(\bigoplus_{k=1}^{n(k)}(a_{j_k},u_{j_k}),b_j\Big) & \overset{a}{\longmapsto} & \bigoplus_{j=1}^{m}\bigoplus_{k=1}^{n(k)}(a_{j_k},(u_{j_k},b_j)) \\
{\scriptstyle r1}\searrow & & \swarrow{\scriptstyle 1\ell} \\
\bigoplus_{j=1}^{m}\Big(\bigoplus_{k=1}^{n(k)}a_{j_k},b_j\Big) & \cong & \bigoplus_{j=1}^{m}\bigoplus_{k=1}^{n(k)}(a_{j_k},b_j)
\end{array}$$

requires a canonical but non-identical isomorphism to commute. However, μ has a mate which is the identity: it can in fact easily be checked that the square

$$\begin{array}{ccc}
(A\otimes\mathbb{1})\otimes B & \xrightarrow{\;a\;} & A\otimes(\mathbb{1}\otimes B) \\
{\scriptstyle r^{-1}1}\Big\uparrow & & \Big\downarrow{\scriptstyle 1\ell} \\
A\otimes B & = & A\otimes B
\end{array}$$

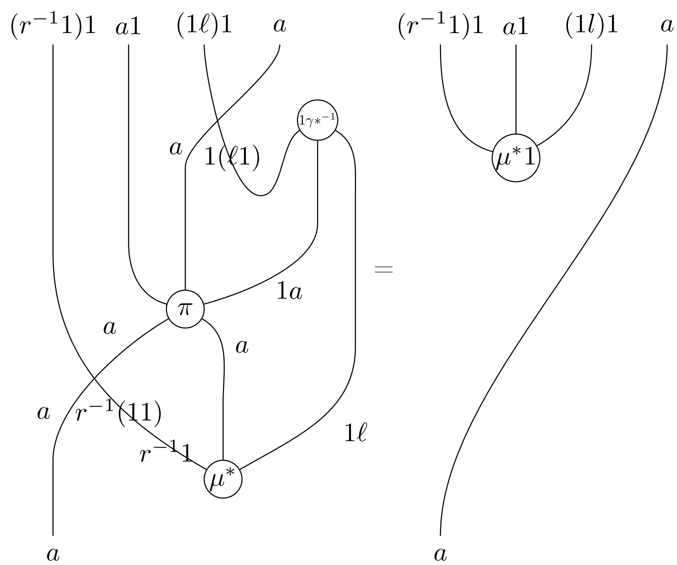
strictly commutes, and then that one can consider id =



The very same argument can be carried out for γ and δ, the two unitors which can be considered part of the structure, and which are useful in order to state the axioms. They can't be chosen themselves to be identical, but their mates

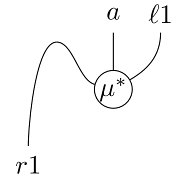
$$\begin{array}{ccc} (\mathbb{1}\otimes A)\otimes B & \xrightarrow{\ a\ } & \mathbb{1}\otimes(A\otimes B) \\ {\scriptstyle \ell^{-1}1}\uparrow & \Downarrow{\scriptstyle\gamma^*} & \downarrow{\scriptstyle \ell} \\ A\otimes B & = & A\otimes B \end{array} \qquad \begin{array}{ccc} (A\otimes B)\otimes\mathbb{1} & \xrightarrow{\ a\ } & A\otimes(B\otimes\mathbb{1}) \\ {\scriptstyle r^{-1}}\uparrow & \Downarrow{\scriptstyle\delta^*} & \downarrow{\scriptstyle 1r} \\ A\otimes B & = & A\otimes B \end{array}$$

do. This argument comes from a work in progress by Vanessa Miemietz and Fiona Torzewska [MT25], who kindly helped me to sort this thing out by sharing their research with me. Then, one observes in the end that the structure and the axioms for a monoidal bicategory can easily be given in terms of these mates. For example, the left normalization axiom for the structure $\pi, \mu^*, \gamma^*, \delta^*$ becomes



This axiom, and similarly the right normalization, are then trivially true, since all of the 2-cells involved are identical. Also, they are equivalent to our original left and right normalization axioms by simply applying triangular identities for the adjunctions involved. More precisely, the equality above implies the usual left normalization axiom for the structure defined by $\pi = \mathrm{id}$,

μ =



and similarly γ and δ. This concludes that Add has a structure of monoidal bicategory.

Let us now come to the definition of the braiding.

Definition 3.5.8. Given the monoidal bicategory Add, we define a pseudonatural transformation

$$\begin{array}{ccc} \mathrm{Add} \times \mathrm{Add} & \xrightarrow{\quad\sigma\quad} & \mathrm{Add} \times \mathrm{Add} \\ & \swarrow\!\!\!\!\swarrow \beta & \\ {\scriptstyle\otimes}\searrow & & \swarrow{\scriptstyle\otimes} \\ & \mathrm{Add} & \end{array}$$

on a pair of additive categories A, B to be the functor

$$\begin{aligned} A \otimes B &\longrightarrow B \otimes A \\ \bigoplus_{j=1}^{m} (a_j, b_j) &\longmapsto \bigoplus_{j=1}^{m} (b_j, a_j) \end{aligned}$$

and similarly, on a morphism, switching components of each entry of the matrix giving that morphism.

Remark 3.5.9. With the above definition, we can conclude that Add is a braided monoidal bicategory. In fact, the rest of the structure R and S, can be chosen to be identical, and then to automatically satisfy the four braiding axioms of Definition 1.4.1, which only involve R, S and π, also defined to be the identity in the monoidal structure (Remark 3.5.7). The reason why R and S can be chosen to be identities, just reduces to checking the commutativity of the hexagon diagrams. For example, the one defining R is

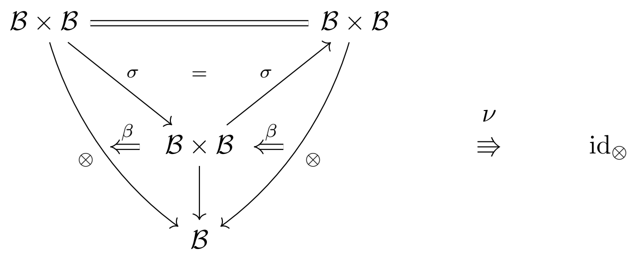
$$\begin{array}{ccc} \bigoplus_k \bigoplus_j \left(a_{k_j}, (b_{k_j}, c_k)\right) & \overset{\beta}{\longmapsto} & \bigoplus_k \bigoplus_j \left((b_{k_j}, c_k), a_{k_j}\right) \\ {\scriptstyle a}\uparrow & & \downarrow{\scriptstyle a} \\ \bigoplus_k \left(\bigoplus_j (a_{k_j}, b_{k_j}), c_k \right) & & \bigoplus_k \bigoplus_j \left(b_{k_j}, (c_k, a_{k_j})\right) \\ {\scriptstyle \beta 1}\downarrow & & \uparrow{\scriptstyle 1\beta} \\ \bigoplus_k \left(\bigoplus_j (b_{k_j}, a_{k_j}), c_k \right) & \underset{a}{\longmapsto} & \bigoplus_k \bigoplus_j \left(b_{k_j}, (a_{k_j}, c_k)\right) \end{array}$$

and similarly for the one defining S.

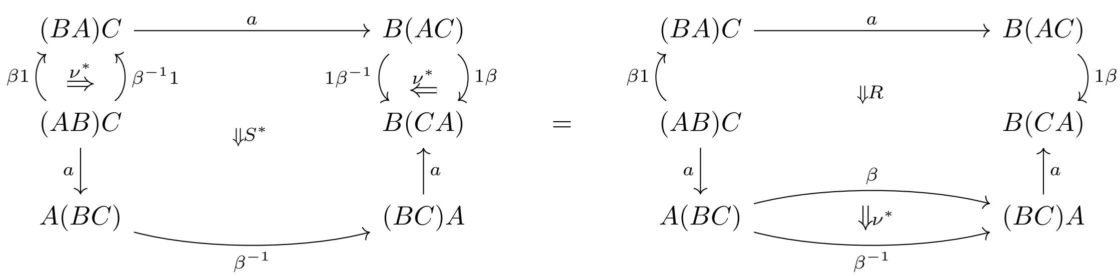
We have hence proved the following proposition.

Proposition 3.5.10. *With the definition above,* Add *is equipped with a structure of braided monoidal bicategory.*

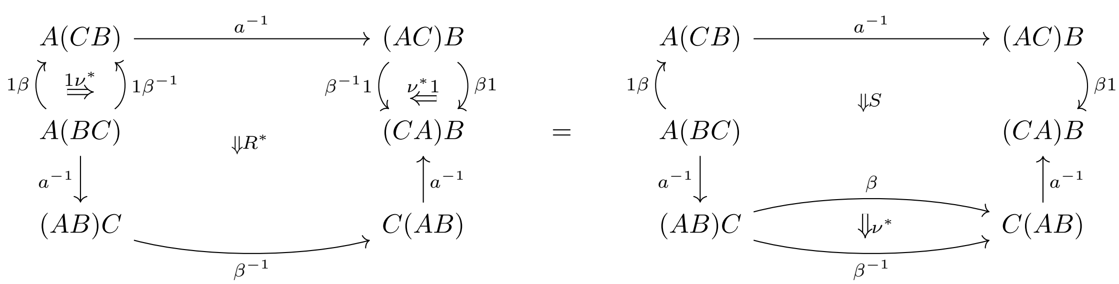
Remark 3.5.11. There is a further structure that one can consider on a braided monoidal bicategory, and is that of a syllepsis $\nu\colon \beta \circ \beta \Rightarrow \mathrm{id}$:



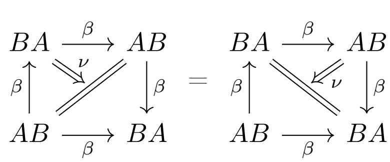
The definition for the syllepsis axioms can be found in [GO13]. They are the following



and



Moreover, for a sylleptic monoidal bicategory, to be *symmetric* means to satisfy the following further axiom



Therefore, one can observe that the identity can be taken as a syllepsis for Add, and the axioms will hold true, since for Add the two braiding structures R and S are also defined to be the identity.

To sum up, we have the following.

Proposition 3.5.12. *The bicategory* Add*, with the structure above, is a symmetric monoidal bicategory.*

3.5.1 Bilimits and bicolimits in Add

As well known, the right notion of limits and colimits in enriched category theory uses the notion of *weights*, which are $\mathcal{V}$-valued $\mathcal{V}$-functors replacing the constant functor $\Delta 1$, when $\mathcal{V}$ is different than Set. In the bidimensional setting we can define the following.

Definition 3.5.13. Let $F\colon \mathcal{P} \to \mathcal{K}$ be a $\mathcal{V}$-pseudofunctor and $W\colon \mathcal{P}^{\mathsf{op}} \to \mathcal{V}$ a weight. The *bicolimit* of F weighted by W is an object $W \star F$ in $\mathcal{K}$ together with an equivalence of $\mathcal{V}$-pseudofunctors

$$\mathcal{K}(W \star F, =) \simeq \mathcal{V}\text{-}\mathsf{PsNat}(W, \mathcal{K}(F(-), =))$$

Dually, one can define a *bilimit* for $W\colon \mathcal{P} \to \mathcal{V}$ to be an object $\{W, F\}$ together with a $\mathcal{V}$-pseudonatural equivalence

$$\mathcal{K}(c, \{W, F\}) \simeq \mathcal{V}\text{-}\mathsf{PsNat}(W, \mathcal{K}(c, G(-)))$$

for all c in $\mathcal{K}$.

The way this definition comes out relates to the non-enriched (and 1-dimensional) case in the following way. For a limit over a diagram $G\colon \mathcal{P} \to \mathcal{K}$, we have

$$\mathcal{K}(a, \mathsf{Lim}G) \cong \mathsf{PsNat}(\mathsf{const}(a), G) \cong \mathsf{PsNat}(\Delta 1, \mathcal{K}(a, G-)),$$

so that it becomes natural to consider any enriched functor F, instead of $\Delta 1\colon \mathcal{P} \to \mathsf{Set}$. The point that is worth to remark is again (as for the definition of extra-naturality) the fact that the constant enriched functor doesn't even exist in general.

The goal of this section is to prove that Add-valued bi(co)ends exist. This can be done by proving that bi(co)ends can be expressed using bi(co)limits, ad that Add admits all bi(co)limits

Remark 3.5.14. The following result should be possibly stated and proved in a more general flavor, but we prove it for the monoidal bicategory $\mathcal{V} = \text{Add}$ because it is easier to work with objects of the hom-objects. However, the proof is done in sufficiently enriched terms in a way that allows to recognize the steps for a proper adaptation to a general $\mathcal{V}$.

Proposition 3.5.15. *Let $\mathcal{V} = \text{Add}$. Suppose $P\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \to \mathcal{V}$ is a $\mathcal{V}$-pseudofunctor. Then its bicoend (in the sense of Definition 2.2.1) can be computed as the hom-weighted bicolimit $\mathcal{B}(-,-) \star P$.*

Proof. By definition of weighted bicolimit, our goal is to prove the equivalence

$$[\int^b P(b,b), a] \simeq \mathcal{V}\text{-}\mathsf{PsNat}(\mathcal{B}(-,-), [P(-,-), a]).$$

The latter is by definition $\int_{b,b'} [\mathcal{B}(b,b'), [P(b',b), a]] \simeq \int_b \int_{b'} [\mathcal{B}(b,b'), [P(b',b), a]]$, via Fubini, while the first is equivalent to $\int_b [P(b,b), a]$. Hence, it suffices to prove an equivalence of $\mathcal{V}$-pseudofunctors

$$[P(-,=), a] \simeq \int_b [\mathcal{B}(-,b), [P(b,=), a]]\colon \mathcal{B}^{\mathsf{op}} \otimes \mathcal{B} \longrightarrow \mathcal{V}$$

On each object (d,c) of $\mathcal{B}^{\mathsf{op}} \otimes \mathcal{B}$, then, one has to prove the equivalence $[P(c,d), a] \simeq \int_b [\mathcal{B}(c,b), [P(b,d), a]]$. Using essential uniqueness of biends, we aim to exhibit a biend stucture

$$i\colon [P(c,d), a] \Rrightarrow [\mathcal{B}(c,-), [P(-,d), a]].$$

The morphism i_b is defined by mapping a morphism $w\colon P(c,d)\to a$ in $\mathcal{V}$ to

$$\mathcal{B}(c,b) \xrightarrow{w\otimes P(-,d)} [P(c,d),a]\otimes[P(b,d),P(c,d)] \xrightarrow{m} [P(b,d),a].$$

From now on we, use the assumption of working in $\mathcal{V} = \mathrm{Add}$. On the 2-dimensional component, i_f, for f object in $\mathcal{B}(c,b)$, is defined by

$$\begin{array}{ccc} [P(c,d),a] & \xrightarrow{\quad i_b \quad} & [\mathcal{B}(c,b),[P(b,d),a]] \\ {\scriptstyle i_c}\big\downarrow & \swarrow i_f & \big\downarrow{\scriptstyle \mathcal{B}(c,f)^*} \\ [\mathcal{B}(c,c),[P(c,d),a]] & \xrightarrow[{[P(f,d),a]_*}]{} & [\mathcal{B}(c,c),[P(b,d),a]] \end{array}$$

having components simply provided by the functoriality for $P(-,d)$: for all $w\colon P(c,d)\to a$

$$\begin{array}{ccccc} & \overset{\mathcal{B}(c,f)}{\nearrow} & \mathcal{B}(c,b) & \overset{i_b(w)}{\searrow} & \\ \mathcal{B}(c,c) & & \Downarrow (i_f)_w & & [P(b,d),a] \\ & \underset{i_c(w)}{\searrow} & [P(c,d),a] & \underset{[P(f,d),a]}{\nearrow} & \end{array}$$

is defined by having each component at h object in $\mathcal{B}(c,c)$

$$((i_f)_w)_h\colon i_b(w)(fh) = w\circ P(fh,d) \xrightarrow{w*\mathsf{fun}^{-1}} w\circ P(h,d)\circ P(f,d) = i_c(w)(h)\circ P(f,d). \quad (3.12)$$

Such an i, being a transposition obtained from the pseudonatural

$$[P(-,d),a]\colon \mathcal{B}(-,-) \Rightarrow [[P(-,d),a],[P(-,d),a]]$$

evaluated at c in the first variable, happens to be extra-pseudonatural. It remains to prove that the extra-pseudonatural transformation i is terminal. Let

$$j\colon x \overset{..}{\Rightarrow} [\mathcal{B}(c,-),[P(-,d),a]]$$

be an extra-pseudonatural transformation. We look for the pair $(\tilde{j},\{J_b\})$

$$\begin{array}{ccc} x & \xrightarrow{\qquad \tilde{j} \qquad} & [P(c,d),a] \\ & {\scriptstyle j_b}\searrow \quad \nearrow J_b \quad \swarrow{\scriptstyle i_b} & \\ & [\mathcal{B}(c,b),[P(b,d),a]] & \end{array}$$

witnessing terminality. The morphism $\tilde{j}$ is defined to be

$$\tilde{j}\colon x \xrightarrow{j_c\otimes u_c} [\mathcal{B}(c,c),[P(c,d),a]]\mathcal{B}(c,c) \xrightarrow{\varepsilon} [P(c,d),a],$$

so that we look for an invertible J_b whose components for each object e in the additive category x will be of the form

$$(J_b)_e\colon j_b(e) \longrightarrow i_b(\varepsilon(j_c(e),\mathrm{id}_c))$$

Now, by definition of i_b and of the counit, to give such a natural isomorphism $(J_b)_e$ of additive functors $\mathcal{B}(c,b) \to [P(b,d),a]$ means to give, for every object f in the additive category $\mathcal{B}(c,b)$, an invertible map

$$((J_b)_e)_f \colon j_b(e)(f) \longrightarrow j_c(e)(\mathrm{id}_c) \circ P(f,d).$$

This is precisely given by the extra-pseudonatural structure of j. By hitting the structural j_{cb} with $f \colon \mathbb{1} \to \mathcal{B}(c,b)$, we find

$$\begin{array}{ccc} x & \xrightarrow{\quad j_b \quad} & [\mathcal{B}(c,b),[P(b,d),a]] \\ {\scriptstyle j_c}\big\downarrow & \swarrow j_f & \big\downarrow{\scriptstyle [\mathcal{B}(f,c),1]} \\ [\mathcal{B}(c,c),[P(c,d),a]] & \xrightarrow[{[1,[P(f,d),1]]}]{} & [\mathcal{B}(c,c),[P(b,d),a]] \end{array}$$

which is indeed a natural transformation whose components

$$(j_f)_e \colon j_b(e) \circ \mathcal{B}(c,f) \to [P(f,d),a] \circ j_c(e)$$

are themselves natural transformations of functors $\mathcal{B}(c,b) \to [P(b,d),a]$. If we evaluate such a natural transformation at the object id_c we find, as desired, an invertible map

$$((J_b)_e)_f \coloneqq ((j_f)_e)_{\mathrm{id}_c} \colon j_b(e)(f) \longrightarrow j_c(e)(\mathrm{id}_c) \circ P(f,d).$$

Then, it remains to prove the biend axioms.

Let us call $Q(-,-) = [\mathcal{B}(c,-),[P(-,d),a]]$ the pseudofunctor in question. The first axiom for i to define a biend then says that the following equality of 2-cells should hold true:

$$\begin{array}{ccccc} \mathcal{B}(c,b) & \xrightarrow{Q(-,b)} & [Q(b,b),Q(c,b)] & & \\ {\scriptstyle Q(c,-)}\big\downarrow & & & \searrow{\scriptstyle j_b^*} & \\ [\mathcal{B}(c,c),[P(c,d),a]] & \swarrow i_{cb} & \big\downarrow{\scriptstyle i_b^*} \quad \swarrow J_b^* & & [x,Q(c,b)] \\ & \searrow{\scriptstyle i_c^*} & & \nearrow{\scriptstyle \tilde{j}^*} & \\ & & [[P(c,d),a],Q(c,b)] & & \end{array}$$

$$=$$

$$\begin{array}{ccccc} \mathcal{B}(c,b) & \xrightarrow{Q(-,b)} & [Q(b,b),Q(c,b)] & & \\ {\scriptstyle Q(c,-)}\big\downarrow & \swarrow j_{cb} & & \searrow{\scriptstyle j_b^*} & \\ [\mathcal{B}(c,c),[P(c,d),a]] & & \xrightarrow{\qquad j_c^* \qquad} & & [x,Q(c,b)] \\ & \searrow{\scriptstyle i_c^*} & \Downarrow J_c^* & \nearrow{\scriptstyle \tilde{j}^*} & \\ & & [[P(c,d),a],Q(c,b)] & & \end{array}$$

That means, we want the square

$$\begin{array}{ccc} [\mathcal{B}(c,f),[P(b,d),a]] \circ j_b & \xrightarrow{\quad j_f \quad} & [\mathcal{B}(c,c),[P(f,d),a]] \circ j_c \\ {\scriptstyle 1 * J_b}\big\downarrow & & \big\downarrow{\scriptstyle 1 * J_c} \\ [\mathcal{B}(c,f),[P(b,d),a]] \circ i_b \circ \tilde{j} & \xrightarrow[i_f * 1]{} & [\mathcal{B}(c,c),[P(f,d),a]] \circ i_c \circ \tilde{j} \end{array} \tag{3.13}$$

to commute in the additive category $[x, [\mathcal{B}(c,c), [P(b,d),a]]]$ for each f in $\mathcal{B}(c,b)$. Let us evaluate the two sides of (3.13), which are natural transformations of functors $x \to [\mathcal{B}(c,c), [P(b,d),a]]$, at an object e in x.

$$\begin{array}{ccc}
j_b(e) \circ \mathcal{B}(c,f) & \xrightarrow{(j_f)_e} & [P(f,d),a] \circ j_c(e) \\
{\scriptstyle (J_b)_e * \mathcal{B}(c,f)} \downarrow & & \downarrow {\scriptstyle [P(f,d),a] * (J_c)_e} \\
i_b(\tilde{j}(e)) \circ \mathcal{B}(c,f) & \xrightarrow[(i_f)_{\tilde{j}(e)}]{} & [P(f,d),a] \circ i_c(\tilde{j}(e))
\end{array}$$

Again, this is a square of natural transformations of (additive) functors $\mathcal{B}(c,c) \to [P(b,d),a]$, and hence we can evaluate it at a morphism $h\colon c \to c$, finding (using the definition of J)

$$\begin{array}{ccc}
j_b(e)(fh) & \xrightarrow{((j_f)_e)_h} & j_c(e)(h) \circ P(f,d) \\
{\scriptstyle ((j_{fh})_e)_{\mathrm{id}_c}} \downarrow & & \downarrow {\scriptstyle ((j_h)_e)_{\mathrm{id}_c} * P(f,d)} \\
j_c(e)(\mathrm{id}_c) \circ P(fh,d) & \xrightarrow[((i_f)_{\tilde{j}(e)})_h]{} & j_c(e)(\mathrm{id}_c) \circ P(h,d) \circ P(f,d)
\end{array} \tag{3.14}$$

Now, by keeping in mind the definition of i_f at (3.12), it should be clear that this equality is precisely given by the functoriality axiom for the extra-pseudonatural transformation j. That axiom gives, in its evaluation at composable arrows $c \xrightarrow{h} c \xrightarrow{f} b$, and then at e object in x, the commutativity of the following square of natural transformations

$$\begin{array}{ccccc}
 & & [P(f,d),a] \circ (j_c)_e \circ \mathcal{B}(c,h) & & \\
 & {\scriptstyle (j_f)_e * \mathcal{B}(c,h)} \nearrow & & \searrow {\scriptstyle [P(f,d),a] * (j_h)_e} & \\
j_b(e) \circ \mathcal{B}(c,f) \circ \mathcal{B}(c,h) & & & & [P(f,d),a] \circ [P(h,d),a] \circ j_c(e) \\
{\scriptstyle j_b(e) * \mathsf{fun}} \downarrow & & & & \uparrow {\scriptstyle \mathsf{fun}^{-1} * j_c(e)} \\
j_b(e) \circ \mathcal{B}(c,fh) & & \xrightarrow[(j_{fh})_e]{} & & [P(fh,d),a] \circ j_c(e)
\end{array}$$

If we evaluate it at id_c, we get precisely the equality (3.14) desired

$$\begin{array}{ccc}
j_b(e)(fh) & \xrightarrow{((j_f)_e)_h} & j_c(e)(h) \circ P(f,d) \\
{\scriptstyle ((j_{fh})_e)_{\mathrm{id}_c}} \downarrow & & \downarrow {\scriptstyle ((j_f)_e)_{\mathrm{id}_c} * P(f,d)} \\
j_c(e)(\mathrm{id}_c) \circ P(fh,d) & \xrightarrow[j_c(e) * \mathsf{fun}^{-1}]{} & j_c(e)(\mathrm{id}_c) \circ P(h,d) \circ P(f,d)
\end{array}$$

Let us now prove the second biend axiom, which in general states that whenever $\ell, k\colon x \to [P(c,d),a]$ are two morphisms and $\{\Gamma_a\colon i_m \ell \Rightarrow i_m k\}$ indexed by objects m in $\mathcal{B}$ is a family of maps such that

$$
\begin{tikzcd}
\mathcal{B}(c,b) \arrow[r, "Q(-,b)"] \arrow[d, "Q(c,-)"'] & {[Q(b,b),Q(c,b)]} \arrow[d, "i_b^*"] \arrow[rd, "i_b^*"] & \\
{[Q(c,c),Q(c,b)]} \arrow[rd, "i_c^*"'] & \Swarrow i_{cb} \quad \Swarrow \Gamma_b^* & {[[P(c,d),a],Q(c,b)]} \arrow[d, "\ell^*"] \\
& {[[P(c,d),a],Q(c,b)]} \arrow[r, "k^*"'] & {[x,Q(c,b)]}
\end{tikzcd}
$$

$$
= \tag{3.15}
$$

$$
\begin{tikzcd}
\mathcal{B}(c,b) \arrow[r, "Q(-,b)"] \arrow[d, "Q(c,-)"'] & {[Q(b,b),Q(c,b)]} \arrow[rd, "i_b^*"] & \\
{[Q(c,c),Q(c,b)]} \arrow[rr, "i_c^*"] \arrow[rd, "i_c^*"'] & \Swarrow i_{cb} & {[[P(c,d),a],Q(c,b)]} \arrow[d, "\ell^*"] \\
& {[[P(c,d),a],Q(c,b)]} \arrow[r, "k^*"'] \quad \Swarrow \Gamma_c^* & {[x,Q(c,b)]}
\end{tikzcd}
$$

There is a unique $\gamma\colon \ell \Rightarrow k$ such that for all m in $\mathcal{B}$ it holds $i_m * \gamma = \Gamma_m$. Now if we evaluate (3.15) at f in $\mathcal{B}(c,b)$, we find a commutative square of natural transformation

$$
\begin{tikzcd}
i_b(\ell(e)) \circ \mathcal{B}(c,f) \arrow[r, "(\Gamma_b)_e * \mathcal{B}(c,f)"] \arrow[d, "(i_f)_{\ell(e)}"'] & i_b(k(e)) \circ \mathcal{B}(c,f) \arrow[d, "(i_f)_{k(e)}"] \\
{[P(f,d),a] \circ i_c(\ell(e))} \arrow[r, "{[P(f,d),a] * (\Gamma_c)_e}"'] & {[P(f,d),a] \circ i_c(k(e))}
\end{tikzcd}
$$

which, evaluated at h, give an equality

$$
\begin{tikzcd}
\ell(e) \circ P(fh,d) \arrow[r, "((\Gamma_b)_e)_{fh}"] \arrow[d, "\ell(e) * \mathsf{fun}^{-1}"'] & k(e) \circ P(fh,d) \arrow[d, "k(e) * \mathsf{fun}^{-1}"] \\
\ell(e) \circ P(h,d) \circ P(f,d) \arrow[r, "((\Gamma_c)_e)_h * P(f,d)"'] & k(e) \circ P(h,d) \circ P(f,d)
\end{tikzcd}
$$

Now if we specialize this commutative square at id_c, we are allowed to glue on both sides the commutative squares provided by the pseudofunctoriality axiom for $P(-,d)$:

$$
\begin{tikzcd}
\ell(e) \circ P(f,d) \arrow[r, "P(\rho,d)^{-1}"] \arrow[d, equal, "\rho = \mathrm{id}"'] & \ell(e) \circ P(f\mathrm{id}_c,d) \arrow[r, "((\Gamma_b)_e)_{f\mathrm{id}_c}"] \arrow[d, "\mathsf{fun}^{-1}"'] & k(e) \circ P(f\mathrm{id}_c,d) \arrow[r, "P(\rho,d)"] \arrow[d, "\mathsf{fun}^{-1}"] & k(e) \circ P(f,d) \arrow[d, equal, "\rho = \mathrm{id}"] \\
\ell(e) \circ P(f,d) \arrow[r, "\mathsf{un} * P(f,d)"'] & \ell(e) \circ P(\mathrm{id}_c,d) \circ P(f,d) \arrow[r, "((\Gamma_c)_e)_h * P(f,d)"'] & k(e) \circ P(\mathrm{id}_c,d) \circ P(f,d) \arrow[r, "\mathsf{un}^{-1} * P(f,d)"'] & k(e) \circ P(f,d)
\end{tikzcd}
\tag{3.16}
$$

Observe how we use the fact that we are dealing with a 2-category $\mathcal{V} = \mathrm{Add}$, so that ρ is the identity and the composition with the identity is strict. If, moreover, we specialize at $c = b$ and $f = \mathrm{id}$, then we can define γ_e to be the resulting composition of the two (equal) morphisms above by the appropriate unitors $\mathsf{un}\colon \ell(e) = \ell(e) \circ \mathrm{id}_{P(c,c)} \to \ell(e) \circ P(\mathrm{id}_c, d)$

and $\mathsf{un}^{-1}\colon k(e)\circ P(\mathrm{id}_c,d)\to k(e)\circ \mathrm{id}_{P(c,c)}=k(e)$. Now, it follows that for all m object in $\mathcal{B}$, we have $(i_m*\gamma)_e=i_m(\gamma_e)$ (as functor applied on a morphism), which evaluated at $f\colon c\to m$ gives $(i_m(\gamma_e))_f=\gamma_e*P(f,d)$, which is precisely the lower side of (3.16), and which, by commutativity of the same, is equal to the upper side (for $b=m$), which is precisely $((\Gamma_m)_e)_f$. □

Definition 3.5.16. A sub-($\mathcal{V}$-)bicategory $\mathcal{K}'\subset\mathcal{K}$ is said to be bireflexive if there exists a left pseudoadjoint to the inclusion ($\mathcal{V}$-)pseudofunctor.

Proposition 3.5.17. *Let* $W\colon\mathcal{P}\to\mathrm{Add}, W'\colon\mathcal{P}^{\mathsf{op}}\to\mathrm{Add}$ *be weights and* $F\colon\mathcal{P}\to\mathrm{Add}$ *an additive pseudofunctor. Then,* F *admits a bilimit* $\{W,F\}$ *and a bicolimit* $W'\star F$.

Proof. We claim that $\{W,F\}$ is given by the object $\mathrm{Add}\text{-}\mathsf{PsNat}(W,F)$, which is the additive category having objects the Add-pseudonatural transformations, and morphisms modifications between them. The proof then follows

$$\begin{aligned}\mathcal{V}(a,\int_p\mathcal{V}(Wp,Fp))&\simeq\int_p\mathcal{V}(a,\mathcal{V}(Wp,Fp))\\&\simeq\int_p\mathcal{V}(a\otimes Wp,Fp)\\&\simeq\int_p\mathcal{V}(Wp,\mathcal{V}(a,Fp))\end{aligned}$$

which exactly means $\mathcal{V}(a,\mathcal{V}\text{-}\mathsf{PsNat}(W,F))\simeq\mathcal{V}\text{-}\mathsf{PsNat}(W,\mathcal{V}(a,F-))$. For what concerns weighted bicolimits the argument consists of observing that the 2-category (*i.e.* Cat-enriched category) Ab-Cat of categories enriched in abelian groups admits 2-colimits (the usual enriched 1-categorical colimits when the base of the enrichment is Cat), and hence in particulars bicolimits. The bicategory Add is a bireflexive 2-full sub-bicategory of Ab-Cat. Therefore, we have a pseudofunctor Ab-Cat $\to$ Add which creates bicolimits. In general, If $L\dashv U\colon\mathcal{K}'\hookrightarrow\mathcal{K}$ is a reflexive sub-bicategory, $F\colon\mathcal{P}\to\mathcal{K}'$ and $W\colon\mathcal{P}^{\mathsf{op}}\to\mathcal{V}$ are $\mathcal{V}$-pseudofunctors, one has that

$$W\star F\simeq L(W\star UF).$$

The proof follows by

$$\begin{aligned}\mathcal{K}'(L(W\star UF),a)&\simeq\mathcal{K}(\mathrm{Colim}^F UF,Ua)\\&\simeq\mathcal{V}\text{-}\mathsf{PsNat}(F,\mathcal{K}(UF(-),Ua))\\&\simeq\mathcal{V}\text{-}\mathsf{PsNat}(F,\mathcal{K}'(F(-),a)).\end{aligned}$$

□

For more detailed references about weighted bi(co)limits, see [GS15]. We have now the vocabulary to express the enriched version of Theorem 3.4.6. Here, of course, the bicategory of motives is intended to be in its additive version.

Theorem 3.5.18. *The cartesian monoidal structure on* $\mathbb{G}$ *extends to a monoidal structure on the* Add*-bicategory* Mot.

Proof. Let's start from the fact that the structure $(\mathsf{Mot},\odot,\mathbb{1},\underline{a},\underline{\ell},\underline{r},\underline{\pi},\underline{\mu})$ defines a monoidal bicategory (that is, satisfies the non-abelian 4-cocycle condition). Thus, it suffices to show that such structural cells in Bicat, that is pseudofunctors $\odot, \mathbb{1}$, pseudonatural transformations $\underline{a}, \underline{\ell}, \underline{r}$ and modifications $\underline{\pi}, \underline{\mu}$, are actually also families of cells in Add-Bicat. Let us denote (as a common convention throughout this work) with the same symbol $\otimes$ both the tensor product in Add and the induced tensor product in Add-Bicat. Then, to say that $\odot\colon \mathsf{Mot} \otimes \mathsf{Mot} \longrightarrow \mathsf{Mot}$ is an enriched pseudofunctor means that it defines on each hom-category an additive functor

$$\phi\colon \mathsf{Mot}(G,G') \otimes \mathsf{Mot}(H,H') \longrightarrow \mathsf{Mot}(G \odot H, G' \odot H').$$

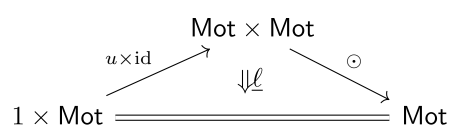
For such a functor ϕ to be additive means to be additive in each variable. So, consider for each let's say $f = (H \overset{f_1}{\leftarrow} P \overset{f_2}{\rightarrow} H')$ in $\mathsf{Mot}(H,H')$, and pair of maps $g = (G \overset{g_1}{\leftarrow} Q \overset{g_2}{\rightarrow} G'), g' = (G \overset{g_1'}{\leftarrow} Q' \overset{g_2'}{\rightarrow} G')$. Since the direct sum of motives is induced by the coproduct in $\mathbb{G}$, we have that the image of $g \oplus g'$ through the functor $\phi(-, f)$ is

$$\phi(g \oplus g', f) = (G \times H \overset{(g_1,g_1')\times f_1}{\longleftarrow} (Q \sqcup Q') \times P \overset{(g_1,g_1')\times f_2}{\longrightarrow} G' \times H')$$

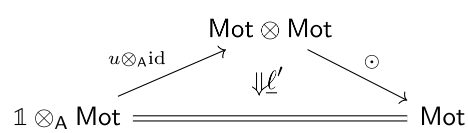
which is, since products distribute over coproducts,

$$(G \times H \overset{(g_1\times f_1, g_1'\times f_1)}{\longleftarrow} Q \times P \sqcup Q' \times P \overset{(g_1\times f_2, g_1'\times f_2)}{\longrightarrow} G' \times H') = \phi(g,f) \oplus \phi(g',f).$$

Let us now consider the associator and the unitors, which we claim to define enriched pseudonatural transformation. That is, the structure of pseudonatural transformation of, let's say $\underline{\ell}$



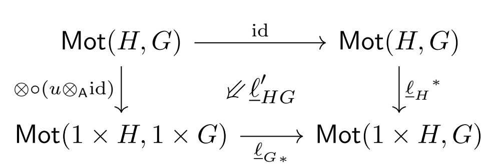
also defines a structure of enriched pseudonatural transformation



where here $\mathbb{1}$ is the unit Add-bicategory, having one object and hom-category $\mathbb{Z}^{\oplus}$. On each 1-cell component $\underline{\ell}'_G\colon \mathbb{Z}^{\oplus} \longrightarrow \mathsf{Mot}(1 \times G, G)$ is defined by mapping the n-tuple object to the direct sum $\bigoplus_n \underline{\ell}_G$. So defined, this is the image of $\underline{\ell}_G$ via the canonical additive functor

$$\mathsf{Mot}(1 \times G, G) \longrightarrow \mathrm{Add}(\mathbb{1}, \mathsf{Mot}(1 \times G, G)) \tag{3.17}$$

This makes $\underline{\ell}'_G$ evidently additive. For what concerns the higher structure $\underline{\ell}'_u$, for $u\colon H \to G$, define the pseudonatural transformation $\ell'_{H,G}$



by letting $(\underline{\ell}'_{HG})_u = \underline{\ell}_u$. Since no other requirement holds for morphisms of additive functors, this is automatically a cell in Add-Bicat. Similarly, it works for right unitor and associator. Then, enriched pseudonatural transformation axioms easily follow from those of natural transformation. For modifications, one can consider again the canonical additive functor 3.17, for the appropriate F and G between which are defined the pseudonatural transformations consisting of domain and codomain of π and μ, and define each component of the enriched modification to be the image through this additive functor (at the level of 1-cells) of the component of the non-enriched one. The modification axiom easily follow, as well as the non-abelian 4-cocycle condition for $\underline{\pi}$. $\square$

3.6 Green pseudofunctors

In this section we deal with the case where a monoidal structure is given on the target $\mathcal{A}$ of a Mackey pseudofunctor for $(\mathbb{G}, \mathbb{J})$. An enhancement of the notion of a Mackey pseudofunctor $\mathcal{F}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ in this context arise then naturally by requiring a monoid structure in the bicategory $\mathsf{PsFun}(\mathbb{G}^{\mathsf{op}}, \mathcal{A})$ which is compatible with the adjunctions. In order to properly express this compatibility, we need to distinguish between two possible ways of considering *pairings* of pseudofunctors. All these notions are to be found in greater detail at [Del22]; the new results are Theorem 3.6.7 and its Corollary 3.6.8, which use the machinery of the Day convolution in order to furnish a conceptually clearer definition of a Green pseudofunctor.

3.6.1 Internal and external pairings

Definition 3.6.1. Let $\mathcal{M}, \mathcal{N}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ be two pseudofunctors valued in a monoidal bicategory $\mathcal{A}$. Then we set

$$\mathcal{M} \boxtimes \mathcal{N}\colon \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} \xrightarrow{\mathcal{M}\times\mathcal{N}} \mathcal{A} \times \mathcal{A} \xrightarrow{\otimes} \mathcal{A},$$
$$\mathcal{M} \otimes \mathcal{N}\colon \mathbb{G}^{\mathsf{op}} \xrightarrow{\Delta} \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} \xrightarrow{\mathcal{M}\boxtimes\mathcal{N}} \mathcal{A}$$

where Δ denotes the diagonal 2-functor. These operations on pseudofunctors are respectively called external and internal tensor product.

Proposition 3.6.2. *Let $\mathcal{A}$ be a braided monoidal bicategory. The mapping $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \otimes \mathcal{N}$ together with the pointwise braiding, define a braided monoidal structure on the bicategory $\mathsf{PsFun}(\mathbb{G}^{\mathsf{op}}, \mathcal{A})$, where all structural equivalences are defined diagonally.*

Proof. It is easy to check that the monoidal structure of $\mathcal{A}$ induce a monoidal structure on the pseudofunctor category, by defining pointwise associators and unitors. The same is true for the braided structure. Axioms of braided monoidal bicategory directly descend from those for $\mathcal{A}$. $\square$

Definition 3.6.3. Let $\mathcal{M}, \mathcal{N}, \mathcal{L}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ be three pseudofunctors. An *internal pairing* is a pseudonatural transformation of the form

$$\odot\colon \mathcal{M} \otimes \mathcal{N} \longrightarrow \mathcal{L}.$$

An *external pairing* is a pseudonatural transformation of the form

$$\boxdot\colon \mathcal{M} \boxtimes \mathcal{N} \longrightarrow \mathcal{L}(- \times -).$$

These two concepts are in fact the same in virtue of the adjunction

$$\mathcal{C} \times \mathcal{C} \underset{\Delta}{\overset{\times}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}.$$

Namely, we have

Proposition 3.6.4. *For any triple of pseudofunctors* $\mathcal{M}, \mathcal{N}, \mathcal{L} \colon \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ *there's a canonical equivalence of categories*

$$\mathsf{PsNat}(\mathcal{M} \otimes \mathcal{N}, \mathcal{L}) \simeq \mathsf{PsNat}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{L}(- \times -)).$$

Proof. This is Proposition 4.10 in [Del19]. □

We are going to use the following definition of Green pseudofunctor.

Definition 3.6.5. An $\mathcal{A}$-valued Green pseudofunctor $\mathcal{M}$ for a cartesian pair $(\mathbb{G}, \mathbb{J})$ is a pseudomonoid object

$$\odot \colon \mathcal{M} \otimes \mathcal{M} \longrightarrow \mathcal{M}$$

in the braided monoidal bicategory $\mathsf{PsFun}(\mathbb{G}^{\mathsf{op}}, \mathcal{A})$ such that the underlying monoid object is a Mackey pseudofunctor for $(\mathbb{G}, \mathbb{J})$, and such that the corresponding (via 3.6.4) exterior product $\mathcal{M} \boxtimes \mathcal{M} \to \mathcal{M}(- \times -)$ is a morphism of quasi-Mackey pseudofunctors in each variable.

Remark 3.6.6. The exterior product $\mathcal{M} \boxtimes \mathcal{N}$ of Mackey pseudofunctors $\mathcal{M}, \mathcal{N} \colon \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ is *not* a Mackey pseudofunctor for $(\mathbb{G} \times \mathbb{G}, \mathbb{J} \times \mathbb{J})$. The problem clearly arise in additivity and for the basic reason that $(a+b)(c+d) \neq ac + bd$, since

$$\begin{aligned}
\mathcal{M} \boxtimes \mathcal{N}((H_1, G_1) \sqcup (H_2, G_2)) &= \mathcal{M} \boxtimes \mathcal{N}(H_1 \sqcup H_2, G_1 \sqcup G_2) \\
&= \mathcal{M}(H_1 \sqcup H_2) \otimes \mathcal{N}(G_1 \sqcup G_2) \\
&\cong (\mathcal{M}(H_1) \oplus \mathcal{M}(H_2)) \otimes (\mathcal{N}(G_1) \oplus \mathcal{N}(G_2))
\end{aligned}$$

is generally different from

$$(\mathcal{M}(H_1) \otimes \mathcal{N}(G_1)) \oplus (\mathcal{M}(H_2) \otimes \mathcal{N}(G_2)) = \mathcal{M} \boxtimes \mathcal{N}(H_1, G_1) \oplus \mathcal{M} \boxtimes \mathcal{N}(H_2, G_2).$$

However, if we just suppose $\mathcal{M}$ and $\mathcal{N}$ to be quasi-Mackey, then

- The exterior product $\mathcal{M} \boxtimes \mathcal{N}$ is quasi-Mackey. On can easily check indeed that there are, for (i, j) in $\mathbb{J} \times \mathbb{J}$, adjunctions

$$i_! \otimes j_! \dashv i^* \otimes j^* \dashv i_* \otimes j_*$$

  as well as that unit and counit are given by tensoring the unit and counit for each adjunction of i and j separately, and hence every pseudopullback (γ_1, γ_2) in $\mathbb{G} \times \mathbb{G}$ will have isomorphic mates given by tensoring isomorphic mates for γ_1 and γ_2.

- As a consequence, since the diagonal pseudofunctor $\Delta \colon \mathbb{G}^{\mathsf{op}} \to \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}}$ evidently maps $\mathbb{J}^{\mathsf{op}}$ in $\mathbb{J}^{\mathsf{op}} \times \mathbb{J}^{\mathsf{op}}$, the internal product $\mathcal{M} \otimes \mathcal{N}$ also happens to be quasi-Mackey, for $(\mathbb{G}, \mathbb{J})$.

- The pseudofunctor $\mathcal{M}(- \times -)\colon \mathbb{G}^{\mathsf{op}} \times \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ is quasi-Mackey for $(\mathbb{G} \times \mathbb{G}, \mathbb{J} \times \mathbb{J})$, since $\mathbb{J}$ is supposed to be closed under products.

For clarifying notations, please observe that what in [BD20] is called a $\mathbb{J}$-bimorphism is the same thing as a morphism of quasi-Mackey pseudofunctors for $(\mathbb{G} \times \mathbb{G}, \mathbb{J} \times \mathbb{J})$. In even other terms, this is also a $(\mathbb{J} \times \mathbb{J})_!$-pseudonatural transformation

The aim is now to prove that for Mackey pseudofunctors $\mathcal{M}, \mathcal{N}, \mathcal{L}$, to have a morphism of quasi-Mackey $t\colon \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{L}(- \times -)$ is necessary and sufficient to extend it to a pseudonatural transformation between the extended pseudofunctors on motives $\widehat{\mathcal{M}} \boxtimes \widehat{\mathcal{N}} \to \widehat{\mathcal{L}}(- \otimes -)$.

Theorem 3.6.7. *Let $\mathcal{M}, \mathcal{N}, \mathcal{L}\colon \mathbb{G}^{\mathsf{op}} \to \mathcal{A}$ be Mackey pseudofunctors and $t\colon \mathcal{M} \boxtimes \mathcal{N} \to \mathcal{L}(- \times -)$ a morphism of quasi-Mackey (Definition 3.2.9). Then t extends to a pseudonatural transformation*

$$\widehat{\mathcal{M}} \boxtimes \widehat{\mathcal{N}} \xrightarrow{\hat{t}} \widehat{\mathcal{L}}(- \otimes -)$$

of pseudofunctors $\mathsf{Mot} \times \mathsf{Mot} \to \mathcal{A}$. Conversely, if $\mathcal{F}, \mathcal{G}, \mathcal{H}\colon \mathsf{Mot} \to \mathcal{A}$ are three pseudofunctors, then any pseudonatural transformation $\mathcal{F} \boxtimes \mathcal{G} \to \mathcal{H}(- \otimes -)$ induces a morphism of quasi-Mackey pseudofunctors via the restriction along $\mathsf{mot} \times \mathsf{mot}$.

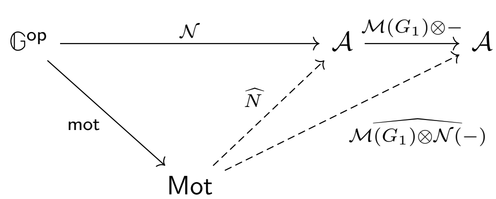
Proof. From the morphisms t of quasi-Mackey pseudofunctors for $\mathbb{G} \times \mathbb{G}$, we get for every G_1 and G_2 in $\mathbb{G}$ two pseudonatural transformation of Mackey pseudofunctors

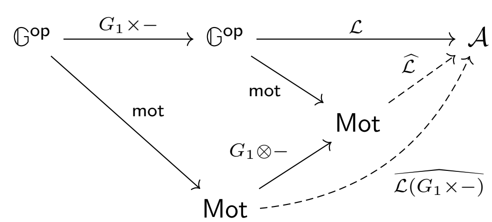
$$\mathcal{M}(G_1) \otimes \mathcal{N}(-) \xrightarrow{t_1} \mathcal{L}(G_1 \times -)$$
$$\mathcal{M}(-) \otimes \mathcal{N}(G_2) \xrightarrow{t_1} \mathcal{L}(- \times G_2)$$

Thus, by the universal property of Mackey motives, these extend bijectively to a pair of pseudonatural transformations

$$\mathcal{M}(G_1) \otimes \widehat{\mathcal{N}}(-) \cong \widehat{M(G_1) \otimes \mathcal{N}(-)} \xrightarrow{\hat{t}_1} \widehat{\mathcal{L}(G_1 \times -)} \cong \widehat{\mathcal{L}}(G_1 \otimes -)$$
$$\widehat{\mathcal{M}}(-) \otimes \mathcal{N}(G_2) \cong \widehat{M(-) \otimes \mathcal{N}(G_2)} \xrightarrow{\hat{t}_2} \widehat{\mathcal{L}(- \times G_2)} \cong \widehat{\mathcal{L}}(- \otimes G_2)$$

Where the above isomorphisms easily follow from the essential uniqueness of the extension, as explained in the two diagrams below





Eventually, this data $(\widehat{t_1}, \widehat{t_2})$ is the same as a pseudonatural transformation

$$\widehat{t}\colon \widehat{\mathcal{M}} \boxtimes \widehat{\mathcal{N}} \to \widehat{L}(- \otimes -).$$

□

Then, we can conclude the following.

Corollary 3.6.8. *$\mathcal{M}$ is a Green pseudofunctor if and only if it is a pseudomonoid for the Day convolution product.*

Proof. From Theorem 3.6.7 we get that a Green pseudofunctor $\odot\colon \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ is the same as a pseudomonoid object $\mathcal{M}$ in $\mathsf{PsFun}(\mathbb{G}^{\mathsf{op}}, \mathcal{A})$ which is a Mackey pseudofunctor whose tensor structure induces a pseudonatural transformation

$$\boxdot\colon \hat{\mathcal{M}} \boxtimes \hat{\mathcal{M}} \to \hat{\mathcal{M}}(- \times -).$$

By Proposition 2.7.2, we get that this is the same as a pseudomonoid Mackey pseudofunctor $\mathcal{M}$ whose tensor structure induces a pseudonatural transformation $\hat{\mathcal{M}} \underset{\text{Day}}{\otimes} \hat{\mathcal{M}} \to \hat{\mathcal{M}}$. Now, is has to be shown that the rest of the pseudomonoid structure of $\mathcal{M}$ induces in this way a pseudomonoid structure on $\hat{\mathcal{M}}$. This is true since the correspondence of pseudonatural transformation described above is functorial. Precisely, there are equivalences of categories

$$\begin{aligned}\mathsf{PsNat}(\hat{\mathcal{M}} \underset{\text{Day}}{\otimes} \hat{\mathcal{M}}, \hat{\mathcal{M}}) &\simeq \mathsf{PsNat}(\hat{\mathcal{M}} \boxtimes \hat{\mathcal{M}}, \hat{\mathcal{M}}(- \times -)) \\ &\simeq q\mathsf{Mack}(\mathcal{M} \boxtimes \mathcal{M}, \mathcal{M}(- \times -)),\end{aligned}$$

and this injects faithfully into the whole category of pseudonatural transformations $\mathsf{PsNat}(\mathcal{M} \otimes \mathcal{M}, \mathcal{M}) \simeq \mathsf{PsNat}(\mathcal{M} \boxtimes \mathcal{M}, \mathcal{M}(- \times -))$ in which axioms for the structure of pseudomonoid hold true by hypothesis. □

Appendices

Appendix A

Pseudoadjunctions

There are various extents to which we can relax the notion of adjunction, which for ordinary categories is expressed in two equivalent ways, for functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

as the existence of set *isomorphisms*

$$\phi\colon \mathcal{D}(FC, D) \cong \mathcal{C}(C, GD) : \psi \tag{A.1}$$

natural in C, D, or as the existence of natural transformations

$$\begin{aligned} \eta &\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF \\ \varepsilon &\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}} \end{aligned} \tag{A.2}$$

satisfying the two *triangle identities*

$$\begin{array}{ccc} F & \xRightarrow{F\eta} & FGF \\ & \diagdown\!\!\diagdown \;/\!/ & \Downarrow \varepsilon F \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xRightarrow{\eta G} & GFG \\ & \diagdown\!\!\diagdown \;/\!/ & \Downarrow G\varepsilon \\ & & G \end{array}$$

Now, it's also useful to have in mind how these two notions coincide. If the isomorphisms of sets are given we can define

$$\eta_C = \phi(\mathrm{id}_{FC}), \qquad \varepsilon_D = \psi(\mathrm{id}_{GD}),$$

while if the two natural transformations η and ε are given, we have the isomorphism of sets given by

$$\begin{aligned} \mathcal{D}(FC, D) &\rightleftarrows \mathcal{C}(C, GD) \\ f &\longmapsto Gf \circ \eta_C \\ \varepsilon_D \circ Fg &\longleftarrow\!\shortmid\; g \end{aligned}$$

In the context of bicategories, there are many ways to generalize the notion of adjunction. At a first attempt one could simply consider the following definition.

Definition A.0.1. $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ a couple of pseudofunctors between bicategories is said to form a *biadjunction* if there exists for every C, D an equivalence of categories

$$\mathcal{D}(FC, D) \simeq \mathcal{C}(C, GD)$$

pseudonatural in C, D.

To give a biadjunction, sometimes together with the adjective *incoherent* is the same as to give, together with pseudonatural transformations η, ε, a pair of invertible modifications s and t, called *triangulators*, filling the triangular identities. The proof of this fact is a fairly straightforward generalization of the usual 1-categorical proof, in which modifications s and t serve as isomorphisms witnessing the equivalence $\mathcal{D}(FC, D) \simeq \mathcal{C}(C, GD)$.

A more *coherent* way of defining this notion is, then, to demand the equivalences to be adjoint equivalences. This translate in our context to a coherence condition that triangulators must satisfy. These conditions are called *swallowtail* equation (see below). This is the notion to which we reserve the name *pseudoadjunction*. It is interesting to observe that at this point one could also require, generalizing the notion in an orthogonal direction, the adjoint equivalence to be just an adjunction. This notion was first developed at [BP88], and many authors call it *lax 2-adjunction*. The translation in terms of triangulators then does not change, except for the fact that the two triangulators satisfying the swallowtail equation are no longer required to be invertible.

Therefore, we directly use the internal notion of pseudoadjunction, defined as follows. This bicategorical definition immediately adapts to the enriched context.

Remark A.0.2. We are working in the tricategory $\mathcal{V}$-Bicat (see [GS15], Section 4 for a detailed description of its structure of tricategory). Thanks to strictification results recalled in Section 1.5, we are allowed to assume this tricategory to be a *Gray category*, which is the multiple object version of a Gray monoid (or semi-strict monoidal 2-category). See [GPS95] for much more on this definition. That means, concretely, a tricategory in which we assume composition and unitality to be strict (like in a strict 3-category), but for every two pairs of composable arrows (H, K), (H', K') and 2-cells $\alpha\colon H \to H'$ and $\beta\colon K \to K'$, *interchangers* are given. Namely, structural isomorphisms

$$\begin{array}{ccc} H \circ K & \xrightarrow{H\beta} & H \circ K' \\ {\scriptstyle \alpha K}\downarrow & \swarrow \Sigma_{\alpha,\beta} & \downarrow{\scriptstyle \alpha K'} \\ H' \circ K & \xrightarrow[H'\beta]{} & H' \circ K' \end{array}$$

satisfying the following axioms.

(i) If $\alpha = \mathrm{id}\colon H \to H$, then $\Sigma_{\alpha,\beta} = \mathrm{id}_{H\beta}$, as well as if $\beta = \mathrm{id}\colon K \to K$, then $\Sigma_{\alpha,\beta} = \mathrm{id}_{\alpha K}$.

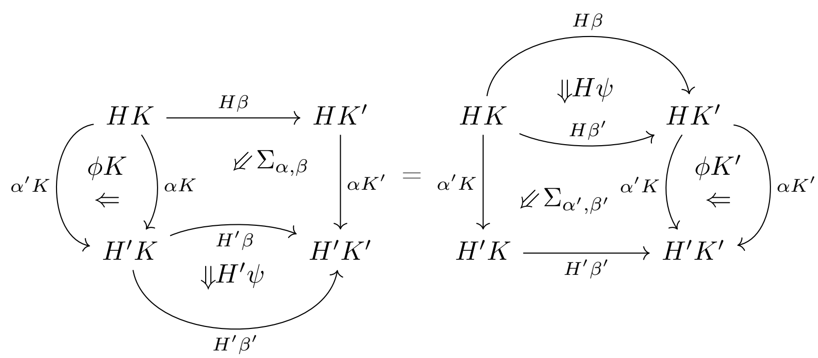
(ii) For all $\alpha\colon H \to H'$, $\beta\colon K \to K'$, $\gamma\colon L \to L'$, it holds

$$\begin{array}{ccc} HKL & \xrightarrow{HK\gamma} & HKL' \\ {\scriptstyle H\beta L}\downarrow & \swarrow H\Sigma_{\beta,\gamma} & \downarrow{\scriptstyle H\beta L'} \\ HK'L & \xrightarrow[HK'\gamma]{} & HK'L' \end{array} = \begin{array}{ccc} HKL & \xrightarrow{HK\gamma} & HKL' \\ {\scriptstyle H\beta L}\downarrow & \swarrow \Sigma_{H\beta,\gamma} & \downarrow{\scriptstyle H\beta L'} \\ HK'L & \xrightarrow[HK'\gamma]{} & HK'L' \end{array}$$

$$\begin{array}{ccc} HKL & \xrightarrow{HK\gamma} & HKL' \\ {\scriptstyle \alpha KL}\downarrow & \Swarrow \Sigma_{\alpha,K\gamma} & \downarrow{\scriptstyle \alpha KL'} \\ H'KL & \xrightarrow[H'K\gamma]{} & H'KL' \end{array} = \begin{array}{ccc} HKL & \xrightarrow{HK\gamma} & HKL' \\ {\scriptstyle \alpha KL}\downarrow & \Swarrow \Sigma_{\alpha K,\gamma} & \downarrow{\scriptstyle \alpha KL'} \\ H'KL & \xrightarrow[H'K\gamma]{} & H'KL' \end{array}$$

$$\begin{array}{ccc} HKL & \xrightarrow{H\beta L} & HK'L \\ {\scriptstyle \alpha KL}\downarrow & \Swarrow \Sigma_{\alpha,\beta}L & \downarrow{\scriptstyle \alpha K'L} \\ H'KL & \xrightarrow[H'\beta L]{} & H'K'L \end{array} = \begin{array}{ccc} HKL & \xrightarrow{H\beta L} & HK'L \\ {\scriptstyle \alpha KL}\downarrow & \Swarrow \Sigma_{\alpha,\beta L} & \downarrow{\scriptstyle \alpha K'L} \\ H'KL & \xrightarrow[H'\beta L]{} & H'K'L \end{array}$$

(*iii*) For all $\phi\colon \alpha \Rightarrow \alpha'$ and $\psi\colon \beta \Rightarrow \beta'$, it holds



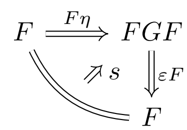
(*iv*) For every composable $H \xrightarrow{\alpha} H' \xrightarrow{\alpha'} H''$ and $K \xrightarrow{\beta} K' \xrightarrow{\beta'} K''$ it holds

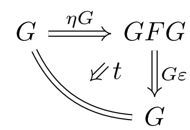
$$\begin{array}{ccccc} HK & \xrightarrow{H\beta} & HK' & \xrightarrow{H\beta'} & HK'' \\ {\scriptstyle \alpha K}\downarrow & \Swarrow \Sigma_{\alpha,\beta} & \downarrow{\scriptstyle \alpha K'} & \Swarrow \Sigma_{\alpha,\beta'} & \downarrow{\scriptstyle \alpha K''} \\ H'K & \xrightarrow{H'\beta} & H'K' & \xrightarrow{H'\beta'} & H'K'' \\ {\scriptstyle \alpha' K}\downarrow & \Swarrow \Sigma_{\alpha',\beta} & \downarrow{\scriptstyle \alpha' K'} & \Swarrow \Sigma_{\alpha',\beta'} & \downarrow{\scriptstyle \alpha' K''} \\ H''K & \xrightarrow[H''\beta]{} & H''K' & \xrightarrow[H''\beta']{} & H''K'' \end{array} = \begin{array}{ccc} HK & \xrightarrow{H(\beta'\beta)} & HK'' \\ {\scriptstyle (\alpha'\alpha)K}\downarrow & \Swarrow \Sigma_{\alpha'\alpha,\beta'\beta} & \downarrow{\scriptstyle (\alpha\alpha)K''} \\ H''K & \xrightarrow[H''(\beta'\beta)]{} & H''K'' \end{array}$$

The rest of this section will highlight, because of our interests, the case of $\mathcal{V}$-Bicat, but the proofs will hold true for general Gray-categories.

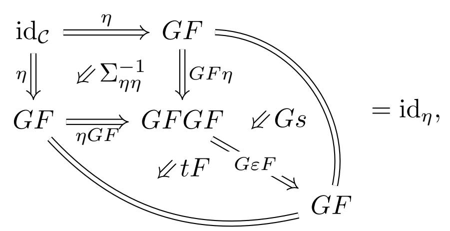
Definition A.0.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be $\mathcal{V}$-pseudofunctors between $\mathcal{V}$-bicategories. We say that they form a *pseudoadjunction*, and that F is left pseudoadjoint to G, if the following data are given:

- pseudonatural transformations, called *unit* and *counit* $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, $\varepsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$
- invertible modifications, called *triangulators*

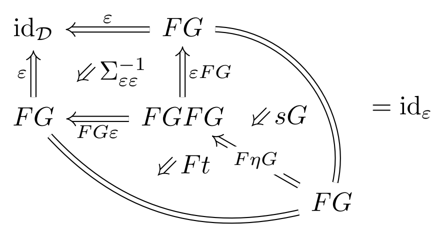




and satisfies the following two axioms, the *swallowtail* equations:



(A.3)



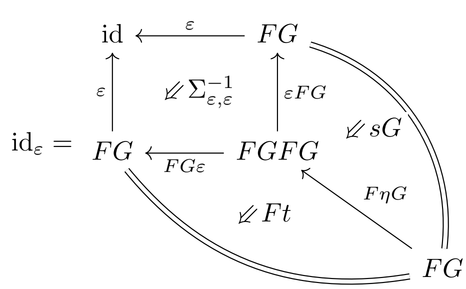
(A.4)

Remark A.0.4. Definition A.0.3 also works for a Gray category, by replacing the words pseudofunctors, pseudonatural transformations and modification with 1-cells, 2-cells and 3-cells, respectively.

Remark A.0.5. Like any equivalence of categories can always be improved to an adjoint equivalence, we get that any biadjunction (Definition A.0.1) can always be improved to a pseudoadjunction, by a specific choice of the triangulators. This result was first proved in [Gur11a] for adjoint equivalences, and then in [Pst14] for general biadjunctions with a different argument, but also in a slightly different context than ours. The following adaptation of Gurski's argument is a tricategorical version of the theorem working in particular in this setting of $\mathcal{V}$-bicategories, once we use the strictification result for tricategories.

Theorem A.0.6 (Strictification Theorem for pseudoadjunctions)**.** *Let $F\colon \mathcal{C} \to \mathcal{D}, G\colon \mathcal{D} \to \mathcal{C}$ be $\mathcal{V}$-pseudofunctors, or, more generally, 1-morphisms of a Gray-category, together with ε, η, s, t the data of a pseudoadjunction, but suppose it only satisfies one of the swallowtail equations. Then, it also satisfy the other one.*

Proof. Suppose that the data satisfy the identity



(A.5)

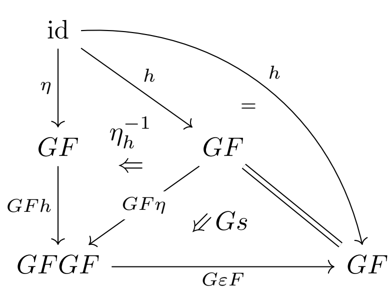
(The other implication is obviously similar.) Then, let us first observe that the data of a pseudoadjunction suffices to have an equivalences of categories

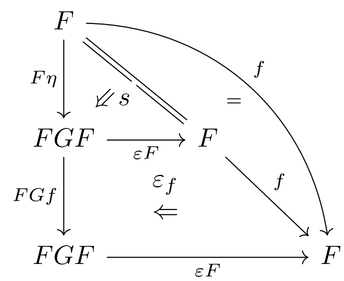
$$\mathcal{V}\text{-}\mathsf{PsNat}(F,F) \longrightarrow \mathcal{V}\text{-}\mathsf{PsNat}(\mathrm{id},GF) \tag{A.6}$$
$$f \longmapsto Gf \circ \eta \tag{A.7}$$

and

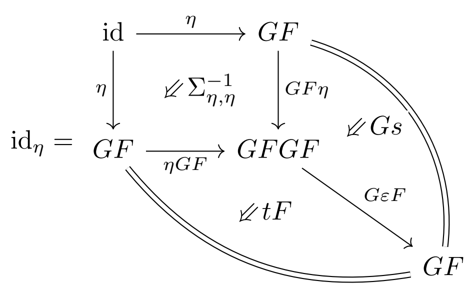
$$\mathcal{V}\text{-}\mathsf{PsNat}(G,G) \longrightarrow \mathcal{V}\text{-}\mathsf{PsNat}(FG,\mathrm{id}) \tag{A.8}$$
$$g \longmapsto \varepsilon \circ Fg. \tag{A.9}$$

In fact, the two have explicit inverses given by mapping $h \mapsto \varepsilon F \circ Fh$ and $k \mapsto Gk \circ \eta G$, and the triangulators are the isomorphisms providing the equivalences: we have indeed the two isomorphisms

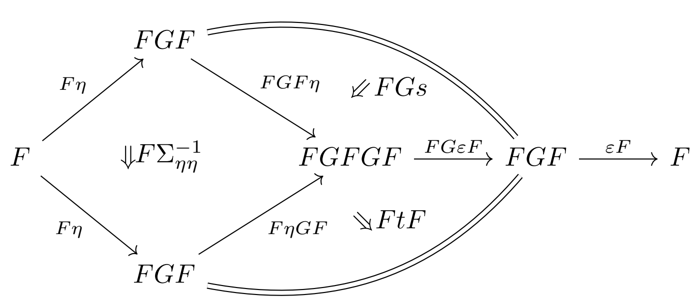




for (A.6), and similarly with t for (A.8). Therefore, in order to prove the second equation

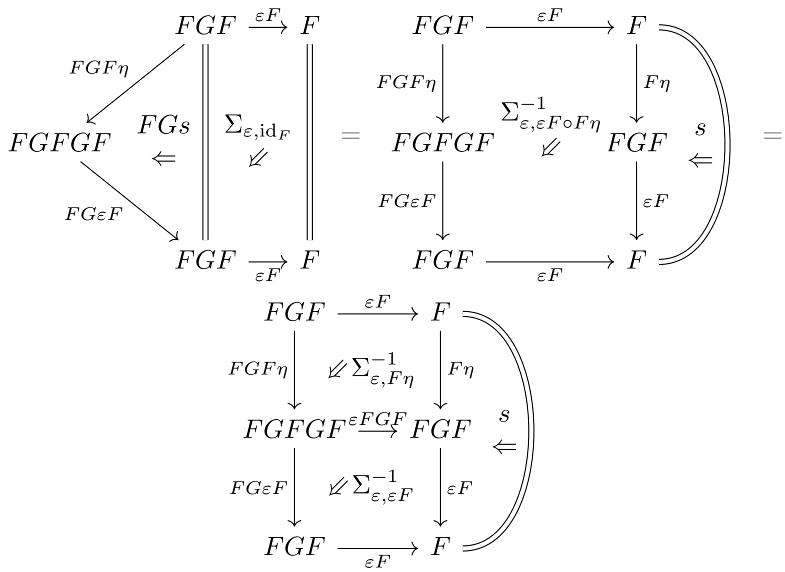


it suffices to prove the equality after the application of the equivalence $\varepsilon * F(-)$. That is, let us start with

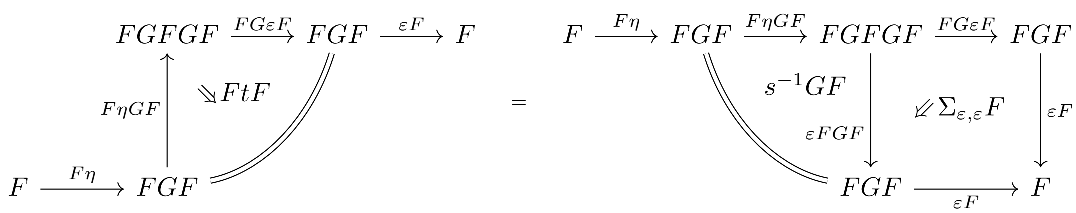


(A.10)

The 2-cell $\varepsilon * FGs$ in the top part of the above (A.10) can now be rewritten, using axioms (i), (iii) and (iv) for the interchanger, as

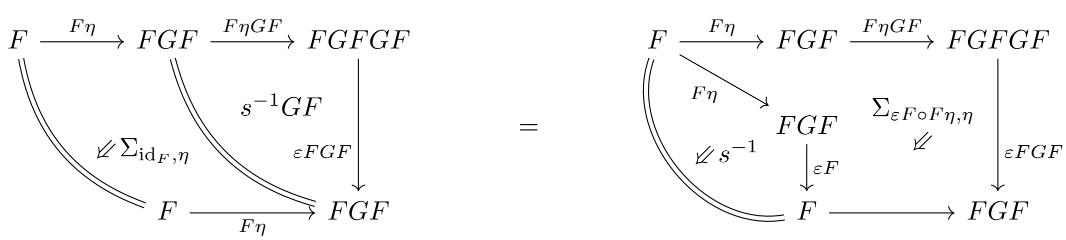


On the other hand, we see that we can rewrite $\varepsilon F * FtF * F\eta$ in (A.10) by first using the swallowtail equation that we are assuming (A.5), which gives

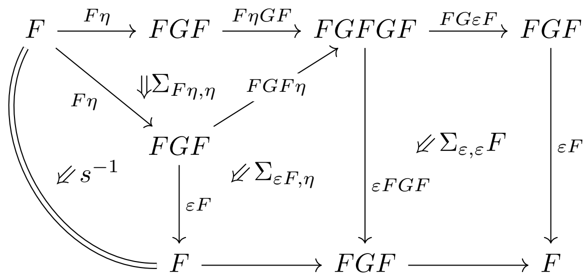


(A.11)

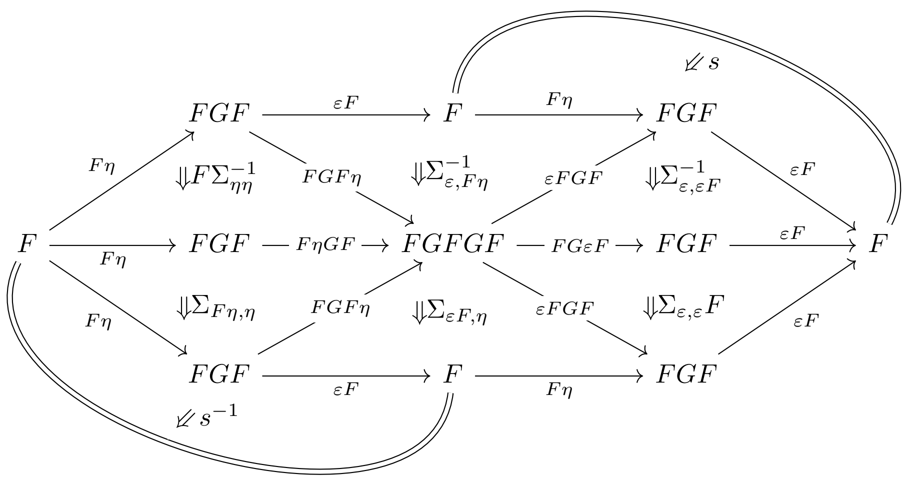
Then, we can consider again axioms (i) and (iii), which allows to rewrite the left part of the latter as



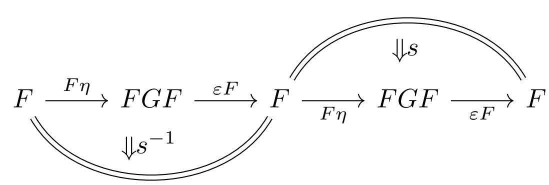
which in turn, thanks to axioms (iv), allows to rewrite (A.11) as



Then, let us reassemble the two computations. The term (A.10) becomes



Axiom (ii), in all of its parts, allows us to recognize that $F\Sigma_{\eta\eta} = \Sigma_{F\eta,\eta}$, $\Sigma_{\varepsilon F,\eta} = \Sigma_{\varepsilon,F\eta}$ and $\Sigma_{\varepsilon,\varepsilon}F = \Sigma_{\varepsilon,\varepsilon F}$. Therefore we can simplify all the interchangers and the above becomes



which eventually is the identity $\mathrm{id}_{\varepsilon F\circ F\eta}$. The last claim follows by the general fact that in a monoidal category, in our case the endomorphism category of F under (horizontal) composition, if $\sigma\colon \mathbb{1} \to A$ is an isomorphism, then one has that the morphisms $A \otimes A \to A \otimes \mathbb{1} \cong A$ and $A \otimes A \to \mathbb{1} \otimes A \cong A$ are equal. This follows from the commutativity of

$$\begin{array}{ccccc} A \otimes A & \xrightarrow{\sigma\otimes\sigma} & \mathbb{1}\otimes\mathbb{1} & \xrightarrow{\sigma^{-1}\otimes\sigma^{-1}} & A\otimes A \\ {\scriptstyle A\otimes\sigma}\downarrow & & \| & & \downarrow{\scriptstyle \sigma\otimes A} \\ A\otimes\mathbb{1} & \xrightarrow{\sigma\otimes\mathbb{1}} & \mathbb{1}\otimes\mathbb{1} & \xrightarrow[\mathbb{1}\otimes\sigma^{-1}]{} & \mathbb{1}\otimes A \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho=\lambda} & & \downarrow{\scriptstyle \lambda} \\ A & \xrightarrow[\sigma]{} & \mathbb{1} & \xrightarrow[\sigma^{-1}]{} & A \end{array}$$

given by naturality of the unitors and the exchange law for the tensor functor. □

A.1 Parametric pseudoadjunctions

In this section we are going to introduce the notion of a *parametric* family of pseudoadjunctions (most notably, this will be the case of a closed monoidal bicategory), and we will see how this is naturally linked to the notion of extra-pseudonaturality. Moreover, we are going to argue via the Yoneda lemma how the family of pseudoadjunction given for a closed monoidal bicategory can indeed be turned in a unique way into a parametric one.

Definition A.1.1. A *parametric family of pseudoadjunctions* is a pair of pseudofunctors $F\colon \mathcal{E}\times\mathcal{D}\to\mathcal{C}$ and $G\colon \mathcal{E}^{\mathsf{op}}\times\mathcal{C}\to\mathcal{D}$ together with, for every e in $\mathcal{E}$ a pseudoadjunction $F(e,-)\dashv G(e,-)$, and the resulting equivalences are part of a pseudonatural transformation:

$$\phi\colon \mathcal{C}(F(-,d),c)\Rightarrow\mathcal{D}(d,G(-,c))$$

for every c in $\mathcal{C}$, d in $\mathcal{D}$.

Theorem A.1.2. *If F,G is a parametric family of pseudoadjunctions, then units and counits define extra-pseudonatural transformations*

$$\eta_d\colon d \ddot{\Rightarrow} G(-,F(-,d))$$

and

$$\varepsilon_c\colon F(-,G(-,c)) \ddot{\Rightarrow} c$$

Proof. From the natural isomorphism

$$\begin{array}{ccc} \mathcal{C}(F(e,d),c) & \xrightarrow{\phi_e} & \mathcal{D}(c,G(e,c)) \\ {\scriptstyle \mathcal{C}(F(f,d),c)}\downarrow & \swarrow\phi_f & \downarrow{\scriptstyle \mathcal{D}(c,G(f,c))} \\ \mathcal{C}(F(e',d),c) & \xrightarrow[\phi_{e'}]{} & \mathcal{D}(c,G(e',c)) \end{array}$$

given by the parametric family, and since $\phi\colon g\mapsto R(c,g)\circ\eta^c_d$, we get for every $g\colon F(e,d)\to c$ an isomorphism

$$(\phi_f)_g\colon G(f,e)\circ G(e,g)\circ\eta^c_d \overset{\sim}{\Rrightarrow} G(e',g\circ F(f,d))\circ\eta^{c'}_d.$$

Now if we take $c=F(e,d)$ and $g=\mathrm{id}$, we specialize the isomorphism above to a pseudonatural transformation

$$\begin{array}{ccc}
d & \xrightarrow{\eta_d^c} & G(c, F(c,d)) \\
{\scriptstyle \eta_d^{c'}}\big\downarrow & \swarrow j_f & \big\downarrow{\scriptstyle G(f, F(c,d))} \\
G(c', F(c', d)) & \xrightarrow[G(c', F(f,d))]{} & G(c', F(c,d))
\end{array}$$

which is indeed the data of an extra-pseudonatural transformation $\eta_d \colon d \ddot{\Rightarrow} G(-, F(-,d))$. Then, unitality, functoriality and naturality axioms for j are easily seen to be true from the correspondent axioms on the pseudonatural transformation ϕ. The proof for ε follows an analogous pattern. □

Remark A.1.3. In a closed monoidal bicategory $\mathcal{V}$, the class of adjunctions $\{- \otimes a \dashv [a, -]\}_a$ can always be improved to a parametric family. This depends on how one defines $[-,-]$ as a pseudofunctor in its first variable (the parametric one). It suffices to observe that if $f \colon a \to a'$, one can consider the morphism imposing the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{V}(b \otimes a', c) & \xrightarrow{\simeq} & \mathcal{V}(b, [a', c]) \\
{\scriptstyle \mathcal{V}(b \otimes f, c)}\big\downarrow & & \vdots \\
\mathcal{V}(b \otimes a, c) & \xrightarrow{\simeq} & \mathcal{V}(b, [a, c])
\end{array}$$

and then define $[f, c] \colon [a', c] \to [a, c]$ to be the map detected by the dashed arrow via the Yoneda lemma.

Appendix B

Additivity for bicategories

Definition B.0.1. A category $\mathcal{C}$ is said to be *semi-additive* if admits finite coproducts, finite products and the canonical morphism

$$C \sqcup D \longrightarrow C \times D$$

is an isomorphism. The term used to refer to both is *biproduct*. A morphism of semi-additive categories is a functor that preserves products. This data, together with natural transformations between these functors, defines a 2-category SAdd.

Remark B.0.2. As proved in [Lac12], it actually suffices for the product and coproduct to be isomorphic via any other map, in order to have a semi-additive category.

Definition B.0.3. A category $\mathcal{C}$ is said to be *additive* if it admits finite products and it's enriched over the category Ab of abelian groups. An *additive* functor is just an Ab-functor. Together with enriched natural transformations, this form a 2-category Add.

Remark B.0.4. It's known that on a semi-additive category we have a natural structure of category enriched in the category of abelian monoids. The monoid structure of each hom-set $\mathcal{C}(X, Y)$ is given setting $f + g$ to be

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

On the other hand, the group structure on hom-sets of any semi-additive category is necessarily equal to this one.

Remark B.0.5. (i) An additive category has coproducts, and they are isomorphic to products, hence it is semi-additive.

(ii) Any additive functor preserves products, and any product preserving functor between additive categories is additive. This is a consequence of Remark B.0.4.

(iii) Eventually, an Ab-natural transformation gives clearly rise to a non-enriched one. This is quite remarkable, and that is the reason why we allows ourselves to call *additive* also the morphisms of semi-additive categories.

This gives a (1- and 2-)fully faithful 2-functor

$$\text{Add} \hookrightarrow \mathsf{SAdd}. \tag{B.1}$$

Definition B.0.6. If $\mathcal{C}$ is a semi-additive category, one can consider the additive category $\mathcal{C}^+$ given by the same objects as those of $\mathcal{C}$, and by setting $\mathcal{C}^+(X,Y) = \mathcal{C}(X,Y)^+$ to be the abelian group completion of the abelian monoid $\mathcal{C}(X,Y)$ (Remark B.0.4). This $(-)^+$ construction provide a section for the bireflexive inclusion B.1.

Now, some constructions are presented for bicategories, in order to fix the terminology used throughout this work.

Definition B.0.7. A bicategory $\mathcal{B}$ is said to be *locally (semi-)additive* if its hom-categories are (semi-)additive and the composition functor is additive (preserves products). A morphism of locally semi-additive bicategories is a pseudofunctor which is additive (preserves products) on each hom-categories, *i.e.* a *locally additive* pseudofunctor. Locally additive pseudofunctors between locally (semi-)additive bicategories $\mathcal{C}$ and $\mathcal{D}$ assemble into a bicategory $\mathsf{PsFun}_{\oplus}(\mathcal{C},\mathcal{D})$.

Definition B.0.8. Given a locally semi-additive category $\mathcal{B}$, its *locally additive completion* is the bicategory $\mathcal{B}^+$ having the same objects as $\mathcal{B}$, and hom-categories $\mathcal{B}^+(X,Y) = \mathcal{B}(X,Y)^+$.

Remark B.0.9. It is straightforward to prove that if $\mathcal{B}$ is a locally semi-additive bicategory, and $\mathcal{C}$ is locally additive, then the canonical comparison

$$\mathsf{PsFun}_+(\mathcal{B}^+,\mathcal{C}) \xrightarrow{\simeq} \mathsf{PsFun}_+(\mathcal{B},\mathcal{C})$$

between bicategories of locally additive pseudofunctors is a biequivalence.

Remark B.0.10. A locally additive bicategory $\mathcal{B}$ obviously has a natural structure of Add-bicategory.

Definition B.0.11. A bicategory $\mathcal{B}$ is said to be *additive* if it is locally additive (equivalently, if it is has a structure of Add-bicategory) and admits biproducts.

Definition B.0.12. A pseudofunctor $F\colon \mathcal{B} \to \mathcal{B}'$ between additive bicategories is additive if it preserves products and is locally additive.

A priori, one could have a pseudofunctor $F\colon \mathcal{B} \to \mathcal{C}$ between additive bicategories which is locally additive but not additive (since local additivity does not require to preserve products). However, this cannot happen. In other terms, additive bicategories injects 1-faithfully into locally (semi-)additive ones

Proposition B.0.13. *Any locally additive pseudofunctor $F\colon \mathcal{B} \to \mathcal{C}$ between additive bicategories is additive.*

Proof. One only need to prove that such a locally additive F preserves products. This follows by applying on each hom-category the fact that Ab-functors preserve direct sums. □

Definition B.0.14. A *zero object* in a bicategory $\mathcal{B}$ is an object 0 such that, for every X, there exist two equivalences of categories (natural in X)

$$\mathcal{B}(0,X) \simeq T \simeq \mathcal{B}(X,0),$$

where T denotes the terminal category with one object $*$ and one morphism.

Definition B.0.15. Let $\mathcal{B}$ be a bicategory with a zero object. Then we define the *initial* and *terminal morphism* for an object X (respectively i_X and t_X) as the images (unique up to isomorphism) of $*$ under the equivalences $T \simeq \mathcal{B}(0, X)$ and $T \simeq \mathcal{B}(X, 0)$ respectively.

Lemma B.0.16. *Given a category $\mathcal{C}$ equivalent to the terminal category T, there is for any pair of functors $F, G\colon T \to \mathcal{C}$ a unique natural transformation $F \Rightarrow G$.*

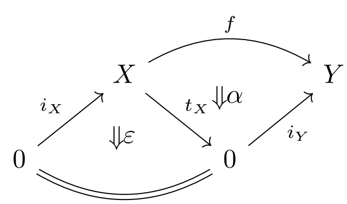
Proof. Call $E\colon \mathcal{C} \to T$ the supposed to exist equivalence. A natural transformation $\alpha\colon F \Rightarrow G$ consists of a unique arrow in $\mathcal{C}$ given by $\alpha_*\colon F* \to G*$, but there's just one such arrow since

$$\mathsf{Hom}_{\mathcal{C}}(F*, G*) \cong \mathsf{Hom}_T(EF*, EG*) = \mathsf{Hom}_T(*, *) = \{*\}.$$

□

Proposition B.0.17. *Let $\mathcal{B}$ be a bicategory with a zero object and suppose the initial and terminal morphisms to be two-sided adjoints. Then for any two objects X, Y in $\mathcal{B}$, the morphism $i_Y t_X\colon X \to Y$ is a zero object in the category $\mathcal{B}(X, Y)$.*

Proof. By the general theory of internal adjunctions applied to $t_X \dashv i_X$, we have for every $f\colon X \to Y$ a bijection $[f, i_Y t_X] \cong [f i_X, i_Y]$, provided by composing a 2-cell α with ε below



From Lemma B.0.16 we have that only one 2-cell exists between $f i_X$ and i_Y, being such given by a pseudonatural transformation between functors $T \to \mathcal{B}(0, Y)$, we conclude that $i_Y t_X$ is terminal. Analogously, we can consider the adjunction $i_Y \dashv t_Y$, providing a bijection $[i_Y t_X, f] \cong [t_X, t_Y f]$, and conclude that $i_Y t_X$ is also initial. □

Example B.0.18. It is *not* in general the case that the pointed object $i_Y t_X$ in $\mathcal{B}(X, Y)$ is a zero object. It suffices, indeed, to consider a locally discrete additive bicategory such as the bicategory of abelian groups Ab. Its hom-categories are $\mathrm{Ab}(H, G)$ in which the zero morphism $z\colon H \to G$ is not initial nor terminal, since no 2-morphism is given between z and any other $f\colon H \to G$.

Remark B.0.19. In a bicategory $\mathcal{B}$ with binary products, coproducts and a zero object, there's a canonical map

$$X_1 \sqcup X_2 \to X_1 \times X_2$$

induced by $(\mathrm{id}, 0_{X_1, X_2})\colon X_1 \to X_1 \times X_2$ and $(0_{X_2, X_1}, \mathrm{id})\colon X_2 \to X_1 \times X_2$.

Definition B.0.20. A bicategory $\mathcal{B}$ is said to have *biproducts* if it has a zero object, binary products and coproducts, and the canonical morphism $X_1 \sqcup X_2 \to X_1 \times X_2$ is an equivalence. In this case, we usually refer to both by $X_1 \oplus X_2$.

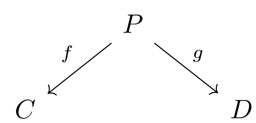
Remark B.0.21. A bicategory with biproducts needs not to be locally semi-additive and a locally semi-additive bicategory needs not to have biproducts. However, in a locally semi-additive bicategory every existing product is also a coproduct. That is an immediate consequence of the definition of product and coproduct and of the Yoneda lemma.

Appendix C

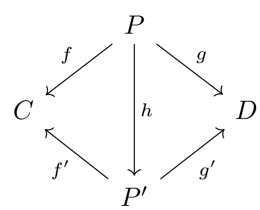
Spans of (bi)categories

In this section we recall the definitions of the span constructions for categories and bicategories.

Definition C.0.1. Let $\mathcal{C}$ be a category with pullbacks. Then its *span category* $\widehat{\mathcal{C}}$ is the category defined by having the same objects of $\mathcal{C}$ and as morphisms $C \to D$ equivalence classes of spans



of morphisms in $\mathcal{C}$. Two such spans $(f, g), (f', g')$ are considered equivalent whether there is a morphism h making



commute. Composition is formed by taking pullbacks. (see RemarkC.0.2 below). We denote

$$(-)_\star \colon \mathcal{C} \longrightarrow \hat{\mathcal{C}} \qquad \text{and} \qquad (-)^\star \colon \mathcal{C}^{\mathsf{op}} \longrightarrow \hat{\mathcal{C}}$$

the two functors mapping $f \mapsto f_\star = [\mathrm{id}, f]$ and $f \mapsto f^\star = [f, \mathrm{id}]$.

Remark C.0.2. The 1-category of spans defined above is the *truncation* $\tau\mathsf{Span}(\mathcal{C}, \mathcal{C})$ of the bicategory of spans in Definition 3.3.1. That means, we first consider $\mathcal{C}$ as a discrete (2,1)-category. Then, together with all of its morphism this defines a spannable pair (Definition 3.2.2, apart from having finite products and coproducts, which are not needed in the definition of the span bicategory), and we hence consider $\mathsf{Span}(\mathcal{C}, \mathcal{C})$. Eventually, we identify every two isomorphic pair of morphisms.

The ordinary category of spans enjoy the following properties.

Lemma C.0.3. *The span construction $\hat{\mathcal{C}}$ of a category with pullbacks $\mathcal{C}$ is such that*

- *Every pullback preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $\hat{F}\colon \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ commuting with $(-)^\star$ and $(-)_\star$, defined by $[f, g] \mapsto [Ff, Fg]$.*
- *Every natural transformation $\alpha\colon F \Rightarrow G$ of pullback preserving functors $\mathcal{C} \to \mathcal{D}$ induces a natural transformation $\hat{\alpha}\colon \hat{F} \Rightarrow \hat{G}$ defined to be $\hat{\alpha}_C = (\alpha_C)_\star$.*
- *commutes with products ($\widehat{\mathcal{C} \times \mathcal{D}} \simeq \hat{\mathcal{C}} \times \hat{\mathcal{D}}$).*

Proof. This is Lemma 1.5.4 in [BD20]. □

Then, we're able to see the requirements needed to perform the same span construction but locally, on a bicategory $\mathcal{B}$.

Definition C.0.4. Let $\mathcal{B}$ be a bicategory such that

a Each hom-category $\mathcal{B}(X, Y)$ has pullbacks.

b Composition functor $- \circ -$ preserves pullbacks in each variable.

The *local span bicategory* $\hat{\mathcal{B}}$ is the bicategory having objects the same of $\mathcal{B}$, and hom-categories $\hat{\mathcal{B}}(X, Y) = \widehat{\mathcal{B}(X, Y)}$. The composition functor is defined via Lemma C.0.3 thanks to condition (b). Explicitly, it is the unique functors making

$$\begin{array}{ccc}
\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) & \xrightarrow{\;\circ\;} & \mathcal{B}(X,Z) \\
{\scriptstyle (-)_\star \times (-)_\star}\big\downarrow & & \big\downarrow{\scriptstyle (-)_\star} \\
\widehat{\mathcal{B}}(Y,Z) \times \widehat{\mathcal{B}}(X,Y) & \overset{\hat{\circ}}{\dashrightarrow} & \widehat{\mathcal{B}}(X,Z) \\
{\scriptstyle (-)^\star \times (-)^\star}\big\uparrow & & \big\uparrow{\scriptstyle (-)^\star} \\
\mathcal{B}(Y,Z)^{\mathsf{op}} \times \mathcal{B}(X,Y)^{\mathsf{op}} & \xrightarrow[\circ^{\mathsf{op}}]{} & \mathcal{B}(X,Z)^{\mathsf{op}}
\end{array}$$

commute. Unitors and associator are the image of those of $\mathcal{B}$ via the canonical $(-)_\star\colon \mathcal{B} \to \hat{\mathcal{B}}$.

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