Hilbert Modules over C^*-categories

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Statement of originality

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Most of the original results in this thesis were first published in a preprint on the arXiv on the 23d May of 2023 (Hilbert modules over C^*-categories, version 1). The results in Section 2 of Chapter 4 were published in the second version of that preprint on the 27th of November of 2023 (Hilbert modules over C^*-categories, version 2).

The results in Appendix A are all collaborative with Benjamin Dünzinger.

Abstract

In this thesis, we investigate the theory of Hilbert modules over C^*-categories and use this theory to prove results on a variety of categories of C^*-categories and C^*-algebras.

In Chapter 1, we introduce the theory of C^*-categories and establish in particular the properties of the additive closure and the multiplier category of a given C^*-category.

In Chapter 2, we look at Hilbert modules over C^*-categories and bounded and compact operators between Hilbert modules. We prove a Yoneda lemma for C^*-categories, and an approximate projectivity property that relates Hilbert modules to free Hilbert modules. We also introduce right Hilbert bimodules, which admit an action from two C^*-categories.

In Chapter 3, we define the tensor product of a right Hilbert bimodule with a Hilbert module, and prove an Eilenberg-Watts theorem characterizing those functors of Hilbert module categories given by tensoring by a right Hilbert bimodule. We also prove a Morita theorem characterizing those bimodules that give equivalences of module categories upon tensoring.

In Chapter 4, we use the Eilenberg-Watts theorem to prove the equivalence of several bicategories and 2-categories of C^*-categories and C^*-algebras. We end by employing the Eilenberg-Watts theorem to exhibit a localization of a category of C^*-categories at the Morita equivalences.

In the Appendix, we look at the maximal tensor product of C^*-categories and prove a tensor-hom adjunction for non-degenerate functors of C^*-categories.

Contents

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Introduction

A pervasive theme in algebra is to consider some appropriately defined category of modules over a fixed type of object, and ask how we might see that two objects share equivalent categories of modules, i.e. that they are *Morita equivalent*. This philosophy of Morita equivalence gets its name from the work of Kiiti Morita who first solved this problem in the case of rings [Mor58]. We refer the reader to [Mey97] for characterizations of Morita equivalences in the cases of rings, C^*- and W^*-algebras, and topological groupoids.

A common step in solving this problem is a result that states that certain functors from the category of A-modules to that of B-modules are determined by their precomposition with some embedding of A into its own module category, or more precisely, are always equivalent to tensoring with the A-B bimodule given by this composition. This sort of result is usually called an *Eilenberg-Watts theorem* after simultaneous discoveries by Eilenberg, Watts and Zisman in the ring case: see [Wat60] for Watts' contribution. Blecher proved an Eilenberg-Watts theorem for Hilbert modules over C^*-algebras in [Ble97]. For an exposition of several more such results, including in homotopical algebra and enriched categories, see [nLa23a]. With an Eilenberg-Watts theorem in hand, the problem of Morita invariance reduces to determining *which* A-B bimodules give, through their tensor product functor, an equivalence between the module categories of A and B.

Finally, a particularly well-behaved modern set-up is a *Morita homotopy category* whose objects are those such as A and B and which has isomorphisms between exactly those objects that are Morita equivalent. We might ask moreover that there is a universal functor from the usual category to the Morita homotopy category which inverts those morphisms between objects which give Morita equivalences (i.e. is a *localization* at the Morita equivalences), and hope that there is a concrete description of the hom-spaces of the category. This has been achieved for example in the case of unital C^*-categories ([DT14]), dg-categories (see [Tab05], [Toë07]), as well as in the case of $(\infty, 1)$-categories ([CG19]). One immediately notices that this only seems to be feasible when our objects are *themselves certain kinds of categories*: one explanation for this phenomenon is that only in this scenario can one invert Yoneda-type functors from each category into its module category.

This thesis solves all three of the above problems (an Eilenberg-Watts theorem, a characterization of Morita equivalent objects, and a Morita homotopy theory) in the case of locally small, not necessarily unital C^*-categories. These are semicategories enriched over complex Banach spaces with a well-behaved involution on the hom-spaces, built to model the topology and involution on spaces of operators between Hilbert spaces. The correct notion of a module here is a Hilbert module over a C^*-category. Proving results on the Morita theory of non-unital C^*-categories is considerably harder than the unital case treated in [DT14], since two Morita equivalent unital C^*-categories must be Morita equivalent as additive categories[1]. This mirrors the fact that unital C^*-algebras are Morita equivalent if and only if they are Morita equivalent as rings ([Bee82, §1.8]). For general C^*-categories there is no hope of such a result, and we must instead explore the theory of Hilbert modules over C^*-categories in earnest.

The first definition of C^*-categories was given in [GLR85]: they are a horizontal categorification of C^*-algebras, meaning one-object C^*-categories are simply C^*-algebras. We will see that many basic results on C^*-algebras carry over straightforwardly to the setting of C^*-categories, but there are a few important constructions from the literature that are specific to C^*-categories. Chapter 1 is devoted to recapitulating these constructions. In particular, it defines for a C^*-category $\mathcal{A}$ its additive closure $\mathcal{A}_\oplus$ (Proposition 1.2.3) and multiplier category $\mathcal{M}\mathcal{A}$ (Proposition 1.3.4), as well as clarifying their universal properties and how they interact (Lemma 1.3.6, Lemma 1.5.5).

Chapter 2 focuses on our notion of module over a C^*-category $\mathcal{A}$, namely right Hilbert $\mathcal{A}$-modules. We define adjointable and compact operators between these, prove a Yoneda lemma (Proposition 2.3.11), and in particular we identify morphisms in $\mathcal{A}$ with compact operators between representable modules over $\mathcal{A}$. We also characterize a 'strong*' topology of pointwise convergence on operators (Definition 2.3.15) and prove that the multiplier category of the compact operators returns the bounded operators (Proposition 2.4.7). We finish by generalizing a result of Blecher's ([Ble97, Theorem 3.1]) which is crucial in his proof of the C^*-algebra Eilenberg-Watts theorem: it says that Hilbert module can asymptotically be viewed as a direct summand in a net of finitely generated free modules (Theorem 2.5.6).

Chapter 3 brings the promised results on functors between module categories, beginning with the Eilenberg-Watts theorem:

Theorem (Theorem 3.2.4)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be locally small C^*-categories, and let $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ be a unital C^*-functor which is strongly* continuous on bounded subsets. Let $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ be the right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule obtained by precomposing F with the Yoneda embedding $\iota_\mathcal{A} : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{A}$. Then there exists a unitary isomorphism $F \cong -\bar{\otimes}_\mathcal{A} E$ of functors.*

Here the strong* topology is the topology of pointwise convergence mentioned earlier (see Definition 2.3.15) and $\bar{\otimes}_\mathcal{A}$ is the tensor product of Hilbert bimodules (see Lemma 3.1.6).

[1] To be precise, they must have the same closure under subobjects and direct sums: in [DT14] this was taken to be the *definition* of Morita equivalence, and we show in Proposition 3.4.8 that our definition coincides with theirs in the unital case.

We build on this result to characterize which C^*-categories are Morita equivalent, and find that the answer is completely analogous to the C^*-algebra case:

Theorem (Theorem 3.4.5)**.** *If $\mathcal{A}$ and $\mathcal{B}$ are locally small C^*-categories, the following are equivalent:*

- *There is a C^*-functor $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ which is an equivalence and strongly* continuous on bounded subsets.*

- *There exists a right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ which is left small, full, non-degenerate, and an isometry onto the compact operators between modules in its image.*

- *There exists a bi-Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule E whose inner products are both full and satisfy the compatibility equation ${}_{\mathcal{A}}\langle e, f\rangle \cdot g = e \cdot \langle f, g\rangle_{\mathcal{B}}$ for all elements $e \in E(x)(y), f \in E(x')(y)$ and $g \in E(x')(y')$ and objects $x, x' \in \mathsf{Ob}\,\mathcal{A}, y, y' \in \mathsf{Ob}\,\mathcal{B}$.*

Here 'left small' is a technical set-theoretic condition and 'full' signifies, as in the algebra case, that the inner products on the bimodule span a dense ideal of $\mathcal{B}$ (Definition 3.3.5). The definition of a bi-Hilbert bimodule is another direct generalization of the algebra case (see Definition 3.3.3).

As an easy corollary of this result, we considerably relax the assumptions on a result of Joachim's [Joa03, Proposition 3.2], which says that a unital C^*-category with countably many objects is always Morita equivalent to a C^*-algebra, and show that in fact every *small* C^*-category is Morita equivalent to a C^*-algebra (Proposition 3.4.9).

Chapter 4 contains the promised categorical framework for our results. First, we use the Eilenberg-Watts theorem and Proposition 3.4.9 to construct biequivalences between four bicategories of C^*-categories and C^*-algebras (Proposition 4.1.4). Finally, we exhibit the localization of the category of locally small C^*-categories at the Morita equivalences:

Theorem (Theorem 4.2.7)**.** *Let $\mathbf{C^*Cat}$ be the 1-category whose objects are locally small C^*-categories, and whose morphisms from $\mathcal{A}$ to $\mathcal{B}$ are natural isomorphism classes of non-degenerate functors from $\mathcal{A}$ to the multiplier category $\mathcal{MB}$ of $\mathcal{B}$.*

Let $\mathcal{K}\mathsf{Hilb}\text{-} : \mathbf{C^*Cat} \to \mathbf{C^*Cat}$ be the endofunctor that takes every C^*-category to its C^*-category of Hilbert modules and compact operators, and every non-degenerate functor to the tensor product with the bimodule obtained by composing with the embedding $\mathcal{MB} \to \mathsf{Hilb}\text{-}\mathcal{B}$. Let $\mathbf{HMod}$ be the full subcategory with objects those in the image of $\mathcal{K}\mathsf{Hilb}\text{-}$.

Then $\mathcal{K}\mathsf{Hilb}\text{-}$ exhibits $\mathbf{HMod}$ as the reflective localization of $\mathbf{C^*Cat}$ at the Morita equivalences, and two C^*-categories become isomorphic in $\mathbf{HMod}$ if and only if they are Morita equivalent.

In Appendix A, we define for two non-unital C^*-categories $\mathcal{A}$ and $\mathcal{B}$ their *maximal tensor product* $\mathcal{A} \otimes_{\max} \mathcal{B}$ and prove a tensor-hom adjunction for non-degenerate functors (Proposition A.20).

Chapter 1

C^*-categories

We begin with an investigation into the theory of C^*-categories. After presenting some basic constructions and results, we introduce the additive closure and the multiplier category of a given C^*-category, proving universal properties for both.

A note on terminology: in this thesis, we will use the term 'category' to refer to a semicategory (a category that doesn't necessarily have units), and 'unital category' to refer to what's usually called a category. Similarly a 'functor' will signify a semifunctor (i.e. a mapping of objects and morphisms that preserves composition but does not necessarily preserve units, even if they exist) and a functor between unital categories that preserves units will be called a 'unital functor'.

1.1 Basic theory of C^*-categories

Throughout this thesis we work in the following commonplace set-up:

Definition 1.1.1. We denote by Set_0 the ordinary category of sets. We'll refer to categories as *small* if their collection of objects is an object of Set_0 and its hom-spaces are objects of Set_0.

We denote by Set_1 the category of large sets, i.e. an enlargement of Set_0 that is large enough to contain as an object the collection of all objects of Set_0. We denote by *locally small* categories those whose hom-sets live in Set_0 and whose collection of objects lives in Set_1, for example the category Set_0.

We denote by Set_2 the category of very large sets, i.e. an enlargement of Set_1 that is large enough to contain as an object the collection of all objects of Set_1. We denote by *large* categories those whose hom-sets live in Set_1 and whose collection of objects lives in Set_2, for example the category Set_1.

Since the categories Set_i embed in each other as i increases we adopt the convention that small categories are locally small and locally small categories are large.

We build on this set-up for our purposes as follows:

Definition 1.1.2. For $i = 0, 1$ we denote by $\mathsf{Vect}_{\mathbb{C},\mathrm{i}}$ the bicomplete closed symmetric monoidal category such that

- Its objects are Set_i-objects endowed with the structure of a complex vector space.
- Its morphisms are complex linear maps.
- Its tensor product is the standard complex tensor product $\otimes_{\mathbb{C}}$.
- Its internal hom object for any two spaces X and Y is the space $\mathrm{Lin}(X, Y)$ of complex linear maps from X to Y.

Definition 1.1.3. For $i = 0, 1$ we denote by Ban_i the bicomplete closed symmetric monoidal category such that:

- Its objects are Set_i-objects endowed with the structure of a complex Banach space.
- Its morphisms are complex linear maps which are in addition *short*, i.e. they do not increase the norm of any element.
- Its tensor product is the standard (projective) tensor product, simply written $\otimes$.
- Its internal hom object for any two spaces X and Y is the Banach space $\mathrm{Map}(X, Y)$ of *bounded* complex linear maps from X to Y.

Finally, we denote by $F_i : \mathsf{Ban}_i \to \mathsf{Vect}_{\mathbb{C},\mathrm{i}}$ the obvious lax monoidal forgetful functors.

The bicompleteness and closed monoidal structure of Ban_0 is treated in [Yua12]. If one wanted to be more careful about showing $\mathsf{Vect}_{\mathbb{C},1}$ and Ban_1 are well-defined, one could use the theory of universe enlargement: for more details on this procedure see for example [Kel05, Section 2.6]. We won't go into this any further here.

Categories enriched over $\mathsf{Vect}_{\mathbb{C},\mathrm{i}}$ or Ban_i are called *complex*, resp. *Banach categories* and we apply labels like *small* and *large* to them in the way suggested by Definition 1.1.1. Since we haven't defined an infinite hierarchy of universes, we'll take care to comment on whether each construction of a new category increases the size or not.

Several recent works on C^*-categories ([Mit02a][DT14][Fer23]) only treat small C^*-categories, while some ([BE21]) work with a 'universes' approach like we do in this first chapter. From Chapter 2 on, we will look only at locally small C^*-categories, and in Chapter 4 we prove some results that hold only for small C^*-categories.

We can characterize which complex categories can be further enriched to Banach categories:

Lemma 1.1.4. *Then $\mathcal{C}$ be a small, locally small, or large complex category. It is the image under change of base (induced by a forgetful functor F_i) of a Banach category if and only if for all $x, y, z \in \mathsf{Ob}\,\mathcal{C}$ the following hold:*

- *The space $\mathcal{C}(x, y)$ can be equipped with a Banach norm.*
- *The composition is* contractive*, i.e. for each $a \in \mathcal{C}(y, z), b \in \mathcal{C}(x, y)$ we have $\|a \circ b\| \leq \|a\|\|b\|$.*

Proof. Suppose the complex enrichment can be lifted to a Banach enrichment: then clearly the first point is satisfied. Furthermore, by definition of the norm on the projective tensor product $\mathcal{C}(y, z) \otimes \mathcal{C}(x, y)$ we have $\|a \otimes b\| = \|a\|\|b\|$. So if the composition map on the $\mathcal{C}(y, z) \otimes_{\mathbb{C}} \mathcal{C}(x, y)$ can be lifted to a short map on the projective tensor product, then we must have $\|a \circ b\| \leq \|a \otimes b\| = \|a\|\|b\|$.

For the converse, we only need to show that under the assumptions in the lemma, the complex composition map

$$\circ : \mathcal{C}(y, z) \otimes_{\mathbb{C}} \mathcal{C}(x, y) \to \mathcal{C}(x, z)$$

lifts to a short map on the projective tensor product. But considering its image under the complex tensor-hom adjunction $\hat{\circ} : \mathcal{C}(y, z) \to \mathrm{Lin}(\mathcal{C}(x, y), \mathcal{C}(x, z))$, the contractive property implies that $\hat{\circ}$ in fact lands in the Banach space $\mathrm{Map}(\mathcal{C}(x, y), \mathcal{C}(x, z))$ and is a short map when the norm on $\mathcal{C}(y, z)$ is considered. So we can pass it back through the tensor-hom adjunction in Ban_i and get a short lift of $\circ$ to the projective tensor product. $\square$

We define a fundamental notion of Banach space theory:

Definition 1.1.5. Let B be a Banach space and Λ be a partially ordered set. A *net* $\{b_\lambda : \lambda \in \Lambda\}$ in B is simply a collection of elements of B indexed over Λ. We say that (b_λ) *converges* to $b \in B$ if for any $\epsilon > 0$ there is an element $\lambda_\epsilon \in \Lambda$ such that

$$\text{for all } \lambda \geq \lambda_\epsilon \text{ we have } \|b_\lambda - b\| < \epsilon.$$

We refer the reader to [Wil70, Chapter 4, Section 11] for details on the theory of nets. Note that a net indexed over the natural numbers is simply a sequence.

Remark 1.1.6. We will often refer to a net such as $\{b_\lambda : \lambda \in \Lambda\}$ simply as (b_λ), leaving the indexing set implicit, and denote 'convergence over Λ' by the symbol $\xrightarrow{\lambda}$.

We prove that in a Banach category the composition maps are continuous in the following sense:

Lemma 1.1.7. *Let $\mathcal{C}$ be a Banach category. If $\{b_\lambda : \lambda \in \Lambda\}$ is a net in $\mathcal{C}(y, z)$ converging to $b \in \mathcal{C}(y, z)$, and $\{c_\mu : \mu \in M\}$ is a net in $\mathcal{C}(x, y)$ converging to $c \in \mathcal{C}(x, y)$, and one of $\{b_\lambda\}$ or $\{c_\mu\}$ is uniformly bounded, then the net $\{b_\lambda c_\mu : (\lambda, \mu) \in \Lambda \times M)\}$ converges to bc. Here the set $\Lambda \times M$ is given the* product order*, i.e. $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$ if $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$.*

Proof. Suppose for some $K \in \mathbb{R}$ that $\|b_\lambda\| \leq K$ for all $\lambda \in \Lambda$. Then note that

$$\|b_\lambda c_\mu - bc\| = \|b_\lambda(c_\mu - c) + (b_\lambda - b)c\| \leq K\|c_\mu - c\| + \|b - b_\lambda\|\|c\|$$

which evidently goes to zero as (λ, μ) varies over $\Lambda \times M$. The case where $\{c_\mu\}$ is uniformly bounded proceeds similarly. □

Definition 1.1.8. A *Banach $*$-category* is a Banach category $\mathcal{C}$ equipped with a conjugate linear involution $(-)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ that fixes objects and squares to the identity functor: that is, for all morphisms f, we have $(f^*)^* = f$.

Before we move on to defining a C^*-category, we remind the reader of some basic concepts which play a crucial role in the theory C^*-algebras.

Definition 1.1.9. If A is a non-unital complex algebra, we denote by A^+ the algebra given as a vector space by $A \oplus \mathbb{C}$ and whose multiplication is given by $(a_1, \lambda_1)(a_2, \lambda_2) = (a_1a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1\lambda_2)$. If A is a unital complex algebra we simply set $A^+ = A$.

It is obvious that if A is non-unital, A^+ has a unit given by $(0, 1) \in A \oplus \mathbb{C}$.

Definition 1.1.10. If A is any (possibly non-unital) complex algebra and $a \in A$, then we denote by $\mathrm{Spec}(a) \subseteq \mathbb{C}$, or by the *spectrum of* a, the set of numbers λ such that $a - \lambda\,\mathrm{id}$ is not invertible in the algebra A^+. We say that a is *positive* if $\mathrm{Spec}(a) \subseteq \mathbb{R}^{\geq 0}$.

Definition 1.1.11. A *C^*-category* is a Banach $*$-category $\mathcal{A}$ such that for all $x, y \in \mathsf{Ob}\,\mathcal{A}$ and $a \in \mathcal{A}(x, y)$ we have:

- the morphism $a^*a \in \mathcal{A}(x, x)$ satisfies the C^*-identity $\|a^*a\| = \|a\|^2$ and
- the morphism a^*a is positive.

Readers versed in operator algebras will know that in the case where $\mathcal{A}$ has a single object x, the first of the above two axioms is enough to specify that $\mathcal{A}(x, x)$ is a C^*-algebra, and the positivity requirement then follows (see e.g. [Mur90, Theorem 2.2.5]). In the case of C^*-categories there are pathological examples satisfying the first but not the second axiom: see [Mit02a, Example 2.10].

The positivity axiom for C^*-categories requires that we make a variety of arguments in this thesis about positivity in C^*-algebras. For this reason we collate here some classical results on this topic, along with citations. In the statements below, A is a C^*-algebra and a and b are elements of A.

Definition 1.1.12. The element a is said to be *normal* if $a^*a = aa^*$ and *self-adjoint* if furthermore $a^* = a$.

Lemma 1.1.13 ([Mur90, Theorem 2.2.5])**.** *The element a is positive if and only if $a = d^*d$ for some $d \in A$.*

Lemma 1.1.14 ([Put19, Corollary 1.4.10]). *If a is normal, then a has real spectrum if and only if it is self-adjoint.*

Lemma 1.1.15 ([Put19, Lemma 1.6.1]). *If a is self-adjoint, it can be written as a difference $a = b - c$ where b and c are positive and $bc = 0$.*

Lemma 1.1.16 ([Mur90, Theorem 2.2.1]). *If a is positive, there is a unique positive element b such that $b^2 = a$.*

Lemma 1.1.17 ([Mur90, p. 44]). *If $A \subseteq B$ is an inclusion of C^*-algebras, then for all $a \in A$, we have $Spec_A(a) \cup \{0\} = Spec_B(a) \cup \{0\}$. In particular, a is positive in A if and only if a is positive in B.*

Lemma 1.1.18 ([Put19, Proposition 1.6.3]). *If a and b are positive, then $a+b$ is positive again.*

Lemma 1.1.19 ([Shi21, Corollary 2.3.2]). *If a and b are positive and $ab = ba$, then ab is again positive.*

Recall for the final lemmas that we say $a \geq b$ when the element $a - b$ is positive.

Lemma 1.1.20 ([Dix77, Lemma 1.6.8]). *If a is any element of a C^*-algebra A and $b \in A$ is positive, then $\|a\|^2 b \geq a^* b a$.*

Lemma 1.1.21 ([Dix77, Lemma 1.6.9]). *If a and b are positive elements and $a \geq b$, then $\|a\| \geq \|b\|$.*

Lemma 1.1.22. *If p_n is a sequence of positive elements in a C^*-algebra with limit p, then p is also positive.*

Proof. Embed A into the algebra of bounded operators $B(H)$ on a Hilbert space H, and recall that an element T of $B(H)$ is positive if and only if the inner product $\langle h, T(h) \rangle_H$ is a positive real number for all $h \in H$. But then by the Cauchy-Schwarz inequality it's clear that $\langle h, p_n(h) \rangle \xrightarrow{n} \langle h, p(h) \rangle$ for all $h \in H$ and we see that if all elements p_n are positive, so is p. $\square$

The final lemma we quote is not about positive elements but about a fundamental construction in C^*-algebra theory called the *functional calculus*:

Lemma 1.1.23 ([Mur90, Theorem 2.1.13]). *Let A be a unital C^*-algebra and a be a normal element. Furthermore let $z : Spec(a) \to \mathbb{C}$ be the inclusion map. Then there is a unique unital $*$-homomorphism $\phi : C(Spec(a)) \to A$ such that $\phi(z) = a$, and furthermore ϕ has image the C^*-subalgebra of A generated by a.*

Returning to C^*-categories, the axiom stating $\|a^* a\| = \|a\|^2$ is called the *C^*-identity*. We record here for posterity a slightly weaker statement that also implies this condition. Note that we will occasionally write $a \in \mathcal{A}(x, y)$ or simply $a \in \mathcal{A}$ for the claim '$a \in \mathcal{A}(x, y)$ for some $x, y \in \mathsf{Ob}\,\mathcal{A}$'.

Lemma 1.1.24. *If $\mathcal{A}$ is a Banach $*$-category such that for all $a \in \mathcal{A}$ we have $\|a^*a\| \geq \|a\|^2$, then in fact $\|a^*\| = \|a\|$ and $\|a^*a\| = \|a\|^2$.*

Proof. If $\|a\| = 0$, then $a = a^* = 0$ and the C^*-equality is obvious. Otherwise, note that by contractivity we get $\|a^*\| \cdot \|a\| \geq \|a^*a\| \geq \|a\|^2$, so dividing out $\|a\|$ we obtain $\|a^*\| \geq \|a\|$. But then we also have $\|a\| = \|a^{**}\| \geq \|a^*\|$, so in fact $\|a^*\| = \|a\|$, and by contractivity $\|a^*a\| \leq \|a\|^2$. Hence we get $\|a^*a\| = \|a\|^2$. □

We give some examples to provide a flavour of the theory.

Example 1.1.25. C^*-algebras are precisely those Banach $*$-algebras that occur as the hom-space of a single-object C^*-category.

It is with this justification that C^*-categories can be regarded as 'multi-object' C^*-algebras.

Example 1.1.26. The category Hilb of small Hilbert spaces and bounded operators between them is a locally small C^*-category with the familiar operator norm, and involution given by taking adjoint operators.

Remark 1.1.27. It is in fact the case that much as every C^*-algebra can be embedded into the algebra of all bounded operators on some Hilbert space, every *small* C^*-category can be embedded isometrically into the C^*-category Hilb: see [Mit02a, Theorem 6.12]. The proof involves a direct analog of the Gelfand-Naimark-Segal construction. We will not use this result in this thesis, and stick with an abstract rather than a concrete notion of C^*-categories.

Example 1.1.28 ([Mit02a, Definition 5.10])**.** If $\mathcal{G}$ is a small discrete groupoid, then there is a *maximal groupoid C^*-category* $C^*_{\max}(\mathcal{G})$ with the same objects as $\mathcal{G}$, whose hom-spaces $\mathrm{C}^*_{\max}(\mathcal{G})(x,y)$ are given by taking the free complex vector space on $\mathcal{G}(x,y)$ and completing in the norm given by $\|a\| = \sup_{F:\mathcal{G}\to\mathsf{Hilb}} \|F(a)\|$ (where F varies over unitary representations of $\mathcal{G}$). The involution is given by $(\lambda \cdot g)^* = \overline{\lambda} \cdot g^{-1}$.

There is also a 'reduced groupoid C^*-category' defined along similar lines in the same publication. For a more general perspective on the maximal groupoid construction, we refer the reader to the series of adjunctions presented on [Bun21, p. 66], which assemble to give a 'free C^*-category' on any category with involution.

A groupoid C^*-category in the case where $\mathcal{G}$ is *not discrete* can be defined using the notion of a topological C^*-category, i.e. one with a topology on the objects. We do not investigate this construction here and instead refer the interested reader to [OSu17, Section 5.1].

We will see in this chapter a number of ways of producing a new C^*-category from a given one, the first being as follows:

Lemma 1.1.29. *If $\mathcal{A}$ is a C^*-category, then $\mathcal{A}^{\mathrm{op}}$ is again a C^*-category with respect to the same complex vector space structure and norms, and involution $(-)^{*op}$ given simply by the opposite functor of the normal involution of $\mathcal{A}$.*

Proof. The only things to check are the C^*-identity and positivity: but clearly requiring that $a^* \circ_{\mathrm{op}} a = aa^*$ is positive for all a is equivalent to requiring that a^*a is positive, and $\|a^* \circ_{\mathrm{op}} a\| = \|aa^*\| = \|(a^{**})a^*\| = \|a^*\|^2 = \|a\|^2$. $\square$

We recall here two important definitions that give us another example of a C^*-category.

Definition 1.1.30. If B is a C^*-algebra, a *right Hilbert B-module* is a right B-module E with a B-valued inner product $\langle -, - \rangle : E \times E \to B$ that satisfies the following requirements for all $e, f \in E, b \in B, \lambda, \mu \in \mathbb{C}$:

- $\langle \lambda e, \mu f \rangle = \overline{\lambda} \mu \langle e, f \rangle$.
- $\langle e, f \rangle = \langle f, e \rangle^*$.
- $\langle e, f \rangle b = \langle e, f \cdot b \rangle$.
- $\langle e, e \rangle$ is a positive element of B and vanishes only if $e = 0$.
- E is complete in the norm $\|e\| := \sqrt{\|\langle e, e \rangle\|_B}$.

We will describe any inner product with the first property as *sesquilinear* from here onwards.

Definition 1.1.31. For two right Hilbert modules E and F over a C^*-algebra B, a B-homomorphism $T : E \to F$ is a *bounded adjointable* map when it is a bounded map in the given norms and there exists an *adjoint transformation* $T^* : F \to E$ such that for all $e \in E, f \in F$ we have

$$\langle T(e), f \rangle_F = \langle e, T^*(f) \rangle_E.$$

It follows easily that the adjoint of a transformation T is unique if it exists. Notice that when we set $B = \mathbb{C}$, the above definition describes Hilbert spaces and bounded adjointable[1] maps between them.

Example 1.1.32. For any C^*-algebra B, the category $\mathsf{Hilb}\text{-}B$ of right Hilbert B-modules and bounded adjointable operators is a unital C^*-category, with the standard operator norm and with adjoints providing the involution.

We do not prove this result here as it is a special case of a result in the next chapter (Proposition 2.2.6). We refer the reader to [Lan95] for a thorough exposition of the theory of Hilbert modules over C^*-algebras.

Functors between C^*-categories get an adapted definition:

[1] In the case $B = \mathbb{C}$ a bounded linear operator necessarily has an adjoint, so we might as well speak simply of 'bounded operators'. For general B, though, the existence of an adjoint does not follow from the other properties here.

Definition 1.1.33. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, a *C^*-functor* from $\mathcal{A}$ to $\mathcal{B}$ is a complex linear functor $F : \mathcal{A} \to \mathcal{B}$ that intertwines the involutions on $\mathcal{A}$ and $\mathcal{B}$.

This is called a linear $*$-functor in parts of the literature, but we prefer 'C^*-functor' for brevity. An example of a functor between C^*-categories that is not a C^*-functor is the involution: the functor $(-)^* : \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ is *not* a C^*-functor since it is not linear but conjugate linear.

One might ask whether there is a continuity requirement missing here, but as in the algebraic case, this is determined by the other structure. The proofs of these results are simple but we include them here for completeness:

Proposition 1.1.34 ([Mit02a, Proposition 2.14])**.** *If $F : \mathcal{A} \to \mathcal{B}$ is a C^*-functor, its associated maps on hom-spaces $F : \mathcal{A}(x,y) \to \mathcal{B}(F(x), F(y))$ are norm-decreasing, so* a fortiori *continuous.*

Furthermore, C^*-functors that give injective maps on hom-spaces are isometric.

Proof. Note by the C^*-identity that for every $a \in \mathcal{A}$ we have $\|a\| = \sqrt{\|a^*a\|}$, so it suffices to prove the lemma holds on (positive) endomorphisms $a \in \mathcal{A}(x,x)$. But clearly F restricts to a $*$-homomorphism $\mathcal{A}(x,x) \to \mathcal{B}(F(x), F(x))$, and it is then a standard result of C^*-algebra theory that such a map is automatically norm-decreasing (see for example [Mur90, Theorem 2.1.7]). Furthermore, an injective $*$-homomorphism of C^*-algebras is isometric (see [Mur90, Theorem 3.1.5]), proving the second result. □

Corollary 1.1.35 ([Mit02a, Corollary 2.16])**.** *The norm on any C^*-category is unique: that is, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a linear $*$-category $\mathcal{C}$ both turning it into a C^*-category, then actually $\|\cdot\|_1 = \|\cdot\|_2$.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two C^*-categories associated to $\mathcal{C}$ by the norms. Then the identity functor $\mathcal{C}_1 \to \mathcal{C}_2$ is injective, so it must be isometric, and in fact $\|a\|_1 = \|a\|_2$ for all $a \in \mathcal{A}$. □

We define several types of subcategory of a C^*-category:

Definition 1.1.36. If $\mathcal{A}$ is a C^*-category, a $\mathbb{C}$-linear subcategory $\mathcal{B} \subseteq \mathcal{A}$ is a *sub-C^*-category* if it is closed in the norm, as well as closed under the involution. A sub-C^*-category $\mathcal{B}$ is a *C^*-ideal* (or simply an *ideal*) if in addition for all composable $f \in \mathcal{A}, g \in \mathcal{B}, h \in \mathcal{A}$ we have $fg \in \mathcal{B}, gh \in \mathcal{B}$. An ideal $\mathcal{B}$ is *essential* if the following statement is true for all $a \in \mathcal{A}$: if $ab = 0$ for all $b \in \mathcal{B}$ such that this composition exist, then $a = 0$.

Note that since there is at least one morphism (the zero morphism) between any two objects in a C^*-category, an ideal of $\mathcal{A}$ necessarily contains all objects of $\mathcal{A}$, i.e. it is necessarily a *wide* subcategory.

We deduce an important property of essential ideals:

Proposition 1.1.37. *If $\mathcal{B} \subseteq \mathcal{A}$ is an essential ideal and $\mathcal{C} \subseteq \mathcal{A}$ is any ideal such that $\mathcal{C} \cap \mathcal{B} = 0$, then $\mathcal{C} = 0$.*

Proof. Suppose $\mathcal{C}$ is another ideal of $\mathcal{A}$ such that $\mathcal{C} \cap \mathcal{B} = 0$. Then certainly for any composable pair $c \in \mathcal{C}, b \in \mathcal{B}$, we have $cb \in \mathcal{C} \cap \mathcal{B}$ by ideal properties, so $cb = 0$, and the stated property gives $c = 0$ for all $c \in \mathcal{C}$. So $\mathcal{B}$ has the stated property. □

In the C^*-algebra case, the above characterization is *equivalent* to essentiality (see e.g. [Mur90, p.82]). As far as we know, it's a little weaker in the C^*-category case.

We prove here for later use an easy technical lemma that helps us *generate* an ideal from any collection of morphisms in a C^*-category.

Proposition 1.1.38. *If B is any collection of morphisms in $\mathcal{A}$, there is an ideal*

$$\mathcal{A}\, B\, \mathcal{A}$$

of $\mathcal{A}$ with the following property:

- *$B \subseteq \mathcal{A}\, B\, \mathcal{A}$*
- *if $\mathcal{C}$ is any ideal of $\mathcal{A}$ and $B \subseteq \mathcal{C}$, then $\mathcal{A}\, B\, \mathcal{A} \subseteq \mathcal{C}$.*

Furthermore, for any two objects v and z of $\mathcal{A}$, the hom-space $\mathcal{A}\, B\, \mathcal{A}(v,z)$ can be described explicitly as the norm-closure of the space of finite sums $\sum_{i=1}^{n} a_i b_i \alpha_i$ where for each i we have $a_i \in \mathcal{A}(y_i, z), \alpha_i \in \mathcal{A}(v, x_i)$ for some $x_i, y_i \in \mathsf{Ob}\, \mathcal{A}$ and either $b_i \in B(x_i, y_i)$ or $b_i^* \in B(y_i, x_i)$.

Proof. Any two-sided ideal containing B must contain elements of the form $a_i b_i \alpha_i$ as above where $b_i \in B(x_i, y_i)$, as well as their involutions, giving us the case where $b_i^* \in B(y_i, x_i)$. As we have defined ideals to be complex linear norm closed subcategories, all norm limits of finite sums of these must be in any ideal containing B.

Conversely, it is easy to see that the finite linear combinations of morphisms of the type appearing in the definition of $\mathcal{A}\, B\, \mathcal{A}$ form a complex linear subcategory closed under involution and multiplication by outside elements, but then by Lemma 1.1.7 so is the subcategory of norm-limits of such combinations. □

For our first example of an essential ideal we define and prove some results on the minimal unitization of a C^*-category, which will be of interest elsewhere.

Definition 1.1.39. For every C^*-category $\mathcal{A}$ there is a unital complex linear $*$-category $\mathcal{A}^+$, termed the *minimal unitization* of $\mathcal{A}$, which has hom-spaces

$\mathcal{A}^+(x,y) = \mathcal{A}(x,y)$ for $x \neq y$ or when $x = y$ and $\mathcal{A}(x,x)$ is unital

$\mathcal{A}^+(x,x) = \mathcal{A}(x,x)^+$, the unitization of $\mathcal{A}(x,x)$ defined in Definition 1.1.9, whenever $\mathcal{A}(x,x)$ is non-unital.

Composition is defined by $(a_1, \lambda_1)(a_2, \lambda_2) = (a_1a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1\lambda_2)$ (where we represent 'unenhanced' elements of $\mathcal{A}$ by setting $\lambda_i = 0$), and involution defined by $(a, \lambda)^* = (a^*, \overline{\lambda})$.

This unitization is in fact a C^*-category:

Proposition 1.1.40. *There is a norm on $\mathcal{A}^+$ making it a C^*-category, given on the enhanced algebras $\mathcal{A}(x,x) \oplus \mathbb{C}$ by the norm*

$$\|(a,\lambda)\| := \sup_{b \in \mathcal{A}(x,x), \|b\|=1} \|ab + \lambda b\|$$

and on all other hom-spaces by the norm from $\mathcal{A}$.

Furthermore, the C^*-functor $\gamma_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{A}^+$ given by $a \mapsto (a, 0)$ on the enhanced algebras and the identity elsewhere embeds $\mathcal{A}$ as an essential ideal in $\mathcal{A}^+$.

Proof. The fact that the unitized endomorphism algebras are C^*-algebras with the given norm is classical, see e.g. [Weg93, Proposition 2.1.7]. Combining this with the fact that $\mathcal{A}$ is a C^*-category we immediately get the completeness of all hom-spaces, the C^*-identity and the positivity axiom on all elements, and the contractive property of composition, *except* in the case of composing an 'enhanced' endomorphism $(a, \lambda) \in \mathcal{A}(x,x) \oplus \mathbb{C}$ with a morphism $b \in \mathcal{A}(y,x)$. In this case we note that

$$\begin{aligned}
\|(a,\lambda)b\|^2 &= \|ab + \lambda b\|^2 \\
&= \|(ab + \lambda b)(ab + \lambda b)^*\| \\
&= \|(abb^* + \lambda bb^*)(a^*, \overline{\lambda})\| \\
&\leq \|(abb^* + \lambda bb^*)\| \|(a^*, \overline{\lambda})\| \\
&\leq \|(a,\lambda)\| \|bb^*\| \|(a^*, \overline{\lambda})\| \\
&= \|(a,\lambda)\|^2 \|b\|^2
\end{aligned}$$

where the two inequalities follows from the definitions of $\|(a^*, \overline{\lambda})\|$, respectively $\|(a,\lambda)\|$, since b^*b and $abb^* + \lambda bb^*$ are endomorphisms of x. Hence $\mathcal{A}^+$ is a C^*-category.

Finally, to see that $\mathcal{A}$ is an essential ideal in $\mathcal{A}^+$, suppose that an element (a_0, λ) of an enhanced endomorphism algebra satisfies

$$(a_0, \lambda)a_1 = a_0 a_1 + \lambda a_1 = 0$$

for all $a_1 \in \mathcal{A}$. Then in particular this is true when a_1 is in the same algebra as a_0, but by [Weg93, p. 29] this already implies $(a_0, \lambda) = (0,0)$. Hence $\mathcal{A}$ is essential in $\mathcal{A}^+$.

□

We now describe a universal property of the C^*-category $\mathcal{A}^+$. To do this we need the following definition:

Definition 1.1.41. Let $\mathcal{A}$ be a C^*-category and $\mathcal{B}$ be a unital complex linear category. We denote by

$$\mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathcal{B})$$

the category of functors from $\mathcal{A}$ to $\mathcal{B}$ that restrict to unital maps on all unital endomorphism algebras in $\mathcal{A}$.

Proposition 1.1.42. *Let $\mathcal{A}$ be a C^*-category and $\mathcal{B}$ be a unital complex linear category. Precomposition with the embedding $\gamma_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{A}^+$ induces an isomorphism of categories*

$$- \circ \gamma_{\mathcal{A}} : \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}^+, \mathcal{B}) \to \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathcal{B}).$$

Proof. We first show that $- \circ \gamma_{\mathcal{A}}$ is surjective on objects, i.e. that if $F : \mathcal{A} \to \mathcal{B}$ is a complex linear functor which acts unitally on any unital endomorphism algebras in $\mathcal{A}$, then it extends to a unital complex linear functor $F^+ : \mathcal{A}^+ \to \mathcal{B}$.

The functor F^+ is defined on the unitized algebras by

$$F^+(a, \lambda) = F(a) + \lambda \cdot \mathrm{id}\,.$$

It is elementary to verify that this gives a unital complex linear functor.

To see that $- \circ \gamma_{\mathcal{A}}$ is injective on objects, note that a unital functor must have

$$F^+(0, \lambda) = \lambda \cdot F(0, 1) = \lambda \cdot \mathrm{id},$$

so by linearity the values on $\mathcal{A}$ determine F^+.

But for any array of $\mathcal{B}$-morphisms indexed by $\mathsf{Ob}\,\mathcal{A}$, naturality with respect to morphisms in $\mathcal{A}$ is clearly equivalent to naturality with respect to morphisms in $\mathcal{A}^+$, so $-\circ\gamma_{\mathcal{A}}$ is also full and faithful, and hence an isomorphism of categories. □

Another example of an essential ideal is the following:

Example 1.1.43. The category Hilb of Hilbert spaces and bounded operators has as an essential ideal the subcategory $\mathcal{K}\mathsf{Hilb}$ of Hilbert spaces and compact operators.

We will not prove here that $\mathcal{K}\mathsf{Hilb}$ is essential in Hilb as this is a special case of a result in the next chapter (Proposition 2.3.6).

Every C^*-ideal gives a new topology on the C^*-category containing it; note that by a topology on a category we will always mean a topology on the *hom-spaces* of the category.

Definition 1.1.44. For a C^*-ideal $\mathcal{B} \subseteq \mathcal{A}$, the *$\mathcal{B}$-relative topology* on $\mathcal{A}(x, y)$ is generated by the seminorms of the form

$$g \mapsto \|fg\| : f \in \mathcal{B}(y, z)$$
$$g \mapsto \|gh\| : h \in \mathcal{B}(w, x)$$

It follows from the contractive property that a net which converges in the norm on $\mathcal{A}$ must converge in the $\mathcal{B}$-relative topology for all $\mathcal{B} \subseteq \mathcal{A}$. Note that even when $\mathcal{B} = \mathcal{A}$, the $\mathcal{A}$-relative topology on $\mathcal{A}$ is not necessarily the same as the norm topology, unless $\mathcal{A}$ is unital.

We prove here that morphism composition is always continuous in the topology relative to an ideal:

Lemma 1.1.45. *Suppose $\mathcal{B} \subseteq \mathcal{A}$ is a C^*-ideal in a C^*-category, and that $(u_\lambda : \lambda \in \Lambda)$ and $(v_\mu : \mu \in M)$ are two norm-bounded nets in $\mathcal{A}(y,z)$ and $\mathcal{A}(x,y)$ respectively. Suppose in addition that there are morphisms $u \in \mathcal{A}(y,z)$ and $v \in \mathcal{A}(x,y)$ such that $u_\lambda \xrightarrow{\lambda} u$ and $v_\mu \xrightarrow{\mu} v$ in the $\mathcal{B}$-relative topology. Then the net $(u_\lambda v_\mu : \lambda \in \Lambda, \mu \in M)$ converges to uv in the $\mathcal{B}$-relative topology.*

Proof. Let b be an element in $\mathcal{B}(v,x)$, and $K \in \mathbb{R}$ be a bound for $\|u_\lambda\|$. Then note that

$$\begin{aligned} \|u_\lambda v_\mu b - uvb\| &\leq \|u_\lambda(v_\mu b - vb)\| + \|u_\lambda vb - uvb\| \\ &\leq K\|(v_\mu b - vb)\| + \|u_\lambda vb - uvb\| \end{aligned}$$

But the first term in the last expression goes to zero as μ varies, and as $vb \in \mathcal{B}$ we see the second term goes to zero as λ varies, so $u_\lambda v_\mu b \xrightarrow{\lambda,\mu} uvb$ in the norm. We can also prove using the bound on (v_μ) that $b' u_\lambda v_\mu \xrightarrow{\lambda,\mu} b'uv$ for $b' \in \mathcal{B}(z,w)$. □

We follow by defining an 'internal hom' for C^*-categories:

Proposition 1.1.46. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are locally small C^*-categories. Then there is a C^*-category*

$$\mathrm{Fun}(\mathcal{A},\mathcal{B})$$

such that

- *Its objects are the C^*-functors $F : \mathcal{A} \to \mathcal{B}$.*
- *Its morphisms $\eta : F \to F'$ are the natural transformations $\eta : F \Rightarrow F'$ such that $\|\eta\| := \sup_{x \in \mathsf{Ob}\,\mathcal{A}} \|\eta_x\| < \infty$.*
- *The involution is taken objectwise, i.e. by setting $(\eta^*)_x = \eta_x^*$.*

In general $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ *is large, but if $\mathcal{A}$ is small, then* $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ *is locally small.*

Proof. Note that *a priori* $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ is a subcategory of the large category of all functors from $\mathcal{A}$ to $\mathcal{B}$, considered simply as Set_0-categories. So $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ is a large category, except in the case where $\mathcal{A}$ is small. Furthermore it has an obvious complex vector space structure on the hom-sets. We show that it's a C^*-category with the given norm and involution.

Contractivity in $\mathrm{Fun}(\mathcal{A},\mathcal{B})$ follows directly from that on $\mathcal{B}$. Completeness on the hom-spaces follows as a Cauchy sequence of natural transformations (η_n) must by definition have Cauchy components $(\eta_{x,n})$, and then by Lemma 1.1.7

we see that $\eta_x := \lim_n \eta_{x,n}$ assembles to a natural transformation η, which is the limit of η_n. So $\mathrm{Fun}(\mathcal{A}, \mathcal{B})$ is a Banach category.

Furthermore, the involution is well-defined: if η is a natural transformation, then for any morphism $a \in \mathcal{A}(x, x')$ we have

$$\eta^*_{x'} \circ F(a) = (F(a^*) \circ \eta_{x'})^* = (\eta_x \circ F'(a^*))^* = F'(a) \circ \eta^*_x$$

by naturality of η, so we can apply the involution at every object and get another natural transformation η^*.

To prove the C^*-identity, note simply that

$$\|\eta^*\eta\| := \sup_{x \in \mathsf{Ob}\,\mathcal{A}} \|\eta^*_x \eta_x\| = \sup_{x \in \mathsf{Ob}\,\mathcal{A}} \|\eta_x\|^2 = (\sup_{x \in \mathsf{Ob}\,\mathcal{A}} \|\eta_x\|)^2 = \|\eta\|^2.$$

To show that $\eta^*\eta$ has positive spectrum, suppose $\eta^*\eta - \lambda$ is not invertible in $\mathrm{Fun}(\mathcal{A}, \mathcal{B})(F, F)^+$. Then at least one of the constituent morphisms

$$\eta^*_x \eta_x - \lambda \in \mathcal{B}(F(x), F(x))^+$$

is not invertible, or else the collection of their inverses would easily be seen to constitute an inverse natural transformation to η. But the morphisms $\eta^*_x \eta_x$ each have positive spectrum, so λ must be positive and real.

□

It is useful to study morphisms whose involution is also a (right or left) inverse.

Definition 1.1.47. In a C^*-category $\mathcal{A}$, a *$*$-monomorphism* or *isometry* is a morphism $a \in \mathcal{A}(x, y)$ such that $a^*a = \mathrm{id}_x$. A *$*$-epimorphism* is a morphism $a \in \mathcal{A}(x, y)$ such that $aa^* = \mathrm{id}_y$. A *unitary isomorphism* is a morphism $a \in \mathcal{A}(x, y)$ such that $a^*a = \mathrm{id}_y$ and $aa^* = \mathrm{id}_x$.

Example 1.1.48. If B is a C^*-algebra, a morphism in the C^*-category of right Hilbert B-modules is a unitary equivalence if and only if it is a bijection and preserves all inner products.

These special isomorphisms allow us to define an appropriate type of equivalence of C^*-categories:

Definition 1.1.49. If $\mathcal{A}$ and $\mathcal{B}$ are two unital C^*-categories, $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are C^*-functors, and $\eta : FG \Rightarrow \mathrm{id}_{\mathcal{B}}$ and $\epsilon : \mathrm{id}_{\mathcal{A}} \Rightarrow GF$ are natural isomorphisms, then this data is called a *unitary equivalence* between $\mathcal{A}$ and $\mathcal{B}$ if η and ϵ are both not just isomorphisms but unitary isomorphisms at every object.

The existence of a unitary isomorphism between two objects is not, however, a stronger condition than that of isomorphism:

Lemma 1.1.50 ([Del12, Proposition 2.6])**.** *If $a \in \mathcal{A}(x, y)$ is an isomorphism, then there is a unitary isomorphism from x to y given by $a(a^*a)^{-\frac{1}{2}}$.*

Corollary 1.1.51. *Two locally small unital C^*-categories are unitarily equivalent if and only if they are equivalent.*

Proof. Simply note that in Definition 1.1.49, the transformation η is an isomorphism in $\mathrm{Fun}(\mathcal{A}, \mathcal{A})$, which is a C^*-category by Proposition 1.1.46. Hence η can be modified to a unitary isomorphism, which will clearly give a unitary isomorphism at every object. One can proceed similarly for ϵ. $\square$

1.2 Finite direct sums in C^*-categories

We turn now to the matter of direct sums in C^*-categories. A standard result about additive categories is that finite products and coproducts coincide. In a C^*-category we ask in addition that their structure maps are related by the involution:

Definition 1.2.1. Given a C^*-category $\mathcal{A}$ and a finite (possibly repeating) list of objects $x_1, \ldots, x_n$ of $\mathcal{A}$ which each have units, we say $\bigoplus_{i=1}^n x_i \in \mathsf{Ob}\mathcal{A}$ is a *direct sum* of $x_1, \ldots, x_n$ with structure maps $\iota_i : x_i \to \bigoplus_{i=1}^n x_i$ if

- All ι_i are isometries.
- $\sum_{i=1}^n \iota_i \iota_i^* = \mathrm{id}_{\bigoplus_{i=1}^n x_i}$.

Remark 1.2.2. As this is a more specific definition than a direct sum in a general additive category, it might be more appropriate to name this a $*$-direct sum, but since the more general sums do not appear in this thesis we stick with 'direct sum' for economy. Readers versed in dagger categories may be interested to know that this notion of direct sum is an example of a dagger limit as defined in [HK19].

It is evident that a C^*-category can only have direct sums for all finite lists of elements if it is unital. Despite this, we can enlarge any C^*-category to include objects that behave something like direct sums, without unitizing the category:

Proposition 1.2.3. *If $\mathcal{A}$ is a C^*-category, there is a C^*-category $\mathcal{A}_\oplus$, termed the* additive hull *or* additive closure *of $\mathcal{A}$, such that:*

- *Objects of $\mathcal{A}_\oplus$ are finite, possibly repeating, lists of $\mathcal{A}$-objects.*
- *$\mathcal{A}_\oplus(\{x_1, .., x_n\}, \{y_1, .., y_k\}) := [\hom_{\mathcal{A}}(x_i, y_j)]$, that is $k \times n$ arrays of morphisms $x_i \to y_j$, which compose by matrix multiplication.*
- *The norm on $\mathcal{A}_\oplus(\{x_1, .., x_n\}, \{y_1, .., y_k\})$ is defined by*

$$\|[f_{ij}]\| := \sup\{\|[f_{ij}][b_i]\| : b_i \in \mathcal{A}(w, x_i), w \in \mathsf{Ob}(\mathcal{A}), \|[b_i]\| = 1\}$$

  where the norms of columns of morphisms are calculated in Hermitian fashion: $\|[b_i]\| := \sqrt{\|\sum_i b_i^* b_i\|}$.

- *The involution is given on* $f = [f_{ij}] \in \mathcal{A}_\oplus(\{x_1,..,x_n\},\{y_1,..,y_k\})$ *by* $f^* := [f^*_{ji}]$.

Proof. It is evident that $\mathcal{A}_\oplus$ is a normed complex linear category with involution so we need to show that its hom-spaces are complete, that multiplication is contractive, and finally that the C^*-identity and the positivity requirement are satisfied.

Contractivity is obvious: if we have $b_i \in \mathcal{A}(w, x_i)$ for $1 \leq i \leq n$ such that $\|[b_i]\| = 1$, then for all

$$[g_{jl}] \in \mathcal{A}_\oplus(\{y_1,..,y_k\},\{z_1,..,z_m\}), [f_{ij}] \in \mathcal{A}_\oplus(\{x_1,..,x_n\},\{y_1,..,y_k\})$$

we have

$$\|[g_{jl}][f_{ij}][b_i]\| \leq \|[g_{jl}]\|\|[f_{ij}][b_i]\| \leq \|[g_{jl}]\|\|[f_{ij}]\|\|[b_i]\|$$

where we have simply used the definition of the matrix norm twice. Hence $\|[g_{jl}][f_{ij}]\| \leq \|[g_{jl}]\|\|[f_{ij}]\|$.

To show the norm makes each hom-space complete, we show it can be sandwiched uniformly by the maximum of the norms of matrix entries. Note that for any matrix $[f_{ij}] \in \mathcal{A}_\oplus(\{x_1,..,x_n\},\{y_1,..,y_k\})$ and any pair of integers $1 \leq i_0 \leq n$, $1 \leq j_0 \leq k$, we can define a vector $[b_i] \in [\mathcal{A}(x_{i_0}, x_i)]$ with entries $b_{i_0} = \frac{f^*_{i_0 j_0}}{\|f_{i_0 j_0}\|}, b_i = 0$ for $i \neq i_0$. We then see that

$$\|f_{i_0 j_0}\| = \|[f_{ij}][b_i]\| \leq \|[f_{ij}]\|$$

as $\|[b_i]\| = 1$ so $\max_{i,j} \|f_{ij}\| \leq \|[f_{ij}]\|$. At the same time, writing $[f_{ij}]$ as a sum of $n \times k$ matrices with only one non-zero entry, one easily obtains that $\|[f_{ij}]\| \leq nk \max_{i,j} \|f_{ij}\|$. Hence we see the norms $\|[f_{ij}]\|$ and $\max_{i,j} \|f_{ij}\|$ are topologically equivalent.

Now suppose we have a sequence of matrices $([f^s_{ij}])_{s \in \mathbb{N}}$ in the hom space $\mathcal{A}_\oplus(\{x_1,\dots,x_n\},\{y_1,\dots,y_k\})$ which is Cauchy in the matrix norm. Then it is Cauchy in the maximum norm, but then clearly each sequence f^s_{ij} of entries is Cauchy and has a limit f_{ij}. Assembling these entries we get a matrix $[f_{ij}]$ which is clearly the limit of $[f^s_{ij}]$ in the maximum norm, and hence in the matrix norm. Hence we see the hom space $\mathcal{A}_\oplus(\{x_1,..,x_n\},\{y_1,..,y_k\})$ is complete in this norm.

We prove the C^*-identity as follows: define a 'sesquilinear form'

$$\langle -, - \rangle : (\mathcal{A}(p, q_1) \oplus \cdots \oplus \mathcal{A}(p, q_n))^2 \to \mathcal{A}(p,p)$$

by $\langle [a_i], [b_i] \rangle = a_1^* b_1 + \cdots + a_n^* b_n$. Note that this form is related to the norm we defined earlier by $\|\langle [a_i], [a_i] \rangle\| = \|[a_i]\|^2$. Note also that by a standard Cauchy-Schwartz argument, $\|\langle [a_i], [b_i] \rangle\| \leq \|[a_i]\|\|[b_i]\|$, and it also follows from the definition of our form that

$$\langle M[a_i], [b_j] \rangle = \langle [a_i], M^*[b_j] \rangle \text{ for all } \begin{array}{l} [a_i] \in \mathcal{A}(p, q_1) \times \cdots \times \mathcal{A}(p, q_n), \\ [b_j] \in \mathcal{A}(p, r_1) \times \cdots \times \mathcal{A}(p, r_k), \\ M \in \mathcal{A}_\oplus(\{q_1,\dots,q_n\},\{r_1,\dots,r_k\}). \end{array}$$

Now whenever we have a matrix $[f_{ij}] \in \mathcal{A}_\oplus(\{x_1,..,x_n\},\{y_1,..,y_k\})$ and a column $[b_i] \in [\mathcal{A}(z,x_i)]$ such that $\|[b_i]\| = 1$, we have

$$\begin{aligned}\|[f_{ij}][b_i]\|^2 = \|\langle [f_{ij}][b_i],[f_{ij}][b_i]\rangle\| = \|\langle [b_i],[f^*_{ji}][f_{ij}][b_i]\rangle\| &\leq \|[f^*_{ji}][f_{ij}]\|\|[b_i]\|^2\\ &= \|[f^*_{ji}][f_{ij}]\|.\end{aligned}$$

It then follows from the definition of the norm that $\|[f_{ij}]\|^2 \leq \|[f^*_{ji}][f_{ij}]\|$. By Lemma 1.1.24 this will be enough to deduce the C^*-equality.

Finally we must show that $[f^*_{ji}][f_{ij}] \in \mathcal{A}_\oplus(\{x_1,..,x_n\},\{x_1,..,x_n\})$ has positive spectrum. Note that since for C^*-algebras the spectrum requirement follows from the C^*-identity (see e.g. [Mur90, Theorem 2.2.5]), the reasoning above shows that the endomorphism algebras of $\mathcal{A}_\oplus$ are C^*-algebras. For brevity, write

$$\{x_1,\ldots,x_n\} = \mathbf{x}, \{y_1,\ldots,y_k\} = \mathbf{y}, \{x_1,\ldots,x_n,y_1,\ldots,y_k\} = \mathbf{x}\sqcup\mathbf{y},$$

and note that the spaces $\mathcal{A}_\oplus(\mathbf{x},\mathbf{y})$, $\mathcal{A}_\oplus(\mathbf{y},\mathbf{x})$, and $\mathcal{A}_\oplus(\mathbf{x},\mathbf{x})$ can each be embedded isometrically in the C^*-algebra $\mathcal{A}_\oplus(\mathbf{x}\sqcup\mathbf{y},\mathbf{x}\sqcup\mathbf{y})$, in a way compatible with the involution and composition. Hence regarding $[f_{ij}]$ as an element of the algebra $\mathcal{A}_\oplus(\mathbf{x}\sqcup\mathbf{y},\mathbf{x}\sqcup\mathbf{y})$, we see that

$$[f^*_{ji}][f_{ij}] \in \mathcal{A}_\oplus(\mathbf{x},\mathbf{x}) \subseteq \mathcal{A}_\oplus(\mathbf{x}\sqcup\mathbf{y},\mathbf{x}\sqcup\mathbf{y})$$

is positive in $\mathcal{A}_\oplus(\mathbf{x}\sqcup\mathbf{y},\mathbf{x}\sqcup\mathbf{y})$ by Lemma 1.1.13, but then it is positive in $\mathcal{A}_\oplus(\mathbf{x},\mathbf{x})$ by Lemma 1.1.17.

This completes the proof that $\mathcal{A}_\oplus$ is a C^*-category. Finally we allay any size-theoretical concerns and show that $\mathcal{A}_\oplus$ is of the same size as $\mathcal{A}$ is: note firstly that if $\mathsf{Ob}\,\mathcal{A}$ belongs to Set_i then so does the collection of finite lists of elements of $\mathsf{Ob}\,\mathcal{A}$. Secondly we have an isomorphism of Banach spaces

$$\mathcal{A}_\oplus(\{x_1,..,x_n\},\{y_1,..,y_k\}) \cong \oplus_{1\leq i\leq n, 1\leq j\leq k} \hom_{\mathcal{A}}(x_i,y_j).$$

So as Ban_i is cocomplete for $i = 0, 1$ and direct sums are coproducts of Banach spaces, the claim follows. □

For all $\mathcal{A}$, there is an evident faithful functor $\mathcal{A} \hookrightarrow \mathcal{A}_\oplus$: it sends each object to its corresponding single-object list and each morphism to a 1×1 matrix. It has the following property:

Lemma 1.2.4. *If $\mathcal{A}$ is a C^*-category and $\mathcal{B}$ is a complex linear category which admits all finite direct sums, then every complex linear functor*

$$F : \mathcal{A} \to \mathcal{B}$$

factors through the embedding $\mathcal{A} \hookrightarrow \mathcal{A}_\oplus$ to a unique extension

$$F_\oplus : \mathcal{A}_\oplus \to \mathcal{B}.$$

If $\mathcal{B}$ is a $*$-category and F is a $*$-functor, then so is $F_\oplus$.

Proof. There is an obvious extension $F_\oplus$ on objects that sends a list of $\mathcal{A}$-objects $\{x_1, \ldots, x_n\} = \mathbf{x}$ to $F(x_1) \oplus \cdots \oplus F(x_n)$. As to the morphisms from $\mathbf{x}$ to $\mathbf{y} = \{y_1, .., y_k\}$, we define the action

$$F_\oplus : \mathcal{A}_\oplus(\mathbf{x}, \mathbf{y}) \to \mathcal{B}(F(x_1) \oplus \cdots \oplus F(x_n), F(y_1) \oplus \cdots \oplus F(y_k))$$

by $[f_{ij}] \mapsto \sum_{i,j} \kappa_j \kappa_j^* \circ F(f_{ij}) \circ \iota_i \iota_i^*$, where ι_i and κ_j are the structure maps of the direct sums of the $F(x_i)$ and the $F(y_j)$ respectively. This map is clearly linear and intertwines the involutions if they exist. The uniqueness and the final statement are left to the reader. □

It was observed in the proof of Proposition 1.2.3 that in particular, for every finite list $\{x_1, .., x_n\}$ of objects there exists a C^*-algebra

$$M_{x_1,..,x_n}(\mathcal{A}) := \mathcal{A}_\oplus(\{x_1, .., x_n\}, \{x_1, .., x_n\}),$$

and that these C^*-algebras embed into each other isometrically along list inclusions. We can therefore form direct limits along these inclusions:

Definition 1.2.5. For a small C^*-category, its *matrix algebra* Mat-$\mathcal{A}$ is the C^*-algebra obtained as the direct limit[2] of the C^*-algebras $M_{x_1,..,x_n}(\mathcal{A})$ along all (not necessarily order-preserving) inclusions $\{x_1, .., x_n\} \hookrightarrow \{y_1, \ldots y_{n+k}\}$ of *non-repeating finite lists* of $\mathcal{A}$-objects.

We note that Mat-$\mathcal{A}$ is unital if and only if $\mathcal{A}$ is unital and has a finite set of objects. Since the inclusions between the matrix algebras are isometric, the inclusions $M_{x_1,..,x_n}(\mathcal{A}) \hookrightarrow$ Mat-$\mathcal{A}$ are isometric too. It is elementary to prove that Mat-$\mathcal{A}$ is isomorphic as a C^*-algebra to the Banach space direct sum $\bigoplus_{x,y \in \mathsf{Ob}\,\mathcal{A}} \mathcal{A}(x, y)$, where this space has multiplication and involution given by 'coordinate-wise' composition and involution, and the obvious ℓ^2-norm. This algebra is described as $A_\mathcal{A}$ in [Joa03] and named the 'category algebra' in [Fer23].

Example 1.2.6. If $\mathcal{A} = \mathrm{C}^*_{\max}(\mathcal{G})$ is the maximal groupoid C^*-category on a discrete groupoid $\mathcal{G}$, then Mat-$\mathcal{A}$ is the classical *full groupoid C^*-algebra* of $\mathcal{G}$.

We will return to the matrix algebra construction in Section 3.4.

We can use the 2×2 matrix C^*-algebras $M_{x,y}(\mathcal{A})$ to generalize a number of results about C^*-algebras to the context of C^*-categories.

Lemma 1.2.7. *For any $u \in \mathcal{A}(x, y)$, there exist morphisms $v \in \mathcal{A}(x, y)$ and $w \in \mathcal{A}(x, x)$ such that $u = vw$, as well as morphisms $s \in \mathcal{A}(y, y), t \in \mathcal{A}(x, y)$ such that $u = st$.*

[2] See e.g. [Mur90, Section 6.1] for an exposition of the theory of direct limits of C^*-algebras.

Proof. Consider the element $u_m = \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} \in M_{x,y}(\mathcal{A})$: then a standard result about C^*-algebras (see for example [Ped18, p. 1.4.6] where we set $\alpha = \frac{1}{2}$) states that there is an element $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \in M_{x,y}(\mathcal{A})$ such that $u_m = v(u_m^* u_m)^{\frac{1}{4}}$. The bottom-left corner of this matrix equation reads $u = v_{21}(u^*u)^{\frac{1}{4}}$ where $v_{21} \in \mathcal{A}(x, y)$. Hence setting $v = v_{21}, w = (u^*u)^{\frac{1}{4}}$ proves the first part of the lemma. Applying this procedure instead to $u^* \in \mathcal{A}(y, x)$ proves the second part. □

Recall that if A is a C^*-algebra, an *approximate unit* is a net (u_λ) of positive elements in A such that for all λ we have $\|u_\lambda\| \leq 1$ and for all $a \in A$ we have $u_\lambda a \xrightarrow{\lambda} a$ and $au_\lambda \xrightarrow{\lambda} a$. We deduce from Lemma 1.2.7 a useful fact about the behaviour of approximate units in C^*-categories:

Corollary 1.2.8. *If (e_λ) is an approximate unit for the C^*-algebra $\mathcal{A}(x, x)$, then for all $a \in \mathcal{A}(x, y)$ we have $\|ae_\lambda - a\| \to 0$ and for all $b \in \mathcal{A}(w, x)$ we have $\|e_\lambda b - b\| \to 0$.*

Proof. Factorizing $a = vw$ as in Lemma 1.2.7, we see

$$\|ae_\lambda - a\| = \|vwe_\lambda - vw\| \leq \|v\|\|we_\lambda - w\| \xrightarrow{\lambda} 0.$$

The other case follows immediately from the first by setting $y = w$ and applying the involution. □

Lemma 1.2.9. *If $b \in \mathcal{B}(y, y)$ is a positive element and $a \in \mathcal{A}(x, y)$ is any morphism, then the inequality*

$$a^*ba \leq \|b\|a^*a$$

holds in $\mathcal{A}(x, x)$.

Proof. By Lemma 1.1.17 the element $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ is positive in $M_{x,y}(\mathcal{A})$. So applying Lemma 1.1.20 to the elements $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$ we see that

$$\begin{bmatrix} a^*ba & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} \|b\|a^*a & 0 \\ 0 & 0 \end{bmatrix}$$

in $M_{x,y}(\mathcal{A})$, but then by Lemma 1.1.17 again we see that $a^*ba \leq \|b\|a^*a$ holds in $\mathcal{A}(x, x)$. □

We can also now deduce two fundamental results on essential ideals, the first being that the relationship of being an essential ideal is transitive:

We end this section by treating briefly a sort of Cauchy completion for unital C^*-categories:

Definition 1.2.10. If $\mathcal{A}$ is a C^*-category, a *projection* on x is a morphism $p \in \mathcal{A}(x,x)$ such that $p = p^*$ and $p^2 = p$. We say p *splits* if for some $y, z \in \mathsf{Ob}\,\mathcal{A}$ there is an isomorphism $u : x \xrightarrow{\cong} y \oplus z$ such that

$$upu^{-1} = \begin{bmatrix} \mathrm{id}_y & 0 \\ 0 & 0 \end{bmatrix}$$

in which case y is termed the *image* of p. We say $\mathcal{A}$ is *idempotent complete* if every projection splits.

Proposition 1.2.11. *For any unital C^*-category $\mathcal{A}$, there is a unital C^*-category*

$$\mathcal{A}^\natural$$

termed the idempotent completion *of $\mathcal{A}$, such that*

- *Its objects are pairs (x,p) where $p \in \mathcal{A}(x,x)$ is a projection.*
- *Its morphisms from (x,p) to (y,q) are those morphisms a in $\mathcal{A}(x,y)$ such that $qa = a = pa$.*

The C^*-category $\mathcal{A}^\natural$ is idempotent complete, contains the isometric image of $\mathcal{A}$ by sending x to (x, id_x), and if $\mathcal{B}$ is an idempotent complete C^*-category, any unital C^*-functor $\mathcal{A} \to \mathcal{B}$ extends to a unital C^*-functor $\mathcal{A}^\natural \to \mathcal{B}$.

Proof. These claims are all proven in [DT14, Section 2.4], save for the one on enrichment. To allay any concerns about size enlargement, note that by definition $\mathcal{A}^\natural((x,p),(y,q))$ is the equalizer of three morphisms $\mathcal{A}(x,y) \to \mathcal{A}(x,y)$, namely postcomposition with q, precomposition with p, and the identity. So as Ban_0 and Ban_1 are complete by assumption, $\mathcal{A}^\natural$ is enriched in the same category as $\mathcal{A}$. □

1.3 The multiplier category

Multiplier algebras are a commonplace construction in C^*-algebra theory that offer a way of unitizing a C^*-algebra in a universal way. In this section we introduce a generalization of the multiplier algebra construction to C^*-categories. This 'multiplier category' was first defined in [Kan01] using different techniques than the ones we use here. It was first formulated in a form alike to the one presented here in [Vas07, Section 2].

We begin by defining the morphisms in the multiplier category.

Definition 1.3.1. If $\mathcal{A}$ is a C^*-category and $x, y \in \mathsf{Ob}\,\mathcal{A}$, a *multiplier morphism* from x to y is a pair of maps (L, R), where:

- $L : \mathcal{A}(x,x) \to \mathcal{A}(x,y)$ is a map of right $\mathcal{A}(x,x)$-modules.
- $R : \mathcal{A}(y,y) \to \mathcal{A}(x,y)$ is a map of left $\mathcal{A}(y,y)$-modules.
- For all $f \in \mathcal{A}(x,x)$, $g \in \mathcal{A}(y,y)$, we have $R(g) \circ f = g \circ L(f)$.

The maps in a multiplier morphism are automatically complex linear and bounded:

Lemma 1.3.2 (c.f. [Weg93, Proposition 2.2.8])**.** *If $L : \mathcal{A}(x,x) \to \mathcal{A}(x,y)$ and $R : \mathcal{A}(y,y) \to \mathcal{A}(x,y)$ form a multiplier morphism from x to y, then L and R are complex linear and bounded with $\|L\| = \|R\|$*

Proof. To show L is a complex linear map, note for all morphisms $b \in \mathcal{A}(y,y)$ and $a_1, a_2 \in \mathcal{A}(x,y)$ and $\lambda \in \mathbb{C}$ that

$$bL(a_1 + \lambda a_2) \quad = R(b)(a_1 + \lambda a_2) \quad = b(L(a_1) + \lambda L(a_2))$$

so letting b vary over an approximate unit and applying Corollary 2.1.9 we see that $L(a_1 + \lambda a_2) = L(a_1) + \lambda L(a_2)$.

To show L and R are bounded we appeal to the closed graph theorem (see e.g. [Con85, Theorem 12.6]) in functional analysis, which states that a map of Banach spaces is continuous (and hence bounded) if and only if its graph is closed. To show the graph of L is closed, suppose that $x_n \in \mathcal{A}(x,x)$ is a sequence converging to x and that $L(x_n) \xrightarrow{n} y$. Then for all $b \in \mathcal{A}(y,y)$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \|b(L(x) - y)\| \quad &\leq \|b(L(x) - L(x_n))\| + \|b(L(x_n) - y)\| \\ &\leq \|R(b)\|\|x - x_n\| + \|b\|\|L(x_n) - y\|. \end{aligned}$$

Hence letting $n \to \infty$ we see $b(L(x) - y) = 0$ so applying Corollary 2.1.9 we see $L(x) = y$. So the graph of L is closed, and that of R is by a similar argument.

To see the norms are equal note that for each $a \in \mathcal{A}(x,x)$,

$$\|L_x(a)\| = \sup_{b \in \mathcal{A}(y,y), \|b\| \leq 1} \|bL(a)\| = \sup_{b \in \mathcal{A}(y,y), \|b\| \leq 1} \|R(b)a\| \leq \|R\|\|a\|.$$

Hence $\|L\| \leq \|R\|$ and of course by a similar argument $\|R\| \leq \|L\|$. □

It will be useful for some purposes to have an alternative definition of multiplier morphisms.

Lemma 1.3.3. *A multiplier morphism from x to y can equivalently be defined as two arrays of maps*

$$\{L_w : \mathcal{A}(w,x) \to \mathcal{A}(w,y)\}$$

and

$$\{R_z : \mathcal{A}(y,z) \to \mathcal{A}(x,z)\},$$

where w and z range over all objects of $\mathcal{A}$, such that:

- *For $f \in \mathcal{A}(w,x), h \in \mathcal{A}(w',w)$, we always have $L_w(f) \circ h = L_{w'}(f \circ h)$.*
- *For $f \in \mathcal{A}(y,z), h \in \mathcal{A}(z,z')$, we always have $h \circ R_z(f) = R_{z'}(h \circ f)$.*
- *For all $f \in \mathcal{A}(w,x)$, $h \in \mathcal{A}(y,z)$, we have $R_w(h) \circ f = h \circ L_z(f)$.*

In addition, all maps in such an array are necessarily complex linear and bounded, and we have

$$\sup_z \|L_z\| = \|L_x\| = \|R_y\| = \sup_w \|R_w\|.$$

Proof. To go from $L : \mathcal{A}(x,x) \to \mathcal{A}(x,y)$ to an array $\{L_w : w \in \mathsf{Ob}\,\mathcal{A}\}$, use Lemma 1.2.7 on a morphism $f \in \mathcal{A}(w,x)$ to get $f = e \circ g$ for $e \in \mathcal{A}(x,x)$, then set $L_w(f) := L(e) \circ f'$. To show this is well-defined, suppose $e \circ g = e' \circ g'$ are two such factorizations of f. Then for all $u \in \mathcal{A}(y,y)$ we have

$$u \circ L_x(e) \circ g = R_y(u) \circ e \circ g = R_y(u) \circ e' \circ g' = u \circ L_x(e') \circ g'$$

so letting u vary over an approximate unit we see by Corollary 2.1.9 that $L_x(e) \circ g = L_x(e') \circ g'$. Hence $\{L_w\}$ is well-defined, and it is easily shown to satisfy the axioms above: if $f = e \circ g$ as above and $h \in \mathcal{A}(w',w)$ then we can factor e out of $f \circ h$ and see that

$$L_w(f) \circ h = L_x(e) \circ g \circ h = L_{w'}(e \circ g \circ h) = L_{w'}(f \circ h).$$

We can similarly go from R to an array $\{R_z\}$ satisfying the given requirements by factoring an endomorphism of y out of any morphism with source y. To see that the final axiom holds for $f \in \mathcal{A}(w,x)$ and $h \in \mathcal{A}(y,z)$, write $f = e \circ g$ where $e \in \mathcal{A}(x,x)$ and $h = k \circ \ell$ where $\ell \in \mathcal{A}(y,y)$. Then

$$R_w(h) \circ f = k \circ R(\ell) \circ e \circ g = k \circ \ell \circ L(e) \circ g = h \circ L_w(f)$$

For the other direction, clearly an array of multipliers as above gives a multiplier morphism by restricting to $w = x$ and $z = y$. Using the factorization lemma as above then tells us $\{L_w\}$ is determined by L_x, and similarly $\{R_z\}$ is determined by R_z: this gives a bijection between arrays and single multipliers.

It follows exactly as in the proof of the previous lemma that these maps are complex linear and bounded that $\|L_x\| = \|R_y\|$. To see that $\|L_w\| \leq \|L_x\|$, note simply that if (u_λ) is an approximate unit for $\mathcal{A}(x,x)$, then we have

$$\|L_w(f)\| = \|\lim_\lambda L_w(u_\lambda \circ f)\| = \|\lim_\lambda (L_x(u_\lambda) \circ f)\| = \lim_\lambda \|L_x(u_\lambda) \circ f\|.$$

But then as $\|u_\lambda\| \leq 1$ for each λ, we see that $\|L_x(u\lambda) \circ f\| \leq \|L_x\|\|f\|$ using contractivity. Hence $\|L_w\| \leq \|L_x\|$. The case $\|R_z\| \leq \|R_y\|$ follows similarly. □

We call the above definition the *multi-object* definition of a multiplier, and the first one the *single-object* definition. We are now ready to define the multiplier category of a C^*-category.

Proposition 1.3.4. *For a C^*-category $\mathcal{A}$, there is a C^*-category $\mathcal{MA}$ called the* multiplier category of $\mathcal{A}$*, such that*

- *The objects of $\mathcal{MA}$ are identical to those of $\mathcal{A}$.*

- *The hom-space $\mathcal{MA}(x, y)$ is the collection of multiplier morphisms (L, R) from x to y.*
- *The norm on $\mathcal{MA}(x, y)$ is defined by $\|(L, R)\| := \|L_x\| = \|R_y\|$.*
- *The involution $\mathcal{MA}(x, y) \to \mathcal{MA}(y, x)$ is defined on multipliers (in their multi-object characterization) by sending $(\{L_w\}, \{R_z\}) \in \mathcal{MA}(x, y)$ to $(\{L_u^*\}, \{R_z^*\}) \in \mathcal{MA}(y, x)$ where*

$$L_u^* : \begin{array}{l} \mathcal{A}(u, y) \to \mathcal{A}(u, x) \\ g \mapsto R_u(g^*)^* \end{array} \quad \textit{and} \quad \begin{array}{l} R_z^* : \mathcal{A}(x, z) \to \mathcal{A}(y, z) \\ f \mapsto L_z(g^*)^*. \end{array}$$

- *The composition law on $\mathcal{MA}$ is defined by setting for two multipliers $(\{L_u\}, \{R_z\}) \in \mathcal{MA}(x, y)$ and $(\{L'_u\}, \{R'_z\}) \in \mathcal{MA}(w, x)$:*

$$(\{L_u\}, \{R_z\}) \circ (\{L'_u\}, \{R'_z\}) := (\{L_u \circ L'_u\}, \{R'_z \circ R_z\}).$$

Proof. The composition is clearly contractive as

$$\sup_{u \in \mathsf{Ob}\,\mathcal{A}} \|L_u \circ L'_u\| \leq \sup_{u \in \mathsf{Ob}\,\mathcal{A}} \|L_u\| \sup_{u \in \mathsf{Ob}\,\mathcal{A}} \|L'_u\|.$$

It is clear from the single-object definition of multiplier morphisms that the space $\mathcal{MA}(x, y)$ is complete, and $\mathcal{MA}$ has obvious identity morphisms given by identity maps on hom-spaces.

Next, we show that for any morphism $T = (L, R) \in \mathcal{MA}(x, y)$ that the morphism $T^*T \in \mathcal{MA}(x, x)$ has norm $\|L\|^2$. Note that for all $a \in \mathcal{A}(x, y)$ such that $\|a\| = 1$, we have

$$\begin{aligned} \|(L^* \circ L)(a)\| &\geq \|a^* \circ (L^* \circ L)_x(a)\| \\ &= \|a^* \circ R_x(L_x(a)^*)^*\| \\ &= \|(R_x(L_x(a)^*) \circ a)^*\| \\ &= \|R_x(L_x(a)^*) \circ a\| \\ &= \|L_x(a)^* \circ L_x(a)\| \\ &= \|L(a)\|^2 \end{aligned}$$

so $\|T^*T\| = \|L^* \circ L\| \geq \|L\|^2 = \|T\|^2$, and by Lemma 1.1.24 we see that $\|T^*T\| = \|T\|^2$.

Finally we show $T^*T \in \mathcal{MA}(x, x)$ is positive. Note that the above arguments already show that $\mathcal{MA}(x, x)$ is a C^*-algebra: in fact, from the first characterization of multiplier morphisms it is clear that $\mathcal{MA}(x, x)$ is simply the multiplier algebra of the C^*-algebra $\mathcal{A}(x, x)$. By the functoriality of the involution, we also see that T^*T is a self-adjoint element of the C^*-algebra $\mathcal{MA}(x, x)$, so by Lemma 1.1.14 it has real spectrum. Suppose $\lambda \in \mathrm{Spec}(T^*T)$. Since T^*T is normal, the continuous functional calculus (see Lemma 1.1.23) states that the unital subalgebra of $\mathcal{MA}(x, x)$ generated by T^*T is isomorphic to the algebra of continuous functions on the open set $\mathrm{Spec}(T^*T)$, so taking bump functions $f_n \in C(\mathrm{Spec}(T^*T)) \subseteq \mathcal{MA}(x, x)$ around λ, we get a sequence (f_n) such that $\|f_n\| = 1$ for each n, and $(T^*T - \lambda 1)f_n \to 0$. Writing $T = (L, R)$, we get $f_n^* T^* T f_n = L(f_n)^* L(f_n)$, so multiplying the above limit on the left by f_n^*, we obtain $L(f_n)^* L(f_n) - \lambda f_n^* f_n \to 0$. But now if $\lambda < 0$, this is the sum of two positive elements, so by Lemma 1.1.18, it is again positive and by Lemma 1.1.21 we have

$$\|L(f_n)^* L(f_n) - \lambda f_n^* f_n\| \geq \max(\|L(f_n)^* L(f_n)\|, \|\lambda f_n^* f_n\|).$$

Hence as $\|\lambda f_n^* f_n\| = |\lambda|$, this contradicts $L(f_n)^* L(f_n) - \lambda f_n^* f_n \to 0$, and we must have $\lambda \geq 0$.

Finally we need to show that if $\mathcal{A}$ is enriched in Ban_i for $i = 0, 1$, then so is $\mathcal{MA}$. But note that the first definition of multiplier morphisms is essentially a description of $\mathcal{MA}(x, y)$ as a pullback of $[\mathcal{A}(x, x), \mathcal{A}(x, y)]$ and $[\mathcal{A}(y, y), \mathcal{A}(x, y)]$ over $[\mathcal{A}(x, x) \otimes \mathcal{A}(y, y), \mathcal{A}(x, y)]$. So as Ban_i is closed and complete, $\mathcal{MA}(x, y)$ lives in Ban_i. □

Just like the multipler C^*-algebra, the multiplier C^*-category receives a natural embedding from the original algebra:

Proposition 1.3.5. *There is a faithful C^*-functor*

$$\kappa_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{MA}$$

which is the identity on objects and sends each morphism a to the pair of multipliers $\kappa_{\mathcal{A}}(a) = (\ell_a, r_a)$ given by pre- and postcomposition with a. The functor $\kappa_{\mathcal{A}}$ embeds $\mathcal{A}$ as an essential ideal in $\mathcal{MA}$ and if $\mathcal{A}$ is unital, $\kappa_{\mathcal{A}}$ is also full and hence an isomorphism.

Proof. To show $\kappa_{\mathcal{A}}$ is faithful, suppose $(\ell_a, r_a) = 0$. Then in particular we have $\|r_a(a^*)\| = \|a^* a\| = 0$ so $a = 0$.

Note that $(L, R) \circ (l_a, r_a) = (l_{L(a)}, r_{L(a)})$ and $(l_a, r_a) \circ (L, R) = (l_{R(a)}, r_{R(a)})$, so $\mathcal{A}$ is an ideal. Finally note that it satisfies the definition of an essential ideal.

If $\mathcal{A}$ is unital then the multiplier axioms immediately give for any multiplier $(L, R) \in \mathcal{MA}(x, y)$ that $(L, R) = \kappa_{\mathcal{A}}(a)$ where $a = L(\mathrm{id}_x) = R(\mathrm{id}_y)$. □

The category $\mathcal{MA}$ also has a universal property akin to that of multiplier C^*-algebras:

Lemma 1.3.6. *If $\mathcal{A} \hookrightarrow \mathcal{D}$ is an embedding of $\mathcal{A}$ as an ideal of a unital C^*-category, there exists a unique extension $F : \mathcal{D} \to \mathcal{MA}$ to $\mathcal{D}$ of the embedding $\kappa_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{MA}$ and this extension is faithful if and only if $\mathcal{A}$ is essential in $\mathcal{D}$. Furthermore, among all C^*-categories containing $\mathcal{A}$ as an ideal, $\mathcal{MA}$ is unique up to isomorphism with this property.*

Proof. The C^*-functor $F : \mathcal{D} \to \mathcal{MA}$ is defined by $F(d) = (L_d, R_d)$ where we set $L_d(a) = da$ and $R_d(b) = bd$: these elements are in $\mathcal{A}$ by the ideal property. F is easily verified to be an extension of the embedding $\kappa_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{MA}$.

If F is faithful, then it acts isometrically on hom-spaces, and clearly $\mathcal{A}$ is essential in $\mathcal{D}$ since it is essential in $\mathcal{MA}$. Conversely, suppose $\mathcal{A}$ is essential in $\mathcal{D}$, and furthermore that two elements $d, d' \in \mathcal{D}(x, y)$ map to the same multiplier in $\mathcal{MA}$; then by essentiality of $\mathcal{A}$ in $\mathcal{D}$ we get $d - d' = 0$, and hence see that the functor $\mathcal{D} \to \mathcal{MA}$ is faithful.

The uniqueness part is straightforward; if $\mathcal{C}$ is another category with the stated property, we get an embedding $\mathcal{MA} \hookrightarrow \mathcal{C}$ since $\mathcal{A}$ is essential in $\mathcal{MA}$. But we also get a map $\mathcal{C} \to \mathcal{MA}$, which can easily be checked to be a left and right inverse by calling on the uniqueness. □

From now on we identify $\mathcal{A}$ with its image under $\kappa_{\mathcal{A}}$ wherever convenient. Recall from Definition 1.1.44 that the topology relative to a subcategory $\mathcal{A}$ of a C^*-category $\mathcal{B}$ is the one generated by the seminorms given by pre- and postcomposing with morphisms in $\mathcal{A}$.

Definition 1.3.7. The $\mathcal{A}$-relative topology on $\mathcal{MA}$ is called the *strict topology.*

This name directly generalizes the analogous topology for C^*-algebras (see e.g. [APT73]) and agrees with that in [AV20] and [BE21].

Lemma 1.3.8. *The morphisms in $\mathcal{A}$ are dense in $\mathcal{MA}$ in the strict topology.*

Proof. We want to show that for any multiplier $T = (L, R) \in \mathcal{MA}(x, y)$, there is a net of morphisms in $\mathcal{A}(x, y) \subseteq \mathcal{MA}(x, y)$ converging to (L, R) in the $\mathcal{A}$-relative topology. Let (e_λ) be an approximate unit for $\mathcal{A}(x, x)$. We're going to show that $\kappa_{\mathcal{A}}(L(e_\lambda)) \xrightarrow{\lambda} T$ in the $\mathcal{A}$-relative topology. We note that for all $w \in \mathsf{Ob}\,\mathcal{A}$ and $a \in \mathcal{A}(w, x)$, we have $L(e_\lambda)a = L(e_\lambda a) \xrightarrow{\lambda} L(a)$ by Corollary 1.2.8 and the continuity of L. For the other direction note that for all $z \in \mathsf{Ob}\,\mathcal{A}$ and $b \in \mathcal{A}(y, z)$, we have $bL(e_\lambda) = R(b)e_\lambda \xrightarrow{\lambda} R(b)$, again by Corollary 1.2.8. □

We note finally that the multiplier operation $\mathcal{M}(-)$ commutes with the additive closure $(-)_\oplus$ defined in Section 2 of this chapter.

Proposition 1.3.9. *There is a canonical isomorphism of C^*-categories*

$$(\mathcal{MA})_\oplus \cong \mathcal{M}(\mathcal{A}_\oplus)$$

Proof. Clearly these two categories have the same collections of objects, that is: finite lists of $\mathcal{A}$-objects. It suffices therefore, to exhibit a natural isomorphism

$$\mathcal{M}(\mathcal{A}_\oplus)(\mathbf{x}, \mathbf{y}) \cong (\mathcal{MA})_\oplus(\mathbf{x}, \mathbf{y}) \tag{1.3.1}$$

for any two lists $\mathbf{x} = \{x_1, \ldots, x_n\}, \mathbf{y} = \{y_1, \ldots, y_k\}$, which is moreover natural in the objects and intertwines the involutions: by Corollary 1.1.35 this isomorphism must then be isometric.

Using the first definition of multiplier morphisms, an element in the left set in 1.3.1 consists of a multiplier pair (L, R) where the left multiplier is a map $L : \mathcal{A}_\oplus(\mathbf{x}, \mathbf{x}) \to \mathcal{A}_\oplus(\mathbf{x}, \mathbf{y})$. An element in the right set consists of a $k \times n$ array of multipliers $(L_{i,j}, R_{i,j}) \in \mathcal{MA}(x_i, x_j)$, where $L_{i,j} : \mathcal{A}(x_i, x_i) \to \mathcal{A}(x_i, y_j)$, or alternatively, by Lemma 1.3.3, a multiplier $L^u_{x_i,y_j} : \mathcal{A}(u, x_i) \to \mathcal{A}(u, y_j)$ for each $u \in \mathsf{Ob}\,\mathcal{A}$.

So starting with an element in the right set, we can let the variable u above range over $\{x_1, \ldots, x_n\}$ to assemble a map $L : \mathcal{A}_\oplus(\mathbf{x}, \mathbf{x}) \to \mathcal{A}_\oplus(\mathbf{x}, \mathbf{y})$, and similarly from all the different right multipliers $\mathcal{A}(y_j, y_l) \to \mathcal{A}(x_i, y_l)$ we can assemble one single right multiplier $R : \mathcal{A}_\oplus(\mathbf{y}, \mathbf{y}) \to \mathcal{A}_\oplus(\mathbf{x}, \mathbf{y})$. Hence we obtain a map in the leftward direction in 1.3.1.

To go in the other direction, we start with a left multiplier from the left set $L : \mathcal{A}_\oplus(\mathbf{x}, \mathbf{x}) \to \mathcal{A}_\oplus(\mathbf{x}, \mathbf{y})$ as above. Since $L(f)h = L(fh)$, letting h vary over matrices in $M_{\mathbf{x}}(\mathcal{A})$ whose only non-zero entry is an approximate unit on the diagonal, we see L must send matrices whose only nonzero entry is in the (i, i)-th place to matrices whose only nonzero column is the i-th. Hence L restricts to multipliers $L_{i,j} : \mathcal{A}(x_i, x_i) \to \mathcal{A}(x_i, y_j)$ for $1 \leq j \leq k$, and the right multiplier R decomposes similarly, giving an element of the right set in 1.3.1. These operations are easily checked to be mutual inverses. □

1.4 Non-degenerate functors

As C^*-functors are often between non-unital categories, it is not always possible to ask for them to be unital. This throws up a variety of problems, which have to be fixed by requiring our C^*-functors to be 'approximately unital'. One way of formulating this requirement is to ask that a C^*-functor $\mathcal{A} \to \mathcal{B}$ preserve approximate units of endomorphism algebras; it is useful, however, to generalize at this point instead to C^*-functors $\mathcal{A} \to \mathcal{MB}$.

Definition 1.4.1. A C^*-functor $F : \mathcal{A} \to \mathcal{MB}$ is said to be *non-degenerate* when for every $x, y \in \mathsf{Ob}\mathcal{A}$, the subspace $F\mathcal{A}(y, y) \circ \mathcal{B}(Fx, Fy)$ (given by the span of all elements of the form $F(a) \circ b$ where $a \in \mathcal{A}(y, y), b \in \mathcal{B}(Fx, Fy)$) is norm-dense in $\mathcal{B}(Fx, Fy)$. A C^*-functor $F : \mathcal{A} \to \mathcal{B}$ is called non-degenerate if its composition with the inclusion $\kappa_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{MB}$ is non-degenerate.

We take this definition from [AV20], where non-degenerate functors form the 1-morphisms in the 2-category C^*-Lin.

We want to provide several equivalent formulations of non-degeneracy, but first we need several technical lemmas on Banach spaces to streamline the proofs.

Lemma 1.4.2. *If V and W are Banach spaces, D is a subset of V whose span is dense, and $(T_\lambda : V \to W)$ is a uniformly bounded net of operators such that $\|T_\lambda(d)\| \xrightarrow{\lambda} 0$ for each $d \in D$, then in fact $\|T_\lambda(v)\| \xrightarrow{\lambda} 0$ for each $v \in V$.*

Proof. Let $\langle D \rangle$ denote the span of V, and note that clearly from the hypothesis $\|T_\lambda(d)\| \xrightarrow{\lambda} 0$ for each $d \in \langle D \rangle$.

Let $K > 0$ be a uniform bound on the net, and for any $v \in V$ pick a sequence (d_n) of elements in $\langle D \rangle$ converging to v. For an arbitrary $\epsilon > 0$ pick an integer n such that $\|d_n - v\| \leq \frac{\epsilon}{K}$. Then for each λ we have

$$\begin{aligned} \|T_\lambda(v)\| &\leq \|T_\lambda(v - d_n)\| + \|T_\lambda(d_n)\| \\ &\leq \epsilon + \|T_\lambda(d_n)\|. \end{aligned}$$

Hence taking $\limsup_\lambda$ on both sides we obtain that $\limsup_\lambda \|T_\lambda(v)\| \leq \epsilon$, hence $\lim_\lambda \|T_\lambda(v)\| = 0$. □

Corollary 1.4.3. *Suppose V is a Banach space, D is a subset of V whose span is dense, and $(U_\lambda : V \to V)$ is a uniformly bounded net of operators such that $U_\lambda(d) \xrightarrow{\lambda} d$ for each $d \in D$. Then we have in fact $U_\lambda(v) \xrightarrow{\lambda} v$ for each $v \in V$.*

Proof. Simply take $W = V$, $T_\lambda = U_\lambda - \mathrm{id}_V$ and apply Lemma 1.4.2. □

The above lemmas will help us with many results involving the strict topology. We will also often need to extend bounded operators from a dense subset to an entire space:

Lemma 1.4.4. *If M and N are Banach spaces, M' is a dense subspace of M and $\phi' : M' \to N$ is a bounded linear map with $\|\phi'\| = L$, then ϕ' extends uniquely to a map $\phi : M \to N$ of Banach spaces with $\|\phi\| = L$. If ϕ' is an isometry then so is ϕ, and if in addition ϕ' has dense image, then ϕ is an isometric isomorphism of Banach spaces.*

Proof. The fact that ϕ extends to a map on M with the same norm is an instance of continuous linear extension, see e.g. [RS80, Theorem I.7]. To show that ϕ is an isometry when ϕ' is, simply write any $m \in M$ as a limit of a sequence of elements $m_n \in M'$, and then note

$$\|\phi(m)\| = \|\phi(\lim_n m_n)\| = \lim_n \|\phi(m_n)\| = \lim_n \|\phi'(m_n)\| = \lim_n \|m_n\| = \|m\|.$$

Finally, we show that if ϕ' has dense image, then ϕ must be surjective. Take for an arbitrary $p \in N$ a sequence (m_n) in M' such that $\phi'(m_n) \to p$. Then by the isometry property it follows that m_n is Cauchy, and letting m be the limit of (m_n) we easily see that $\phi(m) = p$. Hence ϕ is an isometric isomorphism of Banach spaces. □

We can now show the equivalence of several definitions of a non-degenerate functor:

Theorem 1.4.5. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, the following criteria on a C^*-functor $F : \mathcal{A} \to \mathcal{MB}$ are equivalent:*

1. *F is non-degenerate.*
2. *For all $x, y \in \mathsf{Ob}\mathcal{A}$, the subspace $\mathcal{B}(Fx, Fy) \circ F\mathcal{A}(x, x)$ is norm-dense in $\mathcal{B}(Fx, Fy)$.*
3. *If $x, y \in \mathsf{Ob}(\mathcal{A})$ and $(u_\lambda)_{\lambda \in \Lambda}$ is an approximate unit for $\mathcal{A}(y, y)$, then for each morphism $b \in \mathcal{B}(Fx, Fy)$ we have $F(u_\lambda) \circ b \to b$ in the norm.*

4. *If $x, y \in \mathsf{Ob}(\mathcal{A})$ and $(u_\lambda)_{\lambda\in\Lambda}$ is an approximate unit for $\mathcal{A}(x,x)$, then for each morphism $b \in \mathcal{B}(Fx, Fy)$ we have $b \circ F(u_\lambda) \to b$ in the norm.*

5. *F extends (uniquely) to a unital C^*-functor $\bar{F} : \mathcal{MA} \to \mathcal{MB}$, which on norm-bounded subsets is in addition continuous with respect to the strict topologies on $\mathcal{MA}$ and $\mathcal{MB}$, as defined in Definition 1.3.7.*

Proof. Criteria 1 and 2, as well 3 and 4 are obviously equivalent using the involution.

Criterion 3 clearly implies 1 since $F(u_\lambda) \circ b \in F\,\mathcal{A}(y,y) \circ \mathcal{B}(Fx, Fy)$. To see that 1 implies 3, note simply that if $a \in \mathcal{A}(y,y)$ and $b \in \mathcal{B}(Fx, Fy)$, then $F(u_\lambda) \circ F(a) \circ b = F(u_\lambda \circ a) \circ b \xrightarrow{\lambda} F(a) \circ b$ since F is norm-continuous (Proposition 1.1.34).

Criterion 5 clearly implies 3, since approximate units are nets strictly converging to the identity. It remains finally to show that criteria 1 through 4 together imply 5:

We use in this proof the single-object definition of multipliers (see Definition 1.3.1). Take any multiplier $T = (L, R) \in \mathcal{MA}(x, y)$. We define the left component of $\bar{F}(T) = (\bar{L}, \bar{R})$ as follows. For any morphism of the form $F(a) \circ b \in F\mathcal{A}(x,x) \circ \mathcal{B}(Fx, Fx)$, we set

$$\bar{L}(F(a) \circ b) = F(L(a)) \circ b \in \mathcal{B}(Fx, Fy).$$

To show this gives a well-defined map $F\mathcal{A}(x,x) \circ \mathcal{B}(Fx, Fx) \to \mathcal{B}(Fx, Fy)$, suppose $F(a)\circ b = F(a')\circ b'$ and let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathcal{A}(y,y)$. For all $\lambda \in \Lambda$ we have

$$\begin{aligned} F(u_\lambda) \circ F(L(a)) \circ b &= F(R(u_\lambda)) \circ F(a) \circ b \\ &= F(R(u_\lambda)) \circ F(a') \circ b' \\ &= F(u_\lambda) \circ F(L(a')) \circ b'. \end{aligned}$$

Hence by criterion 3, taking limits we get $F(L(a)) \circ b = F(L(a')) \circ b'$. The equality $\bar{L}(F(a) \circ b) = \lim_\lambda F(R(u_\lambda)) \circ F(a) \circ b$ also gives $\|\bar{L}\| \leq \|R\|$. Hence $\bar{L}$ is a bounded map defined on a subset of $\mathcal{B}(Fx, Fx)$ which is norm-dense by criterion 1. So by Lemma 1.4.4 it extends uniquely to a map $\mathcal{B}(Fx, Fx) \to \mathcal{B}(Fx, Fy)$. The property defining right $\mathcal{B}(Fx, Fx)$-module morphisms clearly holds on the norm-dense subset $F\mathcal{A}(x,x) \circ \mathcal{B}(Fx, Fx)$, so again by uniform continuity it holds on all of $\mathcal{B}(Fx, Fx)$. Hence $\bar{L}$ gives a valid left multiplier $\mathcal{B}(Fx, Fx) \to \mathcal{B}(Fx, Fy)$.

Similarly for $b \circ F(a) \in \mathcal{B}(Fy, Fy) \circ F\mathcal{A}(y,y)$, we set

$$\bar{R}(b \circ F(a)) = b \circ F(R(a))$$

and then by criteria 2 and 4 this extends to a right multiplier $\bar{R} : \mathcal{B}(Fy, Fy) \to \mathcal{B}(Fx, Fy)$. The final axiom stating $\bar{R}(b) \circ b' = b \circ \bar{L}(b')$ for $b \in \mathcal{B}(Fy, Fy)$ and $b' \in \mathcal{B}(Fx, Fx)$ is obvious when $b \in F\mathcal{A}(x,x) \circ \mathcal{B}(Fx, Fx)$ and $b' \in \mathcal{B}(Fy, Fy) \circ F\mathcal{A}(y,y)$ and again obtained by uniform continuity of $\bar{L}, \bar{R}$ on the closure. This concludes the verification that $(\bar{L}, \bar{R})$ is a valid multiplier in $\mathcal{MB}(F(x), F(y))$

The assignment $(L,R) \mapsto (\bar{L},\bar{R})$ is easily checked to be complex linear and compatible with the involutions and compositions, hence it extends to a C^*-functor $\bar{F} : \mathcal{MA} \to \mathcal{MB}$, which clearly restricts to F on $\mathcal{A}$.

We show next that this extension is strictly continuous on norm-bounded sets. Suppose $(T_\gamma = (L_\gamma, R_\gamma))_{\gamma\in\Gamma}$ is a net of multipliers in $\mathcal{MA}(x,y)$ whose norm is bounded by K and which strictly converge to 0; that is, for every $a \in \mathcal{A}(x,x)$ we have $L_\gamma(a) \xrightarrow{\gamma} 0$ and for every $a \in \mathcal{A}(y,y)$ we have $R_\gamma(a) \xrightarrow{\gamma} 0$. We have to show for any $b \in \mathcal{B}(Fx,Fx)$ that $\bar{L}_\gamma(b) \xrightarrow{\gamma} 0$ and for any $b \in \mathcal{B}(Fy,Fy)$ that $\bar{R}_\gamma(b) \xrightarrow{\gamma} 0$.

Note that for any element $b = F(a)\circ b' \in F\mathcal{A}(x,x)\circ\mathcal{B}(Fx,Fx)$, we have that $\bar{L}_\gamma(b) = \bar{L}_\gamma(F(a)\circ b') = F(L_\gamma(a))\circ b' \to 0$, and by criterion 1, elements of this form are norm-dense in $\mathcal{B}(Fx,Fx)$. Furthermore we have already established that $\|\bar{L}_\gamma\| \leq \|L_\gamma\|$ for each γ, so as (T_λ) is uniformly bounded, so is the net $\bar{L}_\gamma$. Hence we can apply Lemma 1.4.2 and see that $\bar{L}_\gamma(b) \xrightarrow{\gamma} 0$.

The case of the right multiplier follows similarly from criterion 2. Hence we see that $\bar{F}(T_\gamma)$ goes to 0 in the strict topology. By linearity this proves the case for nets converging strictly to an arbitrary limit and we conclude $\bar{F}$ is strictly continuous on bounded subsets.

Finally, to show this extension is unique, note that by Lemma 1.3.8, any multiplier in $\mathcal{MA}(x,y)$ can be strictly approximated by a bounded net of elements in $\mathcal{A}$, so any other extension of F to $\mathcal{MA}$ which is strictly continuous on bounded subsets must be equal to $\bar{F}$. □

Remark 1.4.6. It follows from the final criterion that if $\mathcal{A}$ is unital, a non-degenerate functor $\mathcal{A} \to \mathcal{MB}$ is simply a unital functor $\mathcal{A} \to \mathcal{MB}$. To see this, note that the strict topology on $\mathcal{MA} \cong \mathcal{A}$ is just the norm topology, any C^*-functor is norm-continuous, and norm convergence in $\mathcal{MB}$ implies strict convergence. So as all C^*-functors are norm-continuous the fourth criterion simply describes a unital functor $\mathcal{A} \to \mathcal{MB}$.

Remark 1.4.7. It is clear from the third criterion that if $G : \mathcal{A} \to \mathcal{MB}$ and $F : \mathcal{B} \to \mathcal{MC}$ are two non-degenerate functors, we can compose their lifts to get $\bar{F} \circ \bar{G} : \mathcal{MA} \to \mathcal{MC}$. This composition is again strictly continuous on bounded subsets as certainly $\bar{F}$ sends bounded subsets to bounded subsets (it is a C^*-functor so norm-decreasing), hence it restricts to a non-degenerate functor, which equals $\bar{F} \circ G$ by uniqueness of lifts.

Example 1.4.8. The embedding $\kappa_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{MA}$ is non-degenerate: we can see this by combining criterion 3 in Theorem 1.4.5 with Corollary 1.2.8.

Example 1.4.9. Any full and faithful C^*-functor $F : \mathcal{A} \to \mathcal{B}$ is non-degenerate; it preserves approximate units of endomorphism algebras so criterion 3 is satisfied. Hence for example, the embedding $\mathcal{A} \hookrightarrow \mathcal{A}_\oplus$ is non-degenerate.

This theory allows us to talk about equivalences between non-unital C^*-categories:

Definition 1.4.10. A non-degenerate C^*-functor $F : \mathcal{A} \to \mathcal{MB}$ is termed a *(unitary) multiplier equivalence* if there is a C^*-functor $G : \mathcal{B} \to \mathcal{MA}$ such that $\bar{F}$ and $\bar{G}$ are inverse (unitary) equivalences between $\mathcal{MA}$ and $\mathcal{MB}$. A C^*-functor $F : \mathcal{A} \to \mathcal{B}$ is termed a (unitary) multiplier equivalence if its composition with the embedding $\kappa_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{MB}$ is non-degenerate and a (unitary) multiplier equivalence.

We end the section with a slight categorification of Theorem 1.4.5. First we need to expand the terminology from Proposition 1.1.46 a little.

Definition 1.4.11. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, we denote by

$$\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, \mathcal{MB})$$

the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{MB})$ spanned by the non-degenerate functors and by

$$\mathrm{Fun}^{\mathrm{strict}}(\mathcal{MA}, \mathcal{MB})$$

the subcategory of $\mathrm{Fun}(\mathcal{MA}, \mathcal{MB})$ spanned by the unital functors which are strictly continuous on bounded subsets.

We can then rephrase Theorem 1.4.5 as follows:

Proposition 1.4.12. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories and $\kappa = \kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{MA}$ is the embedding from Proposition 1.3.5, then the precomposition functor*

$$- \circ \kappa : \mathrm{Fun}^{\mathrm{strict}}(\mathcal{MA}, \mathcal{MB}) \to \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, \mathcal{MB})$$

is an isomorphism of categories.

Proof. The functor $- \circ \kappa$ is surjective on objects by the extension criterion in Theorem 1.4.5, and injective on objects by the uniqueness of the extension.

To show that it is full, we need to show for two non-degenerate functors $F, F' \in \mathsf{Ob}\,\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, \mathcal{MB})$ and a bounded transformation $\epsilon : F \Rightarrow F'$, that $\epsilon = \eta \circ \kappa : F \Rightarrow F'$ for some $\eta : \overline{F} \Rightarrow \overline{F'}$, where $\overline{F}$ and $\overline{F'}$ are the unique lifts of F and F' through κ and $\eta \circ \kappa$ is the whiskering of η through κ.

We can simply set $\eta_x = \epsilon_x$ for all $x \in \mathsf{Ob}\,\mathcal{A}$, but we must show that η_x is natural with respect to multiplier morphisms. Consider a naturality diagram

$$\begin{array}{ccc} F(x) & \xrightarrow{F(T)} & F(y) \\ \downarrow{\scriptstyle \eta_x} & & \downarrow{\scriptstyle \eta_y} \\ F'(x) & \xrightarrow{F'(T)} & F'(y) \end{array}$$

which we know commutes when $T = \kappa(a)$ for some $a \in \mathcal{A}(x, y)$. But then note by Lemma 1.3.8 we can strictly approximate T by a bounded net of multipliers $\kappa(a_\lambda)$, and then by Lemma 1.1.45

$$\eta_y F(\kappa(a_\lambda)) - F'(\kappa(a_\lambda))\eta_x \xrightarrow{\lambda} \eta_y F(T) - F'(T)\eta_x$$

strictly in $\mathcal{MB}(F(x), F'(y))$. So as the left-hand side is zero for all λ, the right-hand side is zero and the diagram commutes for arbitrary T.

The functor is obviously faithful since the whiskered natural transformation ϵ has the same morphisms as η. $\square$

There is in fact a tensor product of C^*-categories that fits into a tensor-hom adjunction with the functor $\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, -)$: to stop this chapter from getting too long we postpone discussion of this adjunction to Appendix A.

1.5 Multiplier direct sums

We end the chapter with a discussion of multiplier direct sums. This construction solves several problems at once: the first is that our definition of direct sums from Section 1.2 involves identities on objects, which may not exist in a general C^*-category. The second is that it allows us to sum an infinite number of objects using the strict topology on the multiplier category. The definition below is taken from [AV20], where it is called simply a 'direct sum'.

In the definition below, recall that a *multiset* is simply a set in which elements are allowed to appear once.

Definition 1.5.1. Given a C^*-category $\mathcal{A}$ and a multiset $\{x_i : i \in I\}$ of objects of $\mathcal{A}$, we say $\bigoplus_{i\in I} x_i \in \mathsf{Ob}\,\mathcal{A}$ is a *multiplier direct sum* of $\{x_i : i \in I\}$ with structure maps $\iota_i \in \mathcal{MA}(x_i, \bigoplus_{i\in I} x_i)$ if:

- All maps ι_i are isometries.
- The net of partial sums $\sum_{j\in J} \iota_j \iota_j^*$ for finite sub-multisets $J \subseteq I$ converges to $\mathrm{id}_{\bigoplus_{i\in I} x_i}$ in the strict topology, as defined in Definition 1.1.44.

Notice firstly that in the case that I is a finite multiset, the above data (together with a choice of total order on the set) just defines a finite direct sum in $\mathcal{MA}$.

Remark 1.5.2. Recall that when $\mathcal{A}$ is unital, then $\mathcal{MA} \cong \mathcal{A}$ and the strict topology is in fact the one given by the norm. But in any normed vector space an uncountable sum can converge in the norm only if all but countably many of its summands are zero: see for instance [HN01, p.136]. Therefore in this case the second requirement can be true only if all but countably many of the morphisms $\iota_i \iota_i^*$ are zero, meaning all but countably many of the objects are zero. Furthermore, even countable infinite direct sums of Hilbert spaces as classically conceived, for example $\ell^2(\mathbb{N})$, are not multiplier direct sums in Hilb. To see this, note simply that the net of partial sums in Definition 1.5.1 is the standard net of n-rank projections in $\mathcal{K}(\ell^2(\mathbb{N}), \ell^2(\mathbb{N}))$, which converges to the identity of $\mathcal{L}(\ell^2(\mathbb{N}), \ell^2(\mathbb{N}))$ in the strong* or pointwise topology, but not in the norm. They are, however multiplier direct sums in the category $\mathcal{K}\mathsf{Hilb}$ of Hilbert spaces with compact operators as morphisms: we will prove this later (see Proposition 2.4.10).

It is easy to see that a non-degenerate functor $F : \mathcal{A} \to \mathcal{MB}$ preserves multiplier direct sums.

Proposition 1.5.3. *If $F : \mathcal{A} \to \mathcal{MB}$ is a non-degenerate functor, it preserves multiplier direct sums: to be precise, its underlying map $\mathsf{Ob}\,\mathcal{A} \to \mathsf{Ob}\,\mathcal{B}$ preserves multiplier direct sums objectwise and its unique, unital, extension which is strictly continuous on bounded subsets $\bar{F} : \mathcal{MA} \to \mathcal{MB}$ preserves their structure maps.*

Proof. This follows immediately from Definition 1.5.1 and the fifth criterion in Theorem 1.4.5. □

Recall from Section 2 of this chapter the awkward fact that the additive hull of a non-unital C^*-category does not possess all direct sums. The notion of a multiplier direct sums allows us to amend this defect:

Lemma 1.5.4. *The C^*-category $\mathcal{A}_\oplus$ admits all finite multiplier direct sums.*

Proof. The structure map $(L_i, R_i) = \iota_i \in \mathcal{M}(\mathcal{A}_\oplus)(\{x_i\}, \{x_1, \ldots, x_n\})$ is given on one coordinate by the map $L_i : \mathcal{A}_\oplus(\{x_i\}, \{x_i\}) \to \mathcal{A}_\oplus(\{x_i\}, \{x_1, \ldots, x_n\})$ that sends a morphism f to the column with f in the i-th place and zeros elsewhere. The right multiplier works similarly. It is elementary to verify that these multipliers exhibit $\{x_1, \ldots, x_n\}$ as the multiplier direct sum of the x_i. □

In fact, we can now formulate a sort of universal property for $\mathcal{A}_\oplus$:

Lemma 1.5.5. *If $\mathcal{B}$ is a C^*-category closed under finite multiplier direct sums, any non-degenerate C^*-functor*

$$F : \mathcal{A} \to \mathcal{MB}$$

extends uniquely to a non-degenerate functor

$$F_\oplus : \mathcal{A}_\oplus \to \mathcal{MB}.$$

Proof. To define $F_\oplus$, consider the unique unital extension $\bar{F} : \mathcal{MA} \to \mathcal{MB}$ which is strictly continuous on bounded subsets. As $\mathcal{MB}$ is closed under finite direct sums, this extends uniquely to a C^*-functor $\bar{F}_\oplus : (\mathcal{MA})_\oplus \to \mathcal{MB}$ by Lemma 1.2.4. By Proposition 1.3.9 we can also write this as a functor $\bar{F}_\oplus : \mathcal{M}(\mathcal{A}_\oplus) \to \mathcal{MB}$. It is easy to deduce that $\bar{F}_\oplus$ is unital and strictly continuous on bounded subsets from the fact that $\bar{F}$ is, hence it restricts to a non-degenerate functor $F_\oplus : \mathcal{A}_\oplus \to \mathcal{MB}$. □

Chapter 2

Hilbert modules over C^*-categories

Hilbert modules over general C^*-algebras, also called *Hilbert C^*-modules*, were first defined in [Pas73]. They generalize Hilbert spaces and vector bundles over topological spaces, and have long been recognized as the 'correct' notion of a module over a C^*-algebra. They play a fundamental role in noncommutative geometry, in particular KK-theory and quantum group theory.

In this chapter we develop the theory of Hilbert modules over C^*-categories, first investigated in [Mit02a] and [Joa03]. After proving some basic results and taking a detour on set-theoretic matters, we characterize the C^*-category of Hilbert modules over a given C^*-category, proving in addition that it is equivalent to the multiplier category of a subcategory called the *compact operators*. We end with some results on Hilbert modules over the additive closure and an approximate projectivity result that relates arbitrary Hilbert modules to finitely generated free ones.

For the rest of this thesis we assume that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc. are *locally small* C^*-categories, although much of the theory in this chapter can be developed for large C^*-categories. Recall that we denote by $\mathsf{Vect}_{\mathbb{C},0}$ the category of small complex vector spaces.

2.1 Hilbert modules and their size

We begin with a 'naive' definition of a pre-Hilbert module over a C^*-category. These naive modules are termed 'large' as they do not satisfy a size requirement that we will later see is important.

Definition 2.1.1. If $\mathcal{A}$ is a (locally small) C^*-category, a *large right pre-Hilbert $\mathcal{A}$-module* is a complex linear functor

$$E : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$$

equipped with sesquilinear 'inner products'

$$\langle -, - \rangle : E(y) \times E(x) \to \mathcal{A}(x, y)$$

for each pair $x, y \in \mathsf{Ob}\,\mathcal{A}$. Furthermore the inner products must satisfy for all $e \in E(y), f \in E(x)$ and $g \in \mathcal{A}(w, x)$ the following:

- $\langle e, f \rangle = \langle f, e \rangle^*$.
- $\langle e, f \rangle \circ g = \langle e, f \cdot g \rangle$, where $f \cdot g := E(g)(f)$.
- $\langle e, e \rangle$ is a positive element of the C^*-algebra $\mathcal{A}(y, y)$.

We say E is a *large right Hilbert $\mathcal{A}$-module* if in addition it satisfies

- $\langle e, e \rangle = 0$ only if $e = 0$.
- For each $x \in \mathsf{Ob}\,\mathcal{A}$, the space $E(x)$ is complete in the norm

$$\|e\| := \sqrt{\|\langle e, e \rangle\|_{\mathcal{A}(x,x)}}.$$

Note that for any large pre-Hilbert module, a sort of conjugate linearity in the first argument follows by combining the first and second axioms:

$$\langle f \cdot g, e \rangle = \langle e, f \cdot g \rangle^* = (\langle e, f \rangle \circ g)^* = g^* \circ \langle e, f \rangle^* = g^* \circ \langle f, e \rangle.$$

We will not be concerned for the moment with left Hilbert $\mathcal{A}$-modules, though if desired these can easily be defined as right Hilbert $\mathcal{A}^{\mathrm{op}}$-modules; alternatively, one can modify the above axioms by making E a covariant functor (so that elements of $\mathcal{A}$ act on the left), requiring that the product $\langle -, - \rangle$ be linear in the first variable and conjugate linear in the second, and that $\langle a \cdot e, f \rangle = a \circ \langle e, f \rangle$.

Example 2.1.2. If $\mathcal{A}$ has only one object x, the large Hilbert modules over $\mathcal{A}$ are exactly the right Hilbert modules over the C^*-algebra $\mathcal{A}(x, x)$ (see Definition 1.1.30). In particular, complex Hilbert spaces are Hilbert modules over the one-object C^*-category with hom-space $\mathbb{C}$.

Example 2.1.3. For any C^*-category $\mathcal{A}$ and any object x of $\mathcal{A}$, there is a 'representable' large Hilbert module

$$h_x : \begin{array}{l} \mathcal{A} \to \mathsf{Vect}_{\mathbb{C},0} \\ y \mapsto \mathcal{A}(y, x) \end{array}$$

with inner product defined for all morphisms $a \in h_x(y), b \in h_x(z)$ by

$$\langle a, b \rangle_{h_x} := a^* b \in \mathcal{A}(z, y).$$

The action of a C^*-category on a Hilbert module is automatically continuous:

Lemma 2.1.4. *If E is a large right pre-Hilbert $\mathcal{A}$-module, then for all $e \in E(x)$ and $a \in \mathcal{A}(w,x)$ the inequality*

$$\|e \cdot a\| \leq \|e\|\|a\|$$

holds. In particular, the action of $\mathcal{A}$ on E is continuous.

Proof. Note that we have

$$\begin{aligned} \|e \cdot a\|^2_{E(w)} &= \|\langle e \cdot a, e \cdot a\rangle\|_{\mathcal{A}(w,w)} && = \|a \circ \langle e, e\rangle \circ a^*\|_{\mathcal{A}(w,w)} \\ &\leq \|a\|_{\mathcal{A}(w,x)}\|\langle e, e\rangle\|_{\mathcal{A}(x,x)}\|a^*\|_{\mathcal{A}(x,w)} && = \|a\|^2_{\mathcal{A}(w,x)}\|e\|^2_{E(x)}, \end{aligned}$$

where we have used subscripts to specify norms. Hence taking square roots we obtain our result. □

There is also a Cauchy-Schwartz lemma we can prove:

Proposition 2.1.5. *If E is a large right pre-Hilbert $\mathcal{A}$-module, then for any pair $e \in E(x)$ and $f \in E(y)$, the following Cauchy-Schwartz inequality holds in the C^*-algebra $\mathcal{A}(x,x)$:*

$$\langle e, f\rangle \circ \langle f, e\rangle \leq \|\langle f, f\rangle\|\langle e, e\rangle.$$

Proof. We proceed as in the proof of [Lan95, Proposition 1.1]. Note that we can assume without loss of generality that $\|\langle f, f\rangle\| = 1$, as then for any f, applying the lemma to $f' = \frac{f}{\sqrt{\|\langle f,f\rangle\|}}$ will do the trick. Now, for any $a \in \mathcal{A}(x,y)$ note that by positivity,

$$\begin{aligned} 0 &\leq \langle e - f \cdot a, e - f \cdot a\rangle \\ &= \langle e, e\rangle - \langle e, f\rangle \circ a - a^* \circ \langle f, e\rangle + a^* \circ \langle f, f\rangle \circ a \\ &\leq \langle e, e\rangle - \langle e, f\rangle \circ a - a^* \circ \langle f, e\rangle + a^* \circ a. \end{aligned}$$

Where in the last inequality we have applied Lemma 1.2.9. Then setting $a = \langle f, e\rangle$, the final two terms cancel out and we get that

$$0 \leq \langle e, e\rangle - \langle e, f\rangle \circ \langle f, e\rangle = \|\langle f, f\rangle\|\langle e, e\rangle - \langle e, f\rangle \circ \langle f, e\rangle$$

proving the required inequality. □

Corollary 2.1.6. *If E is a large right pre-Hilbert $\mathcal{A}$-module, then for any pair $e \in E(x)$ and $f \in E(y)$ the following Cauchy-Schwartz inequality holds in $\mathbb{R}$:*

$$\|\langle f, e\rangle\| \leq \|f\|\|e\|.$$

Proof. Applying norms to Proposition 2.1.5, we get

$$\|\langle f, e\rangle\|^2 = \|\langle f, e\rangle^* \circ \langle f, e\rangle\| = \|\langle e, f\rangle \circ \langle f, e\rangle\| \leq \|\langle f, f\rangle\|\|\langle e, e\rangle\| = \|f\|^2\|e\|^2.$$

□

We apply these results to give a procedure for turning a pre-Hilbert module into a Hilbert module:

Proposition 2.1.7. *If $E : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ is a large right pre-Hilbert $\mathcal{A}$-module, then there is a large right Hilbert $\mathcal{A}$-module*

$$\bar{E} : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$$

obtained at every $x \in \mathsf{Ob}\,\mathcal{A}$ by quotienting out all $e \in E(x)$ such that $\langle e, e\rangle = 0$ and completing the resulting normed vector space.

Proof. Note first that Corollary 2.1.6 implies that for $e, f \in E(x)$ we have $\|e + f\| \leq \|e\| + \|f\|$, therefore $\|\cdot\|$ is a seminorm on each space $E(x)$. Denote by $E_0(x)$ the subspace of elements e such that $\|e\| = 0$. Then as

$$\langle e + e', f + f'\rangle = \langle e, f\rangle + \langle e', f\rangle + \langle e, f'\rangle + \langle e', f'\rangle,$$

we see by Corollary 2.1.6 that

$$\langle e + E_0(x), f + E_0(x)\rangle := \langle e, f\rangle$$

is a well-defined inner product on $E(x)/E_0(x)$. Finally Lemma 2.1.4 tells us the action of $\mathcal{A}$ descends to $E(x)/E_0(x)$, and if we let $\bar{E}(x)$ be the completion of $E(x)/E_0(x)$ in the above norm, the action of $\mathcal{A}$ can be unambiguously extended to $\bar{E}(x)$ by Lemma 2.1.4. We're going to show that $\bar{E} : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ is a large right Hilbert $\mathcal{A}$-module.

We define an inner product on $\bar{E}(y) \times \bar{E}(x)$ by

$$\langle \lim_n e_n, \lim_n f_n\rangle := \lim_n \langle e_n, f_n\rangle$$

for all Cauchy sequences e_n and f_n in $E(x)/E_0(x)$ and $E(y)/E_0(y)$ respectively. To see that this limit exists, note that

$$\langle e_n, f_n\rangle - \langle e_m, f_m\rangle = \langle e_n - e_m, f_n\rangle + \langle e_m, f_n - f_m\rangle$$

and by assumption e_n, f_n are bounded so $\langle e_n, f_n\rangle$ is a Cauchy sequence in $\mathcal{A}(y, x)$, hence has a limit. To see that this limit is independent of the chosen sequence, suppose $\lim_n e_n = \lim_n e'_n$ and $\lim_n f_n = \lim_n f'_n$, we apply Corollary 2.1.6 to both terms on the right-hand side of the previous equation and hence see the product is well-defined. The product is positive definite since it is positive definite on the dense subset $E(x)/E_0(x)$ and by Lemma 1.1.22 a limit of positive elements is again positive. Finally $\bar{E}(x)$ is complete in the norm induced by this inner product, since it agrees with the original norm by definition.

The involution axiom $\langle e, f\rangle = \langle f, e\rangle^*$ holds on $\overline{E}$ since it holds on E/E_0 and the involution is norm-preserving, and similarly the naturality $\langle f, e \cdot g\rangle = \langle f, e\rangle \circ g$ follows from the continuity of the action. This completes the proof that $\bar{E}$ is a right Hilbert module over $\mathcal{A}$. $\square$

We follow by establishing a few basic density results that will come in handy later:

Lemma 2.1.8. *If E is a large right Hilbert $\mathcal{A}$-module, then for each object x of $\mathcal{A}$, the subspace*

$$E(x) \cdot \langle E(x), E(x) \rangle \subseteq E(x)$$

given by the span of elements of the form $e_1 \cdot \langle e_2, e_3 \rangle$ is dense in $E(x)$.

Proof. Note it follows from the Hilbert module axioms that $E(x)$ is a Hilbert module over the C^*-algebra $\mathcal{A}(x,x)$, so this follows from the analogous lemma for C^*-algebras, see e.g. [Lan95, p.5]. □

Corollary 2.1.9. *If (u_λ) is an approximate unit for $\mathcal{A}(x,x)$ and $e \in E(x)$, we have $e \cdot u_\lambda \xrightarrow{\lambda} e$.*

Proof. Note that *a fortiori* from the above lemma we see that the $E(x) \cdot \mathcal{A}(x,x)$ is always dense in $E(x)$. But clearly $(e \cdot a) \cdot u_\lambda = e \cdot (a u_\lambda) \xrightarrow{\lambda} e \cdot a$, and we deduce the corollary using Lemma 1.4.2. □

We can now show that every element of a Hilbert module is in the image of the action of some $\mathcal{A}$-morphism. This is done by combining Corollary 2.1.9 with the following classical result:

Theorem 2.1.10 (Cohen-Hewitt Theorem, see [CLM79, p. 108])**.** *Suppose A is a Banach algebra with approximate unit (u_λ), and M is a right Banach A-module (that is, an A-module with a complete norm making the action of A continuous). Suppose $m \in M$ is such that $m \cdot u_\lambda \xrightarrow{\lambda} m$. Then there exist $n \in M$ and $a \in A$ such that $m = n \cdot a$.*

Corollary 2.1.11. *If $\mathcal{A}$ is a C^*-category and E is a large right Hilbert $\mathcal{A}$-module, then for any $x \in \mathsf{Ob}\,\mathcal{A}, e \in E(x)$ there exists $f \in E(x), a \in \mathcal{A}(x,x)$ such that $e = f \cdot a$.*

We end with a discussion of the *size* of Hilbert modules. Studying Definition 2.1.1, one quickly notices that unless $\mathcal{A}$ has a *set* of objects, we cannot expect the union of all component spaces $E(x)$ to form a set. This will not be a problem for our theory as long as we ask that there is some set of elements *generating* E across all objects of $\mathcal{A}$, as in the theory of small presheaves for locally small categories (see e.g. [nLa23c]). The author first became aware of this notion in the context of Hilbert modules through Benjamin Dünzinger's master's thesis, and later discovered it had also appeared in Simon Henry's paper [Hen15].

Below, we use *set* to mean a set in Set_0 (for example the hom-sets of $\mathcal{A}$) and *collection* to mean a set in Set_1 (which contains $\mathsf{Ob}\,\mathcal{A}$ and the collection of all $\mathcal{A}$-morphisms). We begin with some results on large Hilbert modules.

Definition 2.1.12. For a large right Hilbert $\mathcal{A}$-module $E : \mathcal{A}^{\text{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ and a collection S of elements $s \in E(x_s)$ (where each $x_s \in \mathsf{Ob}\,\mathcal{A}$) we denote by

$$\langle S \rangle : \mathcal{A}^{\text{op}} \to \mathsf{Vect}_{\mathbb{C},0}$$

the smallest subfunctor of E containing S which is closed under scalar multiplication, addition, and action from morphisms in $\mathcal{A}$, and such that its value on each object is closed in the norm inherited from E. Put more succinctly, $\langle S \rangle$ is the smallest sub-Hilbert module of E containing S. We say that S *generates* E when $\langle S \rangle = E$.

Lemma 2.1.13. *For E and S as above, the module $\langle S \rangle$ exists and is given at $x \in \mathsf{Ob}\,\mathcal{A}$ by the norm closure of the space of finite linear combinations $\Sigma_{i=1}^{n} \mu_i s_i + \Sigma_{j=1}^{m} s_j \cdot a_j$, where for all i and j we have $s_i \in S \cap E(x)$, $\mu_i \in \mathbb{C}$, $s_j \in S$, and $a_j \in \mathcal{A}(x, x_{s_j})$.*

Proof. Call the second module in the above statement $[S]$. Note firstly that any finite linear combination of the type defined should be in $\langle S \rangle$, which is also norm-closed, giving us that $[S] \subseteq \langle S \rangle$. Conversely, clearly $[S]$ is a Hilbert submodule of E containing S, giving that $\langle S \rangle \subseteq [S]$, and hence we see that $[S] = \langle S \rangle$. $\square$

Definition 2.1.14. A large right $\mathcal{A}$-Hilbert module $E : \mathcal{A}^{\text{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ is termed a *Hilbert module* if there is a *set* S of elements $s \in E(x_s)$ (where each $x_s \in \mathsf{Ob}\,\mathcal{A}$) such that S generates E.

This definition was first given in [Hen15, Definition 2.2.21]. We easily show that the examples of large Hilbert modules we have looked at so far are in fact Hilbert modules:

Lemma 2.1.15. *If $\mathcal{A}$ is a small C^*-category then every large right Hilbert $\mathcal{A}$-module E is in fact a Hilbert module.*

Proof. As $\mathcal{A}$ has a set of objects we can simply let S be the set $\bigcup_{x \in \mathsf{Ob}\,\mathcal{A}} E(x)$ of all elements of E. $\square$

Lemma 2.1.16. *If $\mathcal{A}$ is locally small then for each $x \in \mathsf{Ob}\,\mathcal{A}$ we have that h_x is a Hilbert module.*

Proof. Take an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ for $\mathcal{A}(x, x)$ where Λ is a set, and note that Corollary 1.2.8 implies that each element $a \in h_x(y) = \mathcal{A}(y, x)$ is the norm limit of $u_\lambda \circ a = u_\lambda \cdot a$. Hence $S = \{u_\lambda : \lambda \in \Lambda\} \subseteq h_x(x)$ generates h_x. $\square$

Remark 2.1.17. We will occasionally use the word 'small' to emphasize the property of being generated by a set: this coincides with the terminology in [Hen15]. As we will not deal with many 'strictly large' Hilbert modules, however, we define Hilbert modules as being necessarily small, for economy of language.

2.2 Bounded adjointable operators

In this section, we aim to define the C^*-category of right Hilbert $\mathcal{A}$-modules. To do this we must define morphisms, an involution, and a norm.

Definition 2.2.1. If $E, F : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ are two right Hilbert $\mathcal{A}$-modules, then an *operator*

$$T : E \to F$$

is simply a natural transformation $T : E \Rightarrow F$ of the underlying functors. We say T is *bounded* if the set

$$\{\|T(e)\| : x \in \mathsf{Ob}\mathcal{A}, e \in E(x), \|e\| = 1\} \subseteq \mathbb{R}$$

is bounded, in which case we denote its supremum by $\|T\|$.

We also ask that our operators are *adjointable*:

Definition 2.2.2. A bounded operator $T^* : F \to E$ of right Hilbert $\mathcal{A}$-modules is an *adjoint* for the bounded operator $T : E \to F$ when we have

$$\langle Te, f\rangle_F = \langle e, T^* f\rangle_E \in \mathcal{A}(x, y) \text{ for all } x, y \in \mathsf{Ob}\,\mathcal{A}, e \in E(x), f \in F(y).$$

We will denote the space of bounded adjointable operators from E to F by $\mathcal{L}(E, F)$, and shorten $\mathcal{L}(E, E)$ to $\mathcal{L}(E)$.

Lemma 2.2.3. *The adjoint T^*, if it exists, is unique.*

Proof. If T_1^* and T_2^* are two adjoints for T then we immediately derive from the adjoint equation that $\langle e, T_1^* f - T_2^* f\rangle_E = \langle Te, f\rangle_F - \langle Te, f\rangle_F = 0$ for all $x, y \in \mathsf{Ob}\,\mathcal{A}, e \in E(x), f \in F(y)$. But then setting $x = y$ and $e = T_1^* f - T_2^* f$ we see that in fact $T_1^* f - T_2^* f = 0$ for all f. □

Remark 2.2.4. We will often drop 'bounded' and 'adjointable' and speak simply of operators, except where we have to prove that an operator is bounded and has an adjoint.

Lemma 2.2.5. *If $\mathcal{A}$ is a C^*-category and $E_1, \dots, E_n$ is a finite list of right Hilbert $\mathcal{A}$-modules, there is a Hilbert module $\oplus_{i=1}^n E_i$, labelled their* direct sum*, given by*

- $(\oplus_{i=1}^n E_i)(x) := \oplus_{i=1}^n E_i(x)$ *and*
- $\langle (e_1, \dots, e_n), (f_1, \dots, f_n)\rangle_{\oplus_{i=1}^n E_i} := \langle e_1, f_1\rangle_{E_1} + \cdots + \langle e_n, f_n\rangle_{E_n}$.

Proof. The axioms for a large pre-Hilbert module are obvious. Positive definitivity is satisfied since by Lemma 1.1.18 and Lemma 1.1.21 the sum of several positive elements can only be zero if all are zero, and completeness follows as

$$\max_i \|\langle e_i, e_i\rangle\| \le \|\langle e_1, e_1\rangle + \cdots + \langle e_n, e_n\rangle\| \le n \max_i \|\langle e_i, e_i\rangle\|,$$

where the first inequality follows from Lemma 1.1.18 and Lemma 1.1.21. Hence we see a sequence in $(\oplus_{i=1}^n E_i)(x)$ is Cauchy if and only if each of its coordinates are, in which case the sequence converges to its coordinate-wise limit.

Finally for the 'smallness' requirement, note that when S_i is a generating set for E_i, then $\oplus_{i=1}^n E_i$ is generated by the set $\sqcup_{i=1}^n \iota_i(S_i)$, where $\iota_i : E_i \Rightarrow \oplus_{i=1}^n E_i$ is the obvious injective transformation. □

Note that since we do not yet have a C^*-category of Hilbert modules, the above construction does not yet have an interpretation as a direct sum in a C^*-category. We move on to show that there *is* in fact a C^*-category of right Hilbert $\mathcal{A}$-modules. The first version of the result below appeared as [Mit02a, Proposition 9.4], and parts of the proof are taken from there.

Proposition 2.2.6. *If $\mathcal{A}$ is a locally small C^*-category, there is a locally small, unital C^*-category*

$$\mathsf{Hilb}\text{-}\mathcal{A}$$

such that

- *Its objects are right Hilbert $\mathcal{A}$-modules.*
- *Its morphisms are bounded, adjointable operators.*
- *Its norm is the one given in Definition 2.2.1.*
- *The involution is given by taking adjoint operators.*

Proof. We first show that each metric space $\mathcal{L}(E, F)$ is complete. Suppose (T_n) is a sequence of operators from E to F which is Cauchy in the norm specified in Definition 2.2.1. Then it follows by definition of this norm that the components $(T_n^x) : E(x) \to F(x)$ each give sequences of maps of Banach spaces which are Cauchy in the individual operator norms. Hence each of these components converge to a limit T^x, the collection of which assembles to a natural transformation $T : E \Rightarrow F$ which is a limit of (T_n) in the operator norm. Linearity and boundedness follow clearly from the convergence, and we also see that T has an adjoint T^* which is the limit of the adjoints T_n^*.

All operators in this category are adjointable by definition so it has an involution, which is well-defined by Lemma 2.2.3 and conjugate linear by the sesquilinearity of $\langle -, - \rangle_E$. It is also clear that the identity operator on every Hilbert $\mathcal{A}$-module is bounded and self-adjoint.

To show that the involution is functorial, we note that for two adjointable operators $T : D \to E, S : E \to F$, we have

$$\langle S(T(d)), f \rangle_F = \langle T(d), S^*(f) \rangle_E = \langle d, T^* S^*(f) \rangle_D$$

so by uniqueness $T^* \circ S^* = (S \circ T)^*$, and similarly $T^{**} = T$.

To show contractivity of composition we note that if $\|e\| = 1$, then

$$\|\langle ST(e), ST(e) \rangle\| \le \|S\| \|\langle T(e), T(e) \rangle\| \le \|S\| \|T\|.$$

Note furthermore that if $\|e\| = 1$, then

$$\begin{aligned}\|T(e)\|^2 &= \|\langle T(e), T(e)\rangle\| \\ &= \|\langle T^*T(e), e\rangle\| \\ &\leq \|T^*T(e)\|\|e\| \\ &\leq \|T^*T\|\|e\|^2 \\ &= \|T^*T\|\end{aligned}$$

so by definition of $\|T\|$ we see $\|T^*T\| \geq \|T\|^2$. By Lemma 1.1.24, this will be enough to prove the C^*-identity in the presence of the other properties.

Note that restricting our attention to operators $T : E \to E$, the above shows $\mathcal{L}(E)$ is a C^*-algebra for each right Hilbert $\mathcal{A}$-module E, so for an arbitrary $T : E \to F$, we can form the direct sum $E \oplus F$ as in Lemma 2.2.5 and the C^*-algebra $\mathcal{L}(E \oplus F)$. Note that the C^*-algebra $\mathcal{L}(E)$ embeds into $\mathcal{L}(E \oplus F)$ as the top left corner in a matrix algebra. Now consider $M_T = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$: we have that $M_T^* M_T = \begin{bmatrix} T^*T & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(E \oplus F)$ is necessarily a positive element of the C^*-algebra $\mathcal{L}(E \oplus F)$, so by Lemma 1.1.17 we see T^*T is a positive element of $\mathcal{L}(E)$. This concludes the proof that $\mathsf{Hilb}\text{-}\mathcal{A}$ is a C^*-category.

We finish by proving local smallness. Suppose E and F are right Hilbert $\mathcal{A}$-modules. If S is a generating set for E, it is clear from Lemma 2.1.13 that any bounded adjointable map $T \in \mathcal{L}(E, F)$ is determined by its image on S. But each element $s \in E(x_s)$ can only be sent to an element in the set $F(x_s)$, and hence we see there is an injective map from $\mathcal{L}(E, F)$ to the set $\Pi_{s \in S} F(x_s)$. $\square$

Lemma 2.2.7. $\mathsf{Hilb}\text{-}\mathcal{A}$ *is closed under finite direct sums and idempotent complete.*

Proof. Firstly, is obvious that the construction in Lemma 2.2.5 gives direct sums, where the structure maps $\iota_i : E_i \to \oplus_{i=1}^n E_i$ are given by inserting an element into the i^{th} coordinate, with adjoint given by projecting.

As for idempotent completeness, take a projection $P \in \mathcal{L}(E, E)$, that is, a bounded adjointable map such that $P^* = P$ and $P^2 = P$. Then it is easily verified that $\ker P$ and $\mathrm{im} P$, taken objectwise, are Hilbert submodules of E and that $E \cong \ker P \oplus \mathrm{im} P$. $\square$

Definition 2.2.8. A right Hilbert $\mathcal{A}$-module E is said to be *finitely generated free* if there is a list of $\mathcal{A}$-objects $x_1, \ldots, x_n$ such that $\oplus_{i=1}^n h_{x_i} \cong E$, where the h_{x_i} are the representable $\mathcal{A}$-modules from Example 2.1.3. We say E is *finitely generated projective* if it is a direct summand of a finitely generated free module.

We close with a characterization of the unitary isomorphisms in $\mathsf{Hilb}\text{-}\mathcal{A}$.

Proposition 2.2.9. *If E and F are right Hilbert $\mathcal{A}$-modules, and $T : E \Rightarrow F$ is any operator (not necessarily bounded or adjointable) from E to F, then T is a bounded adjointable unitary isomorphism of right Hilbert $\mathcal{A}$-modules if and only if T is surjective at every $x \in \mathsf{Ob}\,\mathcal{A}$ and T preserves all inner products.*

Proof. Note that if T preserves inner products it must be an isometry on each single object, so as T is also surjective on each object it must have an object-wise inverse T^{-1}, which is in addition natural since an array of isomorphisms is natural if and only if its inverse is. But then for each $e \in E(x), f \in F(y)$ we have

$$\langle e, T^{-1}(f)\rangle_E = \langle T(e), TT^{-1}(y)\rangle_F = \langle T(e), f\rangle_F$$

showing that $T^{-1} = T^*$. Hence T is a unitary isomorphism of $\mathcal{A}$-modules. The converse is obvious. □

2.3 Compact operators on Hilbert modules

Hilbert spaces have a notion of *compact operators* between them; these are given by the norm-closure of the subspace of all operators with finite-dimensional range. In this section we define compact operators between two Hilbert modules over a C^*-category and characterize the compact-relative topology on the category of bounded operators.

Definition 2.3.1. If E, F are right Hilbert $\mathcal{A}$-modules and $x \in \mathsf{Ob}\,\mathcal{A}$, then for module elements $e \in E(x)$ and $f \in F(x)$, the *single-rank* operator

$$\theta_x^{f,e} : E \to F$$

is defined by setting for all $y \in \mathsf{Ob}\,\mathcal{A}, e' \in E(y)$:

$$\theta_x^{f,e}(e') := f \cdot \langle e, e'\rangle_E \in F(y).$$

It immediately follows that $\theta^{-,-}$ is linear in the first variable, conjugate linear in the second variable, and that for all elements $e \in E(x), f \in F(y)$ and morphisms $a \in \mathcal{A}(x, y)$ we have

$$\theta_y^{f,e\cdot a^*} = \theta_x^{f\cdot a,e}.$$

We will omit the indexing object x where it is implied or not relevant.

Lemma 2.3.2. *The single-rank operator $\theta_x^{f,e}$ is bounded and has adjoint*

$$\theta_x^{e,f} : F \to E$$

Proof. For the bound note that if $e' \in E(y)$ with $\|e'\| = 1$, then

$$\|\theta_x^{f,e}(e')\| = \|f \cdot \langle e, e'\rangle_E\| \le \|f\|\|\langle e, e'\rangle\| \le \|e\|\|f\|$$

where we use Lemma 2.1.4 and Corollary 2.1.6 for the two inequalities. For the adjoint, let $f' \in F(z)$ for some $z \in \mathsf{Ob}\,\mathcal{A}$. Then note

$$\begin{aligned}\langle \theta_x^{f,e}(e'), f'\rangle &= \langle f \cdot \langle e, e'\rangle, f'\rangle \\ &= \langle e, e'\rangle^* \circ \langle f, f'\rangle \\ &= \langle e', e\rangle \circ \langle f, f'\rangle \\ &= \langle e', e \cdot \langle f, f'\rangle\rangle \\ &= \langle e', \theta_x^{e,f}(f')\rangle\end{aligned}$$

hence $\theta_x^{e,f} = (\theta_x^{f,e})^*$. □

Definition 2.3.3. A *finite-rank operator* is a finite sum of single-rank operators. A *compact operator* is a norm-limit of finite-rank operators.

Example 2.3.4. If $\mathcal{A}$ has one object x, our notion of compact operators coincides with the usual notion of compact operators between Hilbert modules over $\mathcal{A}(x,x)$ (see [Lan95, p.10]). Specifically, when $\mathcal{A}(x,x) = \mathbb{C}$, then compact operators of Hilbert $\mathcal{A}$-modules are compact maps between Hilbert spaces.

Note that our definition of compact operators is (at least on the face of it) distinct from that in [Mit02b, Definition 3.2], though it agrees with the definition in [Fer23, Definition 2.3.6].

Lemma 2.3.5. *The single-rank operators are closed under pre- and postcomposition with bounded operators. In particular if $\theta_x^{e,f} : E \to F$ is a single-rank operator and $S : D \to E$ and $T : F \to G$ are bounded operators of right Hilbert $\mathcal{A}$-modules, we have*

$$T \circ \theta_x^{f,e} = \theta_x^{T(f),e} \text{ and } \theta_x^{f,e} \circ S = \theta_x^{f,S^*(e)}.$$

Proof. Let $y \in \mathsf{Ob}\,\mathcal{A}$. Note that for all $e' \in E(y)$ we have

$$(T \circ \theta_x^{f,e})(e') = T(f \cdot \langle e, e' \rangle) = T(f) \cdot \langle e, e' \rangle = \theta_x^{T(f),e}(e')$$

and for all $d \in D(y)$ we have

$$(\theta_x^{f,e} \circ S)(d) = f \cdot \langle e, S(d) \rangle = f \cdot \langle S^*(e), d \rangle = \theta_x^{f,S^*(e)}(d).$$

□

Proposition 2.3.6. *The compact operators form an essential ideal in the C^*-category* Hilb-$\mathcal{A}$.

Proof. By Lemma 2.3.5 it is easy to see that finite sums of single-rank operators are closed under pre- and postcomposition by operators, and then by Lemma 1.1.7 we see that the norm-closure of finite sums of such elements forms a C^*-ideal.

Next we show this ideal is essential: suppose T is a bounded operator $E \to F$ such that $T \circ \theta_x^{e,d} = 0$ for all modules $D \in \mathsf{Hilb}$-$\mathcal{A}$, objects $x \in \mathsf{Ob}(\mathcal{A})$, and module elements $d \in D(x), e \in E(x)$. Let $D = E$, then we have $T \circ \theta_x^{e,e'}(e'') = T(e) \cdot \langle e', e'' \rangle = T(e \cdot \langle e', e'' \rangle) = 0$ for all elements $e, e', e'' \in E(x)$. But by Lemma 2.1.8 the span of such elements is dense in $E(x)$ so we see $T = 0$ on all of $E(x)$. □

Definition 2.3.7. We denote the category of Hilbert modules and compact operators by

$$\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}\,.$$

When $\mathcal{A}$ is implicit we write $\mathcal{K}(E,F)$ rather than $\mathcal{K}\mathsf{Hilb}$-$\mathcal{A}(E,F)$ for the compact operators from E to F, and we further shorten $\mathcal{K}(E,E)$ to $\mathcal{K}(E)$.

Lemma 2.3.8. *If U_λ is an approximate unit for the C^*-algebra $\mathcal{K}(E)$, then for any $x \in \mathsf{Ob}\,\mathcal{A}$ and $e \in E(x)$, we have $U_\lambda(e) \xrightarrow{\lambda} e$.*

Proof. Notice for any $e_1, e_2, e_3 \in E(x)$, that

$$\begin{aligned} \|U_\lambda(e_1 \cdot \langle e_2, e_3\rangle) - e_1 \cdot \langle e_2, e_3\rangle\| &= \|(U_\lambda \circ \theta^{e_1,e_2} - \theta^{e_1,e_2})(e_3)\| \\ &\leq \|U_\lambda \circ \theta^{e_1,e_2} - \theta^{e_1,e_2}\|\|e_3\| \xrightarrow{\lambda} 0 \end{aligned}$$

In other words, $U_\lambda(e_1 \cdot \langle e_2, e_3\rangle) \xrightarrow{\lambda} e_1 \cdot \langle e_2, e_3\rangle$. So since Lemma 2.1.8 tells us the elements of this form generate a dense subspace of $E(x)$ and $\|U_\lambda\| \leq 1$, we see by Corollary 1.4.3 that $U_\lambda(e) \xrightarrow{\lambda} e$ for all $e \in E(x)$. □

We can use compact operators to define a sort of Yoneda embedding for C^*-categories:

Lemma 2.3.9. *The assignment $x \mapsto h_x$ defined in Example 2.1.3 extends to a faithful C^*-functor $\iota : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{A}$, where for any $a \in \mathcal{A}(x, y)$ we let*

$$\iota(a) : h_x \to h_y$$

be the operator with components $\iota(a)_z : \mathcal{A}(z, x) \to \mathcal{A}(z, y)$ given by postcomposition with a. Furthermore, this functor has image $\iota(\mathcal{A}(x, y)) = \mathcal{K}(h_x, h_y)$.

Proof. This functor is clearly linear, faithful and intertwines the involutions on the two categories, so it is an isometric C^*-functor. To show the inclusion $\iota(\mathcal{A}(x, y)) \supseteq \mathcal{K}(h_x, h_y)$, note that for $a \in h_x(z), b \in h_y(z)$, the rank one operator $\theta^{b,a} : h_x \to h_y$ is the one given by postcomposition with ba^*. To show that $\iota(\mathcal{A}(x, y)) \subseteq \mathcal{K}(h_x, h_y)$, factorize any morphism $a \in \mathcal{A}(x, y)$ as in Lemma 1.2.7. Then we see that there exists $d \in \mathcal{A}(x, y)$ such that $\iota(a) = \theta^{(a^*a)^{1/4}, d^*}$, so in fact each operator in the image of ι is single-rank. □

We will occasionally denote ι by $\iota_\mathcal{A}$ when we need to specify $\mathcal{A}$.

Corollary 2.3.10. *If $\mathcal{A}$ is unital, the functor $\iota_\mathcal{A}$ is also full, and furthermore $\iota_\mathcal{A}$ extends to an equivalence*

$$\mathcal{A}_\oplus^\natural \cong \mathsf{fgProj}\text{-}\mathcal{A}$$

where $\mathsf{fgProj}\text{-}\mathcal{A}$ is the category of finitely generated projective right Hilbert $\mathcal{A}$-modules and bounded operators.

Proof. If $\mathcal{A}(x, x)$ is unital a standard Yoneda argument shows that any transformation $T : h_x \to h_y$ is given by postcomposing with the morphism $T_x(\mathrm{id}_x)$: for any $z \in \mathsf{Ob}\,\mathcal{A}, a \in h_x(z)$ we have

$$T_z(a) = T_z(\mathrm{id}_x \circ a) = T_x(\mathrm{id}_x) \circ a.$$

So T is in fact in the image of $\iota_\mathcal{A}$. Hence if $\mathcal{A}$ is unital, $\iota_\mathcal{A}$ is full.

Note for the second equivalence firstly that $\iota_{\mathcal{A}}$ is an equivalence between $\mathcal{A}$ and the category of representable right Hilbert $\mathcal{A}$-modules. Secondly, as $\mathsf{Hilb}\text{-}\mathcal{A}$ is idempotent complete, $\mathsf{fgProj}\text{-}\mathcal{A}$ is by definition the closure of representable right Hilbert $\mathcal{A}$-modules under finite direct sums and direct summands. The stated equivalence follows. □

We can in fact generalize Lemma 2.3.9 to give a Yoneda lemma for C^*-categories.

Proposition 2.3.11. *For each $x \in \mathsf{Ob}\,\mathcal{A}$ and right Hilbert $\mathcal{A}$-module F, there is an isometric isomorphism*

$$\eta : \mathcal{K}(h_x, F) \to F(x).$$

Proof. We construct the map $\eta : \mathcal{K}(h_x, F) \to F(x)$ by defining it first on single-rank operators. For $f \in F(y)$ and $a \in \mathcal{A}(y, x)$, note that for each $b \in \mathcal{A}(z, x)$ we have

$$\theta^{f,a}(b) = f \cdot \langle a, b\rangle = f \cdot a^* b = (f \cdot a^*) \cdot b.$$

Hence we can set $\eta(\theta^{f,a}) := f \cdot a^* \in F(x)$ and extend linearly to finite-rank operators. To show this partial map is well-defined, note that picking $x = y$ and letting b vary over an approximate unit (u_λ) for $\mathcal{A}(x, x)$, we see by Corollary 2.1.9 that

$$\sum_{i=1}^{n} \theta^{f_i, a_i}(u_\lambda) = (\sum_{i=1}^{n} f_i \cdot a_i^*) \cdot u_\lambda \xrightarrow{\lambda} \sum_{i=1}^{n} f_i \cdot a_i^*.$$

So if $\sum_{i=1}^n \theta^{f_i,a_i} = \sum_{j=1}^m \theta^{f'_j, a'_j}$, then $\sum_{i=1}^n f_i \cdot a_i^* = \sum_{j=1}^m f'_i \cdot a'^*_i$. Furthermore, applying norms to the above limit, and noting that $\|u_\lambda\| \le 1$ for each λ, we see that $\|\sum_{i=1}^n \theta^{f_i,a_i}\| \ge \|\sum_{i=1}^n f_i \cdot a_i^*\|$. But conversely it follows from Lemma 2.1.4 that

$$\|\sum_{i=1}^{n} \theta^{f_i,a_i}(b)\| = \|(\sum_{i=1}^{n} f_i \cdot a_i^*) \cdot b\| \le \|(\sum_{i=1}^{n} f_i \cdot a_i^*)\|\|b\|$$

so we see $\|\eta(\sum_{i=1}^n \theta^{f_i,a_i})\| = \|\sum_{i=1}^n f_i \cdot a_i^*\| = \|\sum_{i=1}^n \theta^{f_i,a_i}\|$. But then by Lemma 1.4.4 there is a unique isometric extension of η to all of $\mathcal{K}(h_x, F)$.

To show η is surjective, recall that by Corollary 2.1.11 for any $f \in F(x)$ there exist $a \in \mathcal{A}(x, x)$ and $f' \in F(x)$ such that $f = f' \cdot a$: hence $f = \eta(\theta^{f',a^*})$. □

The map η is natural in F and x, but rather than proving this directly, we find the inverse to the map and show that *it* is natural:

Lemma 2.3.12. *For each $x \in \mathsf{Ob}\,\mathcal{A}$ and right Hilbert $\mathcal{A}$-module F, the inverse $\epsilon : F(x) \to \mathcal{K}(h_x, F)$ to the map η defined above is given by setting $\epsilon : f \mapsto \epsilon_f$ where $\epsilon_f(a) = f \cdot a$ for all $a \in h_x(y)$. The adjoint of ϵ_f is given for each $y \in \mathsf{Ob}\,\mathcal{A}$ and $f' \in F(y)$ by $\epsilon_f^*(f') = \langle f, f'\rangle$.*

This map is natural in both F and x: that is, for any $\phi \in \mathcal{L}(F, F')$, we have $\phi \circ \epsilon_f = \epsilon_{\phi(f)}$, and for any $b \in \mathcal{A}(x', x)$, we have $\epsilon_m \circ \iota(b) = \epsilon_{m \cdot b}$

Proof. Each map ϵ_f is evidently an $\mathcal{A}$-module map and is bounded by Lemma 2.1.4; furthermore, it is simple to verify that its adjoint is as stated.

Note next that for any $b \in h_x(z), \theta^{f,a} \in \mathcal{K}(h_x, F)$ we have

$$(\epsilon \circ \eta)(\theta^{f,a})(b) = (f \cdot a^*) \cdot b = f \cdot (a^* b) = f \cdot \langle a, b \rangle_{h_x} = \theta^{f,a}(b).$$

Hence by the density of the finite-rank operators we see ϵ is the unique inverse for the map η.

We finish by showing the naturality: note for any $a \in h_x(y)$ that

$$\epsilon_{\phi(f)}(a) = \phi(f) \cdot a^* = \phi(f \cdot a^*) = \phi(\epsilon_f(a)).$$

For the second part, simply note for any $b \in \mathcal{A}(x', x), a \in h_{x'}(y)$ we have

$$(\epsilon_f \circ \iota(b))(a) = (\epsilon_f)(ba) = f \cdot (ba) = (f \cdot b)(a) = \epsilon_{f \cdot b}(a).$$

□

This correspondence helps us understand single-rank operators better:

Lemma 2.3.13. *If E and F are right Hilbert $\mathcal{A}$-modules and $x \in \mathsf{Ob}\,\mathcal{A}$, the single-rank operators $\theta_x^{f,e}$ are exactly those operators that factor through h_x as the composite*

$$E \to h_x \to F$$

of two compact operators.

Proof. Note simply that for any $e' \in E(y)$, we have

$$\theta_x^{f,e}(e') = f \cdot \langle e, e' \rangle = \epsilon_f \circ \epsilon_e^*(e').$$

Conversely, by Proposition 2.3.11 all compact sequences $E \to h_x \to F$ are described by an operator $\epsilon_f \epsilon_e^* = \theta_x^{f,e}$ which is single-rank by definition. □

It follows that $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ is, in the sense defined in Proposition 1.1.38, the ideal *generated by* the image of $\iota_{\mathcal{A}}$.

Proposition 2.3.14. *We have an equality*

$$\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} = \mathsf{Hilb}\text{-}\mathcal{A}\, im(\iota_{\mathcal{A}})\, \mathsf{Hilb}\text{-}\mathcal{A}$$

of subcategories of $\mathsf{Hilb}\text{-}\mathcal{A}$.

Proof. Recall from Proposition 1.1.38 that $\mathsf{Hilb}\text{-}\mathcal{A}\,\mathrm{im}(\iota_{\mathcal{A}})\,\mathsf{Hilb}\text{-}\mathcal{A}$ is the norm-closure of the span of all morphisms that factor through some morphism in $\mathrm{im}(\iota_{\mathcal{A}})$. Note that every such morphism must be compact, as compact morphisms are an ideal and by Lemma 2.3.9 all morphisms in the image of ι are compact. Hence we get an inclusion $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} \supseteq \mathsf{Hilb}\text{-}\mathcal{A}\,\mathrm{im}(\iota_{\mathcal{A}})\,\mathsf{Hilb}\text{-}\mathcal{A}$.

For the opposite inclusion, consider the single-rank map $\theta_x^{f,e} : E \to F$ for $e \in E(x), f \in F(x)$, and write $f = f' \cdot a$ for $f' \in F(x), a \in \mathcal{A}(x,x)$ using Corollary 2.1.11. Then

$$\theta^{f,e} = \epsilon_f \circ \epsilon_e^* = \epsilon_{f'} \circ \iota(a) \circ \epsilon_e^*.$$

Hence any single-rank operator can be factorized through a morphism in the image of ι, and hence we see any compact operator is a norm-limit of sums of such operators. ☐

We close by describing a topology of 'pointwise convergence' on $\mathsf{Hilb}\text{-}\mathcal{A}$ that will come to play a significant role.

Definition 2.3.15. The *strong* topology* on $\mathsf{Hilb}\text{-}\mathcal{A}$ is the topology such that a net of operators $T_\lambda : E \to F$ converges to T in the strong* topology when for every object $x \in \mathsf{Ob}\,\mathcal{A}$ and elements $e \in E(x), f \in F(x)$, we have

$$\|T_\lambda(e) - T(e)\| \xrightarrow{\lambda} 0 \text{ and } \|T_\lambda^*(f) - T^*(f)\| \xrightarrow{\lambda} 0.$$

This is a legitimate definition for a topology since it is the one generated by the seminorms $T \mapsto \|T(e)\|$, $T \mapsto \|T^*(f)\|$ for all e and f. We will use the adverb "strongly*" to mean "in the strong* topology".

We now generalize a result from the theory of Hilbert modules over C^*-algebras.

Proposition 2.3.16 (c.f. [Lan95, Proposition 8.1])**.** *The strong* topology coincides with the $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$-relative topology on norm-bounded subsets.*

Proof. Suppose that (T_λ) is a norm-bounded net of operators in $\mathcal{L}(E,F)$: say their norm is bounded by $H \in \mathbb{R}$. We have to show that $T_\lambda \to 0$ in the strong* topology if and only if $T_\lambda \to 0$ in the $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$-relative topology.

Assume first that $T_\lambda \to 0$ strongly*; that is, for each $x \in \mathsf{Ob}\,\mathcal{A}$, we have $T_\lambda(e) \to 0$ for each $e \in E(x)$ and $T_\lambda^*(f) \to 0$ for each $f \in F(x)$. We have to show that $\|T_\lambda \circ h\| \to 0$ for each $h \in \mathcal{K}(D,E)$ and $\|g \circ T_\lambda\| \to 0$ for each $g \in \mathcal{K}(F,G)$.

Note first that for $\theta_w^{f,e} \in \mathcal{K}(D,E)$ we have

$$\|T_\lambda \circ \theta_w^{f,e}\| = \|\theta_w^{T_\lambda(f),e}\| \leq \|T_\lambda(f)\|\|e\|$$

hence $\|T_\lambda \circ \theta_w^{f,e}\| \xrightarrow{\lambda} 0$. Hence as the span of single-rank operators is dense in $\mathcal{K}(D,E)$, and precomposition with the operators T_λ gives a bounded net of operators $\mathcal{K}(D,E) \to \mathcal{K}(D,F)$, we can apply Lemma 1.4.2 to deduce that $T_\lambda \circ h \to$ for all $h \in \mathcal{K}(D,E)$.

Similarly, $T_\lambda^*(f) \to 0$ for each $f \in F(x)$ implies that $\|g \circ T_\lambda\| \to 0$ for each $g \in \mathcal{K}(F,G)$.

For the converse, suppose $T_\lambda \to 0$ relative to $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$. Then in particular $\|T_\lambda \circ h\| \to 0$ for each $h \in \mathcal{K}(E, E)$ and $\|g \circ T_\lambda\| \to 0$ for each $g \in \mathcal{K}(F, F)$. Hence we see for any for any $e, e', e'' \in E(x)$ that

$$T_\lambda(e \cdot \langle e', e'' \rangle) = T_\lambda(\theta_x^{e,e'}(e'')) = 0.$$

But by Lemma 2.1.8 the span of such elements is dense in $E(x)$ so using Lemma 1.4.2 we see $T_\lambda(e) \xrightarrow{\lambda} 0$ for all $e \in E(x)$.

Similarly, the fact that $\|g \circ T_\lambda\| \to 0$ for each $g \in \mathcal{K}(F, G)$ gives us that $T_\lambda^*(f) \to 0$ for all $f \in F(x), x \in \mathsf{Ob}\,\mathcal{A}$. This concludes our proof. □

To see that these topologies do not necessarily coincide on unbounded subsets of operators, see [Lan95, p.76] for an example of an unbounded net of operators on a Hilbert space (i.e. a Hilbert module over $\mathcal{A} = \mathbb{C}$), which goes to zero in the strong* topology but not in the topology relative to the compact operators.

2.4 Bimodules and the equivalence $\mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}) \cong \mathsf{Hilb}\text{-}\mathcal{A}$

A standard result on Hilbert modules is that if A is a C^*-algebra and E is a right Hilbert A-module, then $\mathcal{M}(\mathcal{K}(E)) = \mathcal{L}(E)$ (see e.g. [Lan95, Theorem 2.4]). In this section we categorify this result in two ways: the first is that we replace A by a C^*-category $\mathcal{A}$. The second is that we replace the algebras $\mathcal{K}(E)$ and $\mathcal{L}(E)$ by the C^*-categories $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ and $\mathsf{Hilb}\text{-}\mathcal{A}$.

Definition 2.4.1. If $\mathcal{A}$ and $\mathcal{B}$ are C^*- categories, a *right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule* is a C^*-functor

$$F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}\,.$$

The bimodule F is said to be *non-degenerate* if the linear span of

$$\bigcup_{y \in \mathsf{Ob}\,\mathcal{A}} F\mathcal{A}(y, x)(F(y))$$

is dense in $F(x)$ for all $x \in \mathsf{Ob}\,\mathcal{A}$; that is, dense in $F(x)(z)$ for each $z \in \mathsf{Ob}\,\mathcal{B}$.

Remark 2.4.2. The astute reader may be concerned that we have already defined what it means for a functor to be non-degenerate in Definition 1.4.1, but by the end of this section we will see that the definitions coincide in a precise sense.

Lemma 2.4.3. *The inclusion $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{A}$ is non-degenerate.*

Proof. Recall from Lemma 2.1.8 that for any right Hilbert $\mathcal{A}$-module E and every $x \in \mathsf{Ob}\,\mathcal{A}$, we have that $E(x) \cdot \langle E(x), E(x) \rangle$ is dense in $E(x)$. Note that this is a subspace of $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}(E, E)(E(x))$, which *a fortiori* must then also be dense, satisfying Definition 2.4.1. □

We provide an alternate characterization of non-degeneracy, analogous to criteria 3 and 4 in Definition 1.4.1.

Lemma 2.4.4. *F is non-degenerate if and only if for any $x \in \mathsf{Ob}\,\mathcal{A}$ and any approximate unit (u_λ) for $\mathcal{A}(x,x)$, any $z \in \mathsf{Ob}\,\mathcal{B}$, and any $e \in F(x)(z)$, we have $F(u_\lambda)(e) \xrightarrow{\lambda} e$.*

Proof. For the rightward implication, note that by the continuity of F, for all $F(a)(d) \in F\mathcal{A}(y,x)(F(y)(z))$ we have that

$$F(u_\lambda)(F(a)(d)) = F(u_\lambda a)(d) \xrightarrow{\lambda} F(a)(d).$$

So since by non-degeneracy the span of such elements gives a norm-dense subset of $F(x)(z)$, we can apply Lemma 1.4.2 to see that $F(u_\lambda)(d) \xrightarrow{\lambda} d$ for all morphisms $d \in F(x)(z)$.

For the converse, note that if $F(u_\lambda)(e) \xrightarrow{\lambda} e$ for $e \in F(x)(z)$, then the set

$$\{F(u_\lambda)(e) : \lambda \in \Lambda\} \subseteq F(\mathcal{A}(x,x))(F(x)(z))$$

has e as a limit point, so the density requirement is satisfied. □

We now begin building up to the main theorem of this section.

Lemma 2.4.5. *Suppose $F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a non-degenerate bimodule, and $\mathcal{C}$ is any C^*-category containing $\mathcal{A}$ as a C^*-ideal. Then F extends uniquely to a C^*-functor*

$$\bar{F} : \mathcal{C} \to \mathsf{Hilb}\text{-}\mathcal{B},$$

which is faithful whenever both F is faithful and *$\mathcal{A}$ is essential in $\mathcal{C}$.*

Proof. Note that $\mathcal{A}$ and $\mathcal{C}$ have the same objects by the remark below Definition 1.1.36, so on objects we can simply set $\bar{F} = F$. Take any morphism $c \in \mathcal{C}(x,x')$. By nondegeneracy, at any object $z \in \mathsf{Ob}\,\mathcal{B}$, the span of $\cup_{y \in \mathsf{Ob}\,\mathcal{A}} F\mathcal{A}(y,x)F(y)(z)$ is dense in $F(x)(z)$. On this dense subset we define the transformation

$$\bar{F}(c) : \begin{array}{l} \sum_{y \in \mathsf{Ob}\,\mathcal{A}} F\mathcal{A}(y,x)F(y)(z) \to F(x')(z) \\ \Sigma_{i=1}^n F(a_i)e_i \mapsto \Sigma_{i=1}^n F(ca_i)e_i. \end{array}$$

Note the expression is well-defined since $ca_i \in \mathcal{A}$. We firstly need to show that this is a well-defined function on the given sum of subspaces, i.e. that if $\Sigma_{i=1}^n F(a_i)e_i = \Sigma_{i=1}^m F(b_i)f_i$, then $\Sigma_{i=1}^n F(ca_i)e_i = \Sigma_{i=1}^m F(cb_i)f_i$. So let u_λ be an approximate unit for $\mathcal{A}(x,x)$, then by Corollary 1.2.8 we have $u_\lambda a_i \xrightarrow{\lambda} a_i$, so $cu_\lambda a_i \xrightarrow{\lambda} ca_i$ and by the continuity of F we have

$$\begin{aligned} \Sigma_{i=1}^n F(ca_i)e_i &= \lim_\lambda \textstyle\sum_{i=1}^n F(cu_\lambda a_i)e_i &&= \lim_\lambda F(cu_\lambda) \textstyle\sum_{i=1}^n F(a_i)e_i \\ & &&= \lim_\lambda F(cu_\lambda) \textstyle\sum_{i=1}^m F(b_i)f_i \\ & &&= \lim_\lambda \textstyle\sum_{i=1}^m F(cu_\lambda b_i)f_i \\ & &&= \Sigma_{i=1}^m F(cb_i)f_i. \end{aligned}$$

Hence $\overline{F}(c)$ is well-defined, and applying norms to the third term above, we deduce $\|\overline{F}(c)\| \leq \sup_\lambda \|F(cu_\lambda)\| \leq \|c\|$ as F is norm-decreasing and $\|u_\lambda\| \leq 1$ for all λ. So by Lemma 1.4.4 we can extend $\overline{F}(c)$ to a map

$$\overline{F}(c) : F(x)(z) \to F(x')(z)$$

which is natural in z, bounded, and is easily seen to be have adjoint $\bar{F}(c^*)$. It is also elementary to check that $\bar{F}$ is complex linear and preserves composition. It follows that $\bar{F}$ defines a C^*-functor $\mathcal{C} \to \mathsf{Hilb}\text{-}\mathcal{A}$.

For the final claim, if F is faithful and $\mathcal{A}$ is essential in $\mathcal{C}$, we see that

$$\ker(\bar{F}) \cap \mathcal{A} = \ker F = 0$$

so by Proposition 1.1.37, $\ker(\bar{F}) = 0$ and $\bar{F}$ is faithful. □

Lemma 2.4.6. *Suppose $F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a faithful non-degenerate bimodule. There is a C^*-subcategory $\mathcal{D}$ of $\mathsf{Hilb}\text{-}\mathcal{B}$, termed the* idealizer under F of $\mathcal{A}$*, with objects being those in the image of F and hom spaces*

$$\mathcal{D}(F(x), F(y)) := \left\{ d \in \mathcal{L}(F(x), F(y)) : \begin{array}{l} d \circ F\mathcal{A}(z, x) \subseteq F\mathcal{A}(z, y) \text{ and} \\ F\mathcal{A}(y, z) \circ d \subseteq F\mathcal{A}(x, z) \\ \text{for all } z \in \mathsf{Ob}\,\mathcal{A} \end{array} \right\}.$$

Furthermore, F extends uniquely to an equivalence $\bar{F} : \mathcal{MA} \xrightarrow{\cong} \mathcal{D}$.

Proof. Notice $\mathcal{D}$ is a C^*-category which contains (the isometric image of) $\mathcal{A}$ as an ideal. Now if $\mathcal{C}$ is any other C^*-category containing $\mathcal{A}$ as an ideal, by Lemma 2.4.5 we get an embedding $\mathcal{C} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{B}$ whose image lands in $\mathcal{D}$ by construction. So $\mathcal{D}$ has the defining universal property of $\mathcal{MA}$ from Lemma 1.3.6. □

We deduce our promised result.

Proposition 2.4.7. *For any C^*-category $\mathcal{A}$, there is an isomorphism of C^*-categories*

$$\mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}) \cong \mathsf{Hilb}\text{-}\mathcal{A}\,.$$

Proof. By Lemma 2.4.3 the inclusion $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{A}$ is nondegenerate, and since $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ is an ideal in $\mathsf{Hilb}\text{-}\mathcal{A}$, we see the idealizer of $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ is all of $\mathsf{Hilb}\text{-}\mathcal{A}$, so by Lemma 2.4.6, we get the desired equivalence, which is in fact an isomorphism as $\mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A})$ and $\mathsf{Hilb}\text{-}\mathcal{A}$ have the same objects. □

The above proposition was proven in [Fer23, Corollary 3.4.5] for small C^*-categories. As an easy corollary we see that the multiplier category can be considered as the full subcategory of $\mathsf{Hilb}\text{-}\mathcal{A}$ on the representable modules:

Corollary 2.4.8. *For any $x, y \in \mathsf{Ob}\,\mathcal{A}$ we have*

$$\mathcal{L}(h_x, h_y) \cong \mathcal{MA}(x, y).$$

Proof. It is obvious from the single-object definition of multipliers that if $\mathcal{A}$ is the inclusion of a full sub-C^*-category of a C^*-category $\mathcal{B}$, then for all objects $x, y \in \mathsf{Ob}\,\mathcal{A}$ we have $\mathcal{M}\mathcal{A}(x, y) \cong \mathcal{M}\mathcal{B}(x, y)$. Applying this to $\mathcal{B} = \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ and combining with Proposition 2.4.7, we get the stated result. □

We hence get the promised disambiguation of non-degeneration.

Corollary 2.4.9. *A functor $F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is non-degenerate in the above sense if and only if, considered as a functor $\mathcal{A} \to \mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B})$, it is non-degenerate in the sense of Definition 1.4.1.*

Proof. The content of Lemma 2.4.4 is that $F : \mathcal{A} \to \mathsf{Hilb}-\mathcal{B}$ is non-degenerate if and only for every approximate unit u_λ of every endomorphism algebra $\mathcal{A}(x, x)$, the net $F(u_\lambda)$ approaches id $\in \mathcal{L}(Fx, Fx)$ in the strong* topology. Note that since (u_λ) is bounded, by Proposition 2.3.16 this is equivalent to $F(u_\lambda)$ approaching id in the $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$-relative topology, and in turn by Theorem 1.4.5 this is equivalent to the requirement that F, considered as a functor $F : \mathcal{A} \to \mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B})$, is non-degenerate in the sense of Definition 1.4.1. □

We close with two propositions on multiplier direct sums of Hilbert modules.

Proposition 2.4.10. *The multiplier direct sum in $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ of an arbitrary multiset $(E_i)_{i \in I}$ of right Hilbert $\mathcal{A}$-modules is given on each object $x \in \mathsf{Ob}\,\mathcal{A}$ by*

$$(\bigoplus_{i \in I} E_i)(x) = \{(e_i) \in \prod_{i \in I} E_i(x) : \sum_{i \in I} \langle e_i, e_i \rangle \text{ converges in } \mathcal{A}(x, x)\},$$

with inner products defined by $\langle (e_i), (f_i) \rangle = \sum_{i \in I} \langle e_i, f_i \rangle$, and bounded adjointable structure maps $\iota_i \in \mathcal{L}(E_i, \bigoplus_{i \in I} E_i)$ sending elements to sequences with zeroes in all but one position.

Proof. Clearly ι_i^* is the projection on the i-th coordinate, so each $\iota_i^* \iota_i$ is an isometry. Since for any element $(e_i) \in (\bigoplus_{i \in I} E_i)(x)$ the sum $\sum_{i \in I} \langle e_i, e_i \rangle$ converges in the normed vector space $\mathcal{A}(x, x)$, we see by [HN01, p.136] that all but countably many of the elements $\langle e_i, e_i \rangle$, and hence the entries e_i, are zero. Denote by $J \subseteq I$ the countable 'support' of (e_i), i.e. the coordinates where it is non-zero.

Then by considering the subnet consisting of finite subsets of J, it is clear that the net

$$\{\sum_{i \in F} \iota_i \iota_i^*((e_i)) : F \subseteq I \text{ finite}\}$$

converges to (e_i) in the norm. Hence $\{\sum_{i \in F} \iota_i \iota_i^* : F \subseteq I \text{ finite}\}$ converges strongly* to the identity, so as these maps and their sums clearly all have norm 1, by Proposition 2.3.16 the net converges to the identity in the strict (i.e. the $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$-relative) topology.

Finally, to prove the 'smallness' of this direct sum, we simply proceed as in the finite case. Let S_i be a generating set for each E_i, then as elements with finite support are dense in $(\bigoplus_{i\in I} E_i)(x) = \{(e_i) \in \prod_{i\in I} E_i(x)\}$ one easily sees that $\bigoplus_{i\in I} E_i$ is generated by the set $\sqcup_{i\in I}\iota_i(S_i)$. $\square$

We derive from this a characterization of compact operators between multiplier direct sums:

Proposition 2.4.11. *If $(E_i)_{i\in I}$ and $(F_j)_{j\in J}$ are collections of right Hilbert $\mathcal{A}$-modules with structure maps $\iota_i : E_i \to \bigoplus_{i\in I} E_i$ and $\iota_j : F_j \to \bigoplus_{j\in J} F_j$, then the isometries*

$$\lambda_{ij} : \mathcal{K}(E_i, F_j) \to \mathcal{K}(\bigoplus_{i\in I} E_i, \bigoplus_{j\in J} F_j) : T \mapsto \iota_j T \iota_i^*$$

and their left inverses

$$\mu_{ij} : \mathcal{K}(\bigoplus_{i\in I} E_i, \bigoplus_{j\in J} F_j) \to \mathcal{K}(E_i, F_j) : S \mapsto \iota_j^* T \iota_i$$

exhibit the Banach space $\mathcal{K}(\bigoplus_{i\in I} E_i, \bigoplus_{j\in J} F_j)$ as a direct sum of its subspaces $\mathcal{K}(E_i, F_j)$. In other words, for every operator $T \in \mathcal{K}(\bigoplus_{i\in I} E_i, \bigoplus_{j\in J} F_j)$, the net $\sum_{S\subseteq I\times J finite} \lambda_{ij}(\mu_{ij}(T))$ converges in the norm *to T.*

Proof. Consider firstly an operator $\theta_x^{f,e}$ for some $x \in \mathsf{Ob}\,\mathcal{A}$ and some pair $(e_i) \in \bigoplus_i E_i, (f_i) \in \bigoplus_j F_j$, where both (e_i) and (f_j) have only a single non-zero coordinate e_k and f_ℓ. Then we have $\iota_j^*\iota_j\theta^{e,f} = \theta^{e,f}$ when $j = \ell$ and $\iota_j^*\iota_j\theta^{e,f} = 0$ otherwise. Similarly $\theta^{e,f}\iota_i^*\iota_i = \theta^{e,f}$, when $i = k$, and $\theta^{e,f}\iota_i^*\iota_i = 0$ otherwise. So the stated convergence result is clearly true for this $\theta^{e,f}$. But it is easy to see that the span of such operators $\theta^{e,f}$ includes all single-rank operators whose two arguments have finitely many non-zero entries, and we can also show these operators are norm-dense in all single-rank operators: write an arbitrary $(e_i) \in \bigoplus_i E_i$ and $(f_j) \in \bigoplus_j F_j$ as norm-limits of sequences $(e_i^n), (f_j^n)$ whose elements each have finitely many non-zero coordinates, as well as norm less than or equal to that of (e_i), respectively (f_j). But now note that we have

$$\begin{aligned}
\|\theta^{(e_i^n),(f_j^n)} - \theta^{(e_i),(f_j)}\| &\leq \|\theta^{(e_i^n),(f_j)} - \theta^{(e_i),(f_j)}\| + \|\theta^{(e_i^n),(f_j^n)} - \theta^{(e_i^n),(f_j)}\| \\
&\leq \|\theta^{(e_i^n)-(e_i),(f_j)}\| + \|\theta^{(e_i^n),(f_j^n)-(f_j)}\| \\
&\leq \|(e_i^n) - (e_i)\|\|(f_j)\| + \|(e_i^n)\|\|(f_j^n) - (f_j)\| \\
&\leq \|(e_i^n) - (e_i)\|\|(f_j)\| + \|(e_i)\|\|(f_j^n) - (f_j)\| \xrightarrow{n} 0
\end{aligned}$$

and hence $\theta^{(e_i^n),(f_j^n)} \to \theta^{(e_i),(f_j)}$ in the operator norm. Hence the single-rank operators with both module elements having finitely many non-zero coordinates are dense in all single-rank operators, so as the convergence holds for the former, by Corollary 1.4.3 it holds for the latter.

Another application of Corollary 1.4.3 tells us that the convergence holds for arbitrary compact operators. $\square$

2.5 Extending modules to the additive hull and approximate projectivity

In this section we show that all Hilbert modules over a C^*-category $\mathcal{A}$ extend to Hilbert modules over the additive hull $\mathcal{A}_\oplus$. We use for this the following lemma of Mitchener's, whose proof we include for completeness:

Lemma 2.5.1 ([Mit02b, Lemma 3.13])**.** *If E is a Hilbert module over a C^*-category $\mathcal{A}$, then an operator $T : E \to E$ is a positive element of the C^*-algebra $\mathcal{L}(E, E)$ if and only if for every $x \in \mathsf{Ob}\,\mathcal{A}$ and $e \in E(x)$, the inner product $\langle e, Te\rangle \in \mathcal{A}(x, x)$ is positive.*

Proof. For the rightward implication: by Lemma 1.1.13 if T is positive we can write it as $T = S^*S$ for some $S \in \mathcal{L}(E, E)$, so then $\langle e, Te\rangle = \langle Se, Se\rangle$ for all e, and this second term is always positive.

For the converse, suppose T has the stated property, then T is self-adjoint since $\langle Te, e\rangle = \langle e, Te\rangle^* = \langle e, Te\rangle$ for all e. Hence by Lemma 1.1.15, T decomposes as the difference of two positive operators $T = S - Q$ such that $SQ = 0$. But then for all e we have

$$0 \le \langle Qe, TQe\rangle = \langle Qe, -Q^2e\rangle = \langle e, -Q^3e\rangle.$$

But Q, and hence Q^3 is positive (by Lemma 1.1.19) so by the first part this implies $Q = 0$ and we see T is positive. □

To begin building the extension of E to $\mathcal{A}_\oplus$, we use the above lemma to prove in isolation the positivity axiom, which is of independent interest elsewhere:

Lemma 2.5.2 (c.f. [Lan95, Lemma 4.2])**.** *If E is a right Hilbert module over $\mathcal{A}$ and we have a sequence of elements $e_1 \in E(x_1), \dots, e_n \in E(x_n)$, then the $n \times n$ matrix μ_e with (i, j)-th entry $\langle e_i, e_j\rangle$ is a positive element of the C^*-algebra $M_{x_1,..,x_n}(\mathcal{A})$*

Proof. Consider the right Hilbert $\mathcal{A}$-module $F = h_{x_1} \oplus ... \oplus h_{x_n}$. By Proposition 2.4.11 and Lemma 2.3.9 we have $M_{x_1,..,x_n}(\mathcal{A}) \cong \mathcal{K}(F)$. So if we can show that for every $y \in \mathsf{Ob}\,\mathcal{A}$ and $f \in F(y)$, the inner product $\langle f, \mu_e f\rangle_F \in \mathcal{A}(y, y)$ is positive, then by Lemma 2.5.1, μ_e is a positive element of $\mathcal{L}(F)$, and hence of $M_{x_1,..,x_n}(\mathcal{A})$.

To do this, let $f = (f_1, \dots, f_n)$ where $f_i \in \mathcal{A}(y, x_i)$. Then

$$\begin{aligned}\langle f, \mu_e f\rangle_F = \sum_i (f_i^* \sum_j \langle e_i, e_j\rangle_E f_j) &= \sum_i (f_i^* \sum_j \langle e_i, e_j f_j\rangle_E) \\ &= \sum_i \sum_j \langle e_i f_i, e_j f_j\rangle_E \\ &= \langle \sum_i e_i f_i, \sum_j e_j f_j\rangle_E.\end{aligned}$$

This is a positive element of $\mathcal{A}(y, y)$ by the assumption that E is a Hilbert module. Hence it follows that μ_e is positive. □

The above lemma first appeared as [Mit02b, Lemma 3.14], but as far as we can see the proof given there does not quite work: it is unclear how to define the action of the matrix algebra on the supposed Hilbert module.

To extend an arbitrary Hilbert module from $\mathcal{A}$ to $\mathcal{A}_\oplus$, note firstly that by Lemma 1.2.4, any Hilbert module $E : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ lifts to a complex linear functor $E_\oplus : (\mathcal{A}_\oplus)^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$. We need to show that if E is a right Hilbert module, then $E_\oplus$ can be given an inner product making it into a right Hilbert $\mathcal{A}_\oplus$-module.

We do this as follows:

Proposition 2.5.3. *Let $E : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ be a right Hilbert $\mathcal{A}$-module. Then there is a unique extension $E_\oplus : (\mathcal{A}_\oplus)^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ of E to $\mathcal{A}_\oplus$ with*

$$E_\oplus(\{x_1, .., x_n\}) := E(x_1) \oplus \cdots \oplus E(x_n)$$

and matrices of $\mathcal{A}$-morphisms mapping sums to sums in the obvious way.

Furthermore, $E_\oplus$ has the structure of a right $\mathcal{A}_\oplus$-Hilbert module with inner products

$$\langle -, - \rangle : E_\oplus(\{y_1, .., y_k\}) \times E_\oplus(\{x_1, .., x_n\}) \to \mathcal{A}_\oplus(\{x_1, .., x_n\}, \{y_1, .., y_k\})$$

defined by

$$\langle (f_1, .., f_k), (e_1, .., e_n) \rangle_{E_\oplus} := [\langle f_j, e_i \rangle_E].$$

Proof. The unique extension $E_\oplus : (\mathcal{A}_\oplus)^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ is defined by Lemma 1.2.4, so we only need to show the Hilbert module axioms.

The involution and naturality axioms both follow immediately from the definition of $\langle -, - \rangle_{E_\oplus}$. The positivity axiom is the content of Lemma 2.5.2.

To prove the completeness property, note that from the arguments about norms in the proof of Proposition 1.2.3, we have

$$\max_i \|e_i\| \leq \|(e_i)\| \leq n^2 \max_i \|e_i\|$$

so a Cauchy sequence of vectors in $E_\oplus((x_i)) = \bigoplus_i E(x_i)$ must be a Cauchy sequence in each coordinate and converge to the coordinate-wise limit.

Finally, to find a set generating $E_\oplus$, suppose that $S = \{e \in E(x_e)\}$ is a set generating E across all objects, and that $S_\oplus = \{e \in E_\oplus(\{x_e\})\}$ is the 'same set' in $E_\oplus$. Then by considering 'column' matrices in $\mathcal{A}_\oplus(x_e, x_1 \oplus \cdots \oplus x_n)$ with one non-zero entry, we easily see that the subspaces $E(x_i) \subseteq E_\oplus(\{x_1, \ldots, x_n\})$ are in $\langle S_\oplus \rangle$. But then clearly $\langle S_\oplus \rangle$ is all of $E_\oplus$.

Hence $E_\oplus$ satisfies all axioms of a right Hilbert module over $\mathcal{A}_\oplus$. □

We prove a lemma clarifying the operation of finite-rank operators on right $\mathcal{A}_\oplus$-Hilbert modules.

Lemma 2.5.4. *Suppose E, F are right Hilbert modules over $\mathcal{A}_\oplus$. For all lists $\mathbf{x} = \{x_1, \ldots, x_n\} \in \mathsf{Ob}(\mathcal{A}_\oplus)$ and tuples of elements $(e_i) \in E(\mathbf{x}), (f_i) \in F(\mathbf{x})$, we have $\theta_{\mathbf{x}}^{(f_i),(e_i)} = \sum_{i=1}^n \theta_{x_i}^{f_i,e_i}$.*

Proof. This follows from the definition of the inner product: for any $z \in \mathsf{Ob}\,\mathcal{A}$ and $d \in E(z)$ we have

$$\begin{aligned}
\theta_{(x_i)}^{(f_i),(e_i)}(d) \quad &:= (f_i) \cdot \langle (e_i), d\rangle = (f_i) \cdot [\langle e_1, d\rangle \cdots \langle e_n, d\rangle] \\
&= F_\oplus([\langle e_1, d\rangle \cdots \langle e_n, d\rangle])((f_i)) \\
&= [F(\langle e_1, d\rangle) \cdots F(\langle e_n, d\rangle)]((f_i)) \\
&= F(\langle e_1, d\rangle)(f_1) + \cdots + F(\langle e_n, d\rangle)(f_n) \\
&= f_1 \cdot \langle e_1, d\rangle + \cdots + f_n \cdot \langle e_n, d\rangle \\
&= \textstyle\sum_{i=1}^n \theta_{x_i}^{f_i,e_i}(d).
\end{aligned}$$

□

We end this chapter with a powerful result which says that any Hilbert module E has an approximate identity consisting of self-adjoint operators that factor through finitely generated free modules. The results here are a direct adaptation and generalization of those in [Ble97, Section 3].

Lemma 2.5.5. *If E is a right Hilbert $\mathcal{A}$-module, then the C^*-algebra $\mathcal{K}(E)$ has an approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ consisting of operators $u_\lambda = \sum_{e \in I(\lambda)} \theta_{x_e}^{e,e}$ where each $I(\lambda)$ is a finite list of module elements $e \in E(x_e)$, where $x_e \in \mathsf{Ob}\mathcal{A}$.*

Proof. Note that the right ideal of finite-rank operators is dense in $\mathcal{K}(E)$, so by [Bro77, Theorem 2.1], there is an increasing approximate unit consisting of elements $u_\lambda = \sum_{r \in R_\lambda} r^* r$, where each R_λ is a finite list of finite-rank operators. Take one such operator $r = \sum_{i=1}^n \theta_{x_i}^{f_i,e_i}$, then we have

$$r^* r = \sum_{1 \le i,j \le n} \theta_{x_i}^{e_i,f_i} \theta_{x_j}^{f_j,e_j} = \sum_{1 \le i,j \le n} \theta_{x_i}^{e_i, \theta_{x_j}^{e_j,f_j}(f_i)} = \sum_{1 \le i,j \le n} \theta_{x_i}^{e_i, e_j \cdot \langle f_j, f_i\rangle}.$$

Note by Lemma 2.5.2 the matrix $\mu_f \in M_{x_1,\ldots,x_n}(\mathcal{A})$ with (i,j)-th entry $\langle f_i, f_j\rangle$ is positive, so by Lemma 1.1.13 there is a matrix N with entries n_{ij} such that $NN^* = \mu_f$. Hence extending E over $\mathcal{A}_\oplus$ and using Lemma 2.5.4 we get

$$r^* r = \theta_{\oplus_i x_i}^{(e_i),(e_i)\cdot NN^*} = \theta_{\oplus_i x_i}^{(e_i)\cdot N,(e_i)\cdot N} = \sum_{i=1}^n \theta_{x_i}^{\sum_j e_j \cdot n_{ij}, \sum_j e_j \cdot n_{ij}},$$

giving an approximate unit of the described form. □

We can then deduce the following generalization of one direction in [Ble97, Theorem 3.1].

Theorem 2.5.6. *If E is a right Hilbert $\mathcal{A}$-module, then there is a set of operators*

$$\{\phi_\lambda : \bigoplus_{x\in S_\lambda} h_x \to E : \lambda \in \Lambda\},$$

where each S_λ is a finite list of $\mathcal{A}$-objects, such that the net of operators $\phi_\lambda\phi_\lambda^*$ converges strongly to id_E.*

Proof. By the previous lemma, $\mathcal{K}(E)$ has an approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ consisting of operators $u_\lambda = \sum_{x\in S_\lambda} \theta_x^{e_x,e_x}$. It then follows from Lemma 2.3.12 that the map $\phi_\lambda : \bigoplus_{x\in S_\lambda} h_x \to E$ defined on $h_x(y) = \mathcal{A}(y,x)$ by $a \mapsto e_x \cdot a$ is bounded and has adjoint ϕ_λ^* with components

$$\phi_\lambda^* : \begin{array}{l} E(y) \to h_x(y) \\ e' \mapsto \langle e_x, e'\rangle \end{array}$$

and one easily verifies $\phi_\lambda\phi_\lambda^*(e) = u_\lambda(e)$. Hence Lemma 2.3.8 gives us the required convergence result. □

Remark 2.5.7. Note that we do not have that $S_\lambda \subset S_{\lambda'}$ for $\lambda < \lambda'$, and even if we did, it would not follow that $\phi_{\lambda'}$ restricts to ϕ_λ. Hence Theorem 2.5.6 does not allow us to embed E as a direct summand of a single infinite-dimensional free module: to do so would give an untenable generalization of the Kasparov stabilization theorem, contradicting the results in e.g. [Li10].

As an easy consequence we deduce that a Hilbert module with compact identity must be finitely generated projective:

Proposition 2.5.8. *If a right Hilbert $\mathcal{A}$-module E has $\mathrm{id}_E \in \mathcal{K}(E)$, then E must be finitely generated and projective. If $\mathcal{A}$ is unital, the converse holds, i.e. any finitely generated projective module has compact unit.*

Proof. As noted before, an approximate unit in a unital algebra converges in the norm to the identity. Therefore if $\mathcal{K}(E)$ is unital, then by Lemma 2.5.5 the module E must have a finite set of elements $e_1 \in E(x_1), \dots, e_n \in E(x_n)$ such that $\|\,\mathrm{id}_E - \sum_{i=1}^n \theta_{x_i}^{e_i,e_i}\| < 1$. But then as $\theta_{x_i}^{e_i,e_i}$ is normal, a simple functional calculus argument shows that $\sum_{i=1}^n \theta_{x_i}^{e_i,e_i}$ is invertible: a function with distance less than 1 from the constant function at 1 clearly has a multiplicative inverse. So decomposing as in Theorem 2.5.6 we see that in fact there is a compact operator $\phi : \bigoplus_{i=1}^n h_{x_i} \to E$ such that $\phi\phi^* : E \to E$ is an isomorphism. Suppose it has inverse ψ, then one easily verifies that $\phi^*\psi\phi : \bigoplus_{i=1}^n h_{x_i} \to \bigoplus_{i=1}^n h_{x_i}$ is a projection with image isomorphic to E.

For the partial converse, suppose $\mathcal{A}$ is unital, then for any $x \in \mathsf{Ob}\,\mathcal{A}$ we have $\mathcal{L}(h_x, h_x) \cong \mathcal{MA}(x,x) \cong \mathcal{A}(x,x) \cong \mathcal{K}(h_x,h_x)$ so we see that for any list $x_1 \dots x_n \in \mathsf{Ob}\,\mathcal{A}$ the module $\bigoplus_{i=1}^n h_{x_i}$ has a compact unit given by a diagonal matrix of identities id_{x_i}. So if E is a direct summand of $\bigoplus_{i=1}^n h_{x_i}$, we immediately see we can factor the identity of E through that of $\bigoplus_{i=1}^n h_{x_i}$, meaning id_E is compact. □

Chapter 3

Tensor products of bimodules

In this chapter we define the tensor product of two right Hilbert bimodules. In particular, we define for a right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule E the tensor functor

$$-\overline{\otimes}_{\mathcal{A}}E : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}.$$

We prove that this functor is strongly* continuous on bounded subsets and prove an Eilenberg-Watts theorem stating that all unital functors satisfying this continuity requirement are in fact given by tensor products with bimodules. We finish with a Morita theorem, i.e. a result characterizing those bimodules that tensor to give *equivalences* of Hilbert module categories.

As in the previous chapter, $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are locally small C^*-categories.

3.1 Tensoring bimodules

In this section we present some basic results on right Hilbert bimodules of C^*-categories and their tensor products.

Recall from Chapter 2 that a right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule is simply a C^*-functor $F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$, and that we say F is *non-degenerate* when the linear span of $\bigcup_{y \in \mathsf{Ob}\,\mathcal{A}} F\mathcal{A}(y, x)(F(y))$ is dense in $F(x)$ for all $x \in \mathsf{Ob}\,\mathcal{A}$, that is: the linear span of $\bigcup_{y \in \mathsf{Ob}\,\mathcal{A}} F\mathcal{A}(y, x)(F(y)(z))$ is dense in $F(x)(z)$ for all $x \in \mathsf{Ob}\,\mathcal{A}$ and $z \in \mathsf{Ob}\,\mathcal{B}$.

For any $a \in \mathcal{A}(x, y), z \in \mathsf{Ob}\,\mathcal{B}$ and $e \in F(x)(z)$, we write $a \cdot e$ for $F(a)(e)$ when the action is unambiguous.

Notice that by Proposition 1.1.46, there is a (large) C^*-category

$$\mathrm{Fun}(\mathcal{A}, \mathsf{Hilb}\text{-}\mathcal{B})$$

of right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodules. A morphism $T : E \to E'$ in this category is an natural transformation of adjointable maps with a uniform bound $\|T\|$ given by $\sup_{x \in \mathsf{Ob}\,\mathcal{A}} \|T_x\|_{\mathcal{L}(E(x), E'(x))}$. There is also a *strong* topology* on the hom-space $\mathrm{Fun}(\mathcal{A}, \mathsf{Hilb}\text{-}\mathcal{B})(E, E')$ in which $T_\lambda \to T$ if and only if $(T_\lambda)_x \to T_x$ strongly* for each $x \in \mathsf{Ob}\,\mathcal{A}$.

We can easily generalize Proposition 2.2.9 to characterize unitary isomorphisms of bimodules:

Proposition 3.1.1. *If E and F are right Hilbert $\mathcal{A}$-modules, and $T : E \Rightarrow F$ is a bimodule map from E to F, then T is a unitary isomorphism of right Hilbert $\mathcal{A} - \mathcal{B}$ bimodules if and only if for all objects $x \in \mathsf{Ob}\,\mathcal{A}$, the map of $\mathcal{B}$-modules $T(x) : E(x) \to F(x)$ is surjective at every $y \in \mathsf{Ob}\,\mathcal{B}$ and preserves all inner products.*

Proof. It is obvious that a transformation of functors on C^*-categories is a unitary isomorphism in the functor C^*-category if and only if it is a unitary isomorphism at any object. So this follows from Proposition 2.2.9. $\square$

We continue with our first example of a non-degenerate bimodule.

Lemma 3.1.2. *The Yoneda functor $\iota_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{A}$ is a non-degenerate $\mathcal{A} - \mathcal{A}$ bimodule.*

Proof. Recall from Lemma 2.4.4 that a functor $F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is non-degenerate if and only if for every approximate unit (u_λ) for $\mathcal{A}(x,x)$ and for every $z \in \mathsf{Ob}\,\mathcal{B}$ and $e \in F(x)(z)$ we have $u_\lambda \cdot e \xrightarrow{\lambda} e$. But in the case of the Yoneda functor this follows from Corollary 2.1.9. $\square$

Lemma 3.1.3. *If $\mathcal{A}$ is a unital C^*-category, then a right Hilbert bimodule $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is non-degenerate if and only if it is a unital functor.*

Proof. Recall from Corollary 2.4.9 that E is a non-degenerate bimodule if and only if it is non-degenerate as a functor $E : \mathcal{A} \to \mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B})$, and from Remark 1.4.6 that a functor from a unital C^*-category into a multiplier category is non-degenerate if and only if it is unital. The result follows. $\square$

Corollary 3.1.4. *Every right Hilbert $\mathcal{A}$-module E can be given the structure of a non-degenerate right Hilbert $\mathbb{C} - \mathcal{A}$ bimodule by the unique unital C^*-functor $\mathbb{C} \to \mathsf{Hilb}\text{-}\mathcal{A}$ that sends the single object of the former category to E.*

We now move on to a discussion of tensor products of bimodules.

Definition 3.1.5. If $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $F : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{C}$ are right Hilbert bimodules, their *algebraic tensor product* $E \otimes_{\mathcal{B}} F : \mathcal{A} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Vect}_{\mathbb{C},1})$ is defined at any two objects $x \in \mathsf{Ob}\,\mathcal{A}, z \in \mathsf{Ob}\,\mathcal{C}$ by taking the coend

$$E \otimes_{\mathcal{B}} F(x)(z) := \bigoplus_{y \in \mathsf{Ob}\,\mathcal{B}} E(x)(y) \otimes_{\mathbb{C}} F(y)(z) \Big/ \sim$$

where the relation $\sim$ is generated by identifying any pair of simple tensor elements $e \otimes f \in E(x)(y) \otimes_{\mathbb{C}} F(y)(z)$ and $e' \otimes f' \in E(x)(y') \otimes_{\mathbb{C}} F(y')(z)$ such that there exists an element $b \in \mathcal{B}(y, y')$ with $e' \cdot b = e$ and $b \cdot f = f'$. The covariant action of $\mathcal{A}$ on this space is defined by $a \cdot (e \otimes f) := (a \cdot e) \otimes f$ and the contravariant $\mathcal{C}$-action is given by $(e \otimes f) \cdot c := e \otimes (f \cdot c)$.

Note above that since the definition of $E \otimes_{\mathcal{B}} F(x)(z)$ involves a colimit over $\mathsf{Ob}\,\mathcal{B}$, we've had to define it to lie in thet category $\mathsf{Vect}_{\mathbb{C},1}$ of large complex vector spaces. But defining an inner product on the space helps us see that it is in fact small:

Lemma 3.1.6. *For any two right Hilbert bimodules $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $F : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{C}$ and for every $x \in \mathsf{Ob}\,\mathcal{A}$, the value of the algebraic tensor product $E \otimes_{\mathcal{B}} F(x) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},1}$ can be given an inner product satisfying the axioms of a large right pre-Hilbert $\mathcal{C}$-module, defined on simple tensors $e \otimes f \in E(x)(y) \otimes F(y)(z), e' \otimes f' \in E(x)(y') \otimes F(y')(z')$ as follows:*

$$\langle e \otimes f, e' \otimes f' \rangle = \langle f, \langle e, e' \rangle_E \cdot f' \rangle_F.$$

Moreover, for all $x \in \mathsf{Ob}\,\mathcal{A}$, the functor $E \otimes_{\mathcal{B}} F(x)$ and its completion $E \bar{\otimes}_{\mathcal{B}} F(x)$ in the norm resulting from the above inner product are in fact small vector spaces, and therefore are respectively large pre-Hilbert and Hilbert modules.

Proof. First we show that this inner product is well-defined. Take any tensor $e' \otimes f' \in E(x)(y') \otimes F(y')(z')$ and suppose that $e_0 \otimes f_0 \in E(x)(y_0) \otimes F(y_0)(z)$ and $e_1 \otimes f_1 \in E(x)(y_1) \otimes F(y_1)(z)$ are equivalent, witnessed by $b \in \mathcal{B}(y_0, y_1)$ with $e_1 \cdot b = e_0$ and $b \cdot f_0 = f_1$. Then

$$\begin{aligned}
\langle e_0 \otimes f_0, e' \otimes f' \rangle \quad &= \langle e_1 \cdot b \otimes f_0, e' \otimes f' \rangle \\
&= \langle f_0, \langle e_1 \cdot b, e' \rangle \cdot f' \rangle \\
&= \langle f_0, b^* \langle e_1, e' \rangle \cdot f' \rangle \\
&= \langle b \cdot f_0, \langle e_1, e' \rangle \cdot f' \rangle \\
&= \langle f_1, \langle e_1, e' \rangle \cdot f' \rangle \\
&= \langle e_1 \otimes f_1, e' \otimes f' \rangle.
\end{aligned}$$

It is also clear that $\langle e \otimes f, e' \otimes f' \rangle = \langle e' \otimes f', e \otimes f \rangle^*$, and this tells us the product is well-defined in the second argument too. Furthermore the inner product is clearly complex linear in both arguments, so by definition of the equivalence relation defining the tensor product, the inner product is well-defined.

Naturality under $\mathcal{B}$-morphisms follows from the definition of the action.

Now extend the product linearly to all of $E \otimes_{\mathcal{B}} F(x)$. To show this is positive, note we can extend F to $(\mathcal{B}^{\oplus})^{\mathrm{op}}$, and then for all $y_1, \ldots y_n \in \mathsf{Ob}\,\mathcal{B}$ and $e_i \in E(x)(y_i), f_i \in F(y_i)(z_i)$ we have

$$\begin{aligned}
\langle \textstyle\sum_{i=1}^n e_i \otimes f_i, \sum_{i=1}^n e_i \otimes f_i \rangle \quad &= \textstyle\sum_{i=1,j=1}^{i=n,j=n} \langle e_i \otimes f_i, e_j \otimes f_j \rangle \\
&= \textstyle\sum_{i=1,j=1}^{i=n,j=n} \langle f_i, \langle e_i, e_j \rangle \cdot f_j \rangle \\
&= \langle \mathbf{f}, M \cdot \mathbf{f} \rangle_{\oplus_j F(y_j)}
\end{aligned}$$

where $\mathbf{f} = (f_1, \cdots, f_n)$ and M is the $n \times n$ matrix whose (i,j)-th entry is $\langle e_i, e_j \rangle$. Lemma 2.5.2 tells us M is positive, but notice that it follows immediately from Lemma 1.1.13 that $F(M)$, which gives the action of M on $\mathbf{f}$, is positive too, and again by Lemma 2.5.1 we get that $\langle \sum_{i=1}^n e_i \otimes f_i, \sum_{i=1}^n e_i \otimes f_i \rangle$ is positive.

We show finally that $E \otimes_{\mathcal{B}} F(x)(z)$ is in fact a small vector space for all $x \in \mathsf{Ob}\,\mathcal{A}, z \in \mathsf{Ob}\,\mathcal{C}$. Let S be a set of elements $s \in E(x)(y_s)$ generating

$E(x) \in \mathsf{Hilb}\text{-}\mathcal{B}$. Suppose that $e = \Sigma_{i=1}^{n} \mu_i s_i + \Sigma_{j=1}^{m} s_j \cdot b_j$, where for all i and j we have $s_i \in S \cap E(x)(y), \mu_i \in \mathbb{C}, s_j \in S$, and $b_j \in \mathcal{B}(y, y_{s_j})$. Then for any $f \in F(y)(z)$, using the relationship $\sim$ on the second sum defining e, we see in fact that

$$e \otimes f \in \bigoplus_{y_s \in \mathsf{Ob}\,\mathcal{B}: s \in S} E(x)(y_s) \otimes_{\mathbb{C}} F(y_s)(z) \Big/ \sim$$

noting that this coend is indexed by a *set* and is hence a small vector space. But of course by definition of S generating $E(x)$, elements e of the form described are dense in $E(x)(y)$. Now approximating an arbitrary $e \in E(x)(y)$ by a sequence of finite sums e_n as above, we see by definition of the seminorm on $E \otimes_{\mathcal{B}} F(x)(z)$ that $e_n \otimes f \to e \otimes f$ as $n \to \infty$. Hence the small coend above is, before we complete, dense in the space of finite sums of simple tensors in the given seminorm. But then every element of $E \otimes_{\mathcal{B}} F(x)(z)$ can be identified with some Cauchy sequence of elements in $\bigoplus_{y_s \in \mathsf{Ob}\,\mathcal{B}: s \in S} E(x)(y_s) \otimes_{\mathbb{C}} F(y_s)(z) \Big/ \sim$, and hence $E \otimes_{\mathcal{B}} F(x)(z)$ is a set. It is easy to see that the space obtained by quotienting out elements with seminorm zero and completing is again small, and this is of course exactly $E \bar{\otimes}_{\mathcal{B}} F(x)(z)$. ☐

Lemma 3.1.7. *For each $x \in \mathsf{Ob}\,\mathcal{A}$, the large right Hilbert module $E \bar{\otimes}_{\mathcal{B}} F(x)$ is in fact a right Hilbert module, i.e. is generated by a set of elements. The spaces $E \bar{\otimes}_{\mathcal{B}} F(x)(z)$ assemble to a right Hilbert $\mathcal{A}$-$\mathcal{C}$ bimodule*

$$E \bar{\otimes}_{\mathcal{B}} F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{C}\,.$$

Proof. Let S be a set of elements $s \in E(x)(y_s)$ generating $E(x)$, and for each s let T_s be a set of elements generating $F(y_s) \in \mathsf{Hilb}\text{-}\mathcal{C}$. Then like above, we can use the relation $\sim$ and the definition of the norm to show that the set $\{s \otimes t : s \in S, t \in T_s\}$ generates $E \bar{\otimes}_{\mathcal{B}} F(x)(-)$.

It is trivial to show that the action in Definition 3.1.5 gives a linear action of $\mathcal{A}$ on the modules $E \bar{\otimes}_{\mathcal{B}} F(x)$ which is moreover compatible with the involution, proving the final part. ☐

A basic result on tensor products of bimodules is that they are associative up to unitary isomorphism:

Lemma 3.1.8. *If E, F, and G are respectively $\mathcal{A}$-$\mathcal{B}$, $\mathcal{B}$-$\mathcal{C}$ and $\mathcal{C}$-$\mathcal{D}$ right Hilbert bimodules, there is a canonical unitary isomorphism*

$$E \bar{\otimes}_{\mathcal{B}} (F \bar{\otimes}_{\mathcal{C}} G) \cong (E \bar{\otimes}_{\mathcal{B}} F) \bar{\otimes}_{\mathcal{C}} G.$$

Proof. This isomorphism is given on simple tensors by $e \otimes (f \otimes g) \mapsto (e \otimes f) \otimes g$. It is elementary to verify that it is in fact a unitary isomorphism which is natural in all three modules. ☐

In addition, the Yoneda bimodule functions as a tensor identity for non-degenerate bimodules:

Lemma 3.1.9. *If E is a right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule there is a canonical isometry of right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodules*

$$U_\ell : (\iota_{\mathcal{A}})\bar{\otimes}_{\mathcal{A}} E \xrightarrow{\cong} E$$

which is a unitary isomorphism if E is non-degenerate. Furthermore, there is always a canonical unitary isomorphism

$$U_r : E\bar{\otimes}_{\mathcal{B}}(\iota_{\mathcal{B}}) \xrightarrow{\cong} E$$

of right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodules.

Proof. The first map is given on simple tensors by $U_\ell(a \otimes e) = a \cdot e$, and the second by $U_r(e \otimes b) = e \cdot b$. It is elementary to verify that these maps respect the equivalence relation defining the uncompleted tensor product. They also preserve inner products:

$$\begin{aligned}\langle U_\ell(a \otimes e), U_\ell(a' \otimes e')\rangle \quad &= \langle a \cdot e, a' \cdot e'\rangle \\ &= \langle e, a^* a' \cdot e'\rangle \\ &= \langle e, \langle a, a'\rangle_{h_x} \cdot e'\rangle \\ &= \langle a \otimes e, a' \otimes e'\rangle\end{aligned}$$

and

$$\begin{aligned}\langle U_r(e \otimes b), U_r(e' \otimes b)\rangle \quad &= \langle e \cdot b, e' \cdot b'\rangle \\ &= b^* \langle e, e'\rangle b' \\ &= \langle b, \langle e, e'\rangle \cdot b'\rangle_{h_x} \\ &= \langle e \otimes b, e' \otimes b'\rangle\end{aligned}$$

meaning they are isometric and we can extend them to isometries on the whole module by Lemma 1.4.4. Note that if $e \in E(x)(y)$ and (u_λ) is an approximate unit for $\mathcal{A}(x,x)$, then $u_\lambda \cdot e \xrightarrow{\lambda} e$ if E is non-degenerate, and furthermore if (v_μ) is an approximate unit for $\mathcal{B}(y,y)$ then $e \cdot v_\mu \xrightarrow{\mu} e$ by Corollary 2.1.9. Hence by applying Theorem 2.1.10, we see U_ℓ is surjective when E is non-degenerate and U_r is always surjective. Hence by Proposition 3.1.1 we get the unitary isomorphism claims.

It is elementary to verify that the inverses of these maps are in fact their adjoints, and that they are natural in E. □

We deduce that the tensor product functor is an extension (through the Yoneda embedding) of a bimodule from representable modules to arbitrary modules:

Corollary 3.1.10. *If $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a non-degenerate right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule, then there is for each $x \in \mathsf{Ob}\,\mathcal{A}$ a unitary isomorphism of $\mathcal{B}$-modules $\rho_x : h_x \bar{\otimes}_{\mathcal{A}} E \to E(x)$, given on simple tensors $a \otimes e \in \mathcal{A}(x', x) \otimes E(x')(y)$ by*

$$\rho_x(a \otimes e) = a \cdot e \in E(x)(y).$$

This isomorphism is moreover natural in x.

Proof. The map ρ_x is obtained by taking the component at x of the map U_r from Lemma 3.1.9. Naturality of ρ_x in x is obvious, and naturality of its adjoint/inverse follows from this. □

Before we move onto our next proposition we need an intermediate lemma on positive operators. Recall that if a and b are two elements of a C^*-algebra A, we say $a \geq b$ if the element $a - b$ is positive.

Lemma 3.1.11. *If S is a positive operator on a right Hilbert $\mathcal{D}$-module H, then for any $x \in \mathsf{Ob}\,\mathcal{D}$, $h \in H(x)$ we have $\|S\|\langle h, h\rangle \geq \langle h, Sh\rangle$ in the C^*-algebra $\mathcal{D}(x, x)$. Furthermore, if $S \geq Q \geq 0$, we have $\|\langle h, Sh\rangle\| \geq \|\langle h, Qh\rangle\|$. If S is any operator, we have $\langle Sh, Sh\rangle \leq \|S\|^2\langle h, h\rangle$.*

Proof. Note by functional calculus in $\mathcal{L}(H)$ we have $\|S\| \operatorname{id}_H \geq S \geq 0$ and hence $\|S\| \operatorname{id}_H - S \geq 0$, hence by Lemma 2.5.1 we get

$$\|S\|\langle h, h\rangle - \langle h, Sh\rangle = \langle h, (\|S\| \operatorname{id}_H - S)h\rangle \geq 0.$$

If $S \geq Q \geq 0$ then by applying Lemma 2.5.1 to $S - Q$ we have

$$\langle h, Sh\rangle \geq \langle h, Qh\rangle \geq 0$$

so $\|\langle h, Sh\rangle\| \geq \langle \|h, Qh\rangle\|$.

The final result is obtained by applying the first result to the positive operator S^*S. □

This allows us to prove the functoriality of the tensor product construction.

Proposition 3.1.12. *For any right Hilbert $\mathcal{B}$-$\mathcal{C}$ bimodule F, a bounded adjointable transformation of $\mathcal{A}$-$\mathcal{B}$ bimodules $T : E \to E'$ induces a bounded adjointable morphism of $\mathcal{A} - \mathcal{C}$ bimodules*

$$T \otimes_{\mathcal{B}} \operatorname{id} : E \bar{\otimes}_{\mathcal{B}} F \to E' \bar{\otimes}_{\mathcal{B}} F$$

by acting on the first coordinate.

Proof. We first define a map $T \otimes_{\mathcal{B}} \operatorname{id} : E \otimes_{\mathcal{B}} F \to E' \otimes_{\mathcal{B}} F$ on the uncompleted Hilbert modules by $(T \otimes_{\mathcal{B}} \operatorname{id})(e \otimes f) := T(e) \otimes f$. It is straightforward to show that $(T \otimes_{\mathcal{B}} \operatorname{id})^* = (T^*) \otimes_{\mathcal{B}} \operatorname{id}$ functions as an adjoint on the uncompleted module. To extend the transformation to the completed module, we must show that $T \otimes_{\mathcal{B}} \operatorname{id}$ is bounded on sums of simple tensors: more specifically, we will prove that $\|T \otimes_{\mathcal{B}} \operatorname{id}\| \leq \|T\|$ on this subspace.

We need to show, for an arbitrary element

$$g = \sum_{i=1}^{n} e_i \otimes f_i \in E \otimes_{\mathcal{B}} F(x)(z)$$

where $e_i \otimes f_i \in E(x)(y_i) \otimes F(y_i)(z)$, that $\|T\|\|g\| \geq \|(T \otimes \operatorname{id})(g)\|$. Note that by Proposition 2.5.3 we can extend the right Hilbert $\mathcal{B}$-module $E(x)$ to a right Hilbert $\mathcal{B}_\oplus$-module $E(x)_\oplus$, and by Lemma 1.2.4 we can extend the C^*-functor $F : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{C}$ to a C^*-functor $F_\oplus : \mathcal{B}_\oplus \to \mathsf{Hilb}\text{-}\mathcal{C}$. So writing $\mathbf{f} := (f_i) \in (F_\oplus(\oplus_i y_i))(z)$ and $\mathbf{e} := E(x)_\oplus(\oplus_i y_i)$, we have

$$\begin{aligned}\|T\|^2\|g\|^2 &= \|T\|^2 \|\textstyle\sum_{i,j}\langle f_i, F(\langle e_i, e_j\rangle_{E(x)})(f_j)\rangle_{F(y_i)}\| \\ &= \|\langle \mathbf{f}, F_\oplus(\langle \|T\|\mathbf{e}, \|T\|\mathbf{e}\rangle_{E(x)_\oplus})(\mathbf{f})\rangle_{F_\oplus(\oplus_i y_i)}\| \\ &\geq \|\langle \mathbf{f}, F(\langle T\mathbf{e}, T\mathbf{e}\rangle)(\mathbf{f})\rangle\| \\ &= \|(T \otimes \mathrm{id})(g)\|^2\end{aligned}$$

where the inequality follows from Lemma 3.1.11.

By Lemma 1.4.4 we can extend $T \otimes_{\mathcal{B}} \mathrm{id}$ to a map on the completed module $T \otimes_{\mathcal{B}} \mathrm{id} : E \bar{\otimes}_{\mathcal{B}} F \to E' \bar{\otimes}_{\mathcal{B}} F$, also bounded by $\|T\|$ and with adjoint $T^* \bar{\otimes}_{\mathcal{B}} \mathrm{id}$. □

We are now going to show that moreover, the tensoring operation above satisfies a strong* continuity requirement.

Lemma 3.1.13. *For any right Hilbert $\mathcal{B}$-$\mathcal{C}$ bimodule F, the right tensor product with F, together with the assignment $T \mapsto T \otimes_{\mathcal{B}} \mathrm{id}$ defined above, gives a unital C^*-functor*

$$-\bar{\otimes}_{\mathcal{B}} F : \mathrm{Fun}(\mathcal{A}, \mathsf{Hilb}\text{-}\mathcal{B}) \to \mathrm{Fun}(\mathcal{A}, \mathsf{Hilb}\text{-}\mathcal{C})$$

which is strongly* continuous on bounded subsets of operators.

Proof. It is easily verified from the definition of $T \otimes \mathrm{id}$ that this assignment defines a C^*-functor.

So we have only left to show that if T_λ is a bounded net of operators in $\mathcal{L}(E, E')$ strongly* converging to 0, then the net $(T_\lambda \otimes \mathrm{id})$ of bimodule operators in $\mathrm{Fun}(\mathcal{A}, \mathsf{Hilb}\text{-}\mathcal{C})$ strongly* converges to 0 too. Note that as the operators are uniformly bounded, by Lemma 1.4.2 it is enough to show convergence on simple tensors. So we note

$$\begin{aligned}\|(T_\lambda \otimes \mathrm{id})(e \otimes f)\|^2 &= \|T_\lambda(e) \otimes f\|^2 \\ &= \|\langle f, \langle T_\lambda(e), T_\lambda(e)\rangle \cdot f\rangle\| \\ &= \|\langle f, F(\langle T_\lambda(e), T_\lambda(e)\rangle)(f)\rangle\| \\ &\leq \|F(\langle T_\lambda(e), T_\lambda(e)\rangle)\|\|\langle f, f\rangle\| \\ &\leq \|T_\lambda(e)\|^2\|f\|^2\end{aligned}$$

where the first inequality follows from the last statement of Lemma 3.1.11, and the second from the norm-decreasing property of F (see Proposition 1.1.34). But by hypothesis the last term goes to zero. A similar argument applies to $(T_\lambda \bar{\otimes} \mathrm{id})^* = (T_\lambda^* \bar{\otimes} \mathrm{id})$. Hence $(T_\lambda \bar{\otimes} \mathrm{id})$ goes to 0 strongly*. □

Remark 3.1.14. From now on we will write 'strong*' instead of 'strongly* continuous on bounded subsets', for brevity.

We follow with a lemma that says the functoriality of tensor products also holds on the left.

Lemma 3.1.15. *If E is a right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule, the left tensor product with E induces a strong* unital C^*-functor*

$$E\bar{\otimes}_{\mathcal{B}}- : \mathrm{Fun}(\mathcal{B}, \mathsf{Hilb}\text{-}\mathcal{C}) \to \mathrm{Fun}(\mathcal{A}, \mathsf{Hilb}\text{-}\mathcal{C}).$$

Proof. This statement is an obvious analogue of the preceding two results and its proof proceeds similarly: for right Hilbert $\mathcal{B}$-$\mathcal{C}$ modules F and F' and a bounded adjointable bimodule map $S : F \to F'$ we define

$$\mathrm{id} \otimes_{\mathcal{B}} S : E\bar{\otimes}_{\mathcal{B}} F \to E\bar{\otimes}_{\mathcal{B}} F'$$

by $(\mathrm{id} \otimes_{\mathcal{B}} S)(e \otimes f) := e \otimes S(f)$ on simple tensors. To show that $\|\mathrm{id} \otimes_{\mathcal{B}} S\| \leq \|S\|$ on finite sums of simple tensors, let

$$g = \sum_{i=1}^{n} e_i \otimes f_i \in E \otimes_{\mathcal{B}} F(x)(z)$$

where $e_i \otimes f_i \in E(x)(y_i) \otimes F(y_i)(z)$. Assume again that $\mathcal{B}$ is closed under finite direct sums, writing $\mathbf{f} := (f_i) \in (F(\oplus_i y_i))(z)$ and $\mathbf{e} := E_{\oplus}(x)(\oplus_i y_i)$. Then

$$\begin{aligned}
\|S\|^2\|g\|^2 &= \|S\|^2 \|\textstyle\sum_{i,j} \langle f_i, F(\langle e_i, e_j\rangle)(f_j)\rangle\| \\
&= \|\langle \mathbf{f}, \|S\|^2 F(\langle \mathbf{e}, \mathbf{e}\rangle)(\mathbf{f})\rangle\| \\
&\geq \|\langle \mathbf{f}, (S^* \circ F(\langle \mathbf{e}, \mathbf{e}\rangle) \circ S)(\mathbf{f})\rangle\| \\
&= \|\langle S(\mathbf{f}), F(\langle e, e\rangle)(S(\mathbf{f}))\rangle\| \\
&= \|(\mathrm{id} \otimes S)(g)\|^2
\end{aligned}$$

where in the inequality we have used Lemma 1.2.9 together with Lemma 3.1.11. Hence $\mathrm{id} \otimes_{\mathcal{B}} S$ is bounded by $\|S\|$ and can be extended to a natural transformation on the whole module; it also clearly has adjoint $\mathrm{id} \otimes_{\mathcal{B}} S^*$.

To prove strong* continuity on bounded subsets of operators, note that

$$\begin{aligned}
\|(\mathrm{id} \otimes S)(e \otimes f)\|^2 &= \|e \otimes S(f)\|^2 \\
&= \|\langle S(f), F(\langle e, e\rangle)(S(f))\rangle\| \\
&\leq \|F(\langle e, e\rangle)\| \|\langle S(f), S(f)\rangle\| \\
&\leq \|e\|^2 \|S(f)\|^2
\end{aligned}$$

where we have used the first part of Lemma 3.1.11 for the first inequality and the norm-decreasing property of F for the second. Hence just as in the previous proof we deduce that if (S_λ) is a bounded net of transformations in $\mathrm{Fun}(\mathcal{C}, \mathsf{Hilb}\text{-}\mathcal{C})$ strongly* converging to zero, then $(\mathrm{id} \otimes S_\lambda)$ converges strongly* to zero too. $\square$

It turns out that if E is non-degenerate, the above functor has image entirely in non-degenerate bimodules:

Lemma 3.1.16. *If $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a non-degenerate right Hilbert bimodule and $F : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{C}$ is any right Hilbert bimodule, then the right Hilbert bimodule $E\bar{\otimes}_{\mathcal{B}} F : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{C}$ is also non-degenerate.*

Proof. By the characterization in Theorem 1.4.5, it suffices to show that for any element $g \in E\bar{\otimes}_{\mathcal{B}}F(x)(z)$, and approximate unit (u_λ) for $\mathcal{A}(x,x)$, that $u_\lambda \cdot g \xrightarrow{\lambda} g$ in norm. Consider first a simple tensor $e \otimes f \in E(x)(y) \otimes F(y)(z)$: note that

$$\begin{aligned} \|u_\lambda \cdot (e \otimes f) - e \otimes f\|^2 &= \|(u_\lambda \cdot e - e) \otimes f\|^2 \\ &= \|\langle f, F(\langle u_\lambda \cdot e - e, u_\lambda \cdot e - e\rangle)(f)\rangle\| \end{aligned}$$

and this last term tends to zero since $u_\lambda \cdot e \xrightarrow{\lambda} e$ in the norm and F is bounded.

Hence by Corollary 1.4.3 we obtain the case for general g, since every g is by definition a norm-limit of finite sums of simple tensors. $\square$

Letting now $\mathcal{A} = \mathbb{C}$ and noting that by Lemma 3.1.3 a non-degenerate right Hilbert bimodule whose left action comes from $\mathbb{C}$ is simply a right Hilbert module, we easily derive the following proposition from Lemma 3.1.15 and Lemma 3.1.16:

Proposition 3.1.17. *Given a right Hilbert $\mathcal{B}$-$\mathcal{C}$ bimodule E, the tensor product above defines a strong* unital C^*-functor*

$$-\bar{\otimes}_{\mathcal{B}}E : \mathsf{Hilb}\text{-}\mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{C}\,.$$

We derive a corollary that characterizes which bimodules preserve compact operators upon tensoring.

Corollary 3.1.18. *For a given right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$, the tensor product functor $-\bar{\otimes}_{\mathcal{A}}E : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ preserves compact morphisms if and only if E has image entirely in compact morphisms.*

Proof. Since by Corollary 3.1.10 we have $E \cong (-\bar{\otimes}_{\mathcal{A}}E)\circ\iota_{\mathcal{A}}$ and by Lemma 2.3.9 the image of $\iota_{\mathcal{A}}$ consists of compact operators, the 'only if' direction is clear.

For the converse, note that by Proposition 2.3.14 the ideal $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ is generated by the image of $\iota_{\mathcal{A}}$, so if E has image in $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$, we have that $-\bar{\otimes}_{\mathcal{A}}E$ sends all of $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ to compact morphisms since it is an extension of E through $\iota_{\mathcal{A}}$. $\square$

The point of the following section is to prove a converse to Proposition 3.1.17: that is, to prove that up to unitary isomorphism, tensor products by right Hilbert $\mathcal{B}$-$\mathcal{C}$ bimodules in fact make up *all strong* unital functors* from $\mathsf{Hilb}\text{-}\mathcal{B}$ to $\mathsf{Hilb}\text{-}\mathcal{C}$.

3.2 The Eilenberg-Watts theorem

In this section we prove the C^*-categorical Eilenberg-Watts theorem, which establishes a 1-1 correspondence between bimodules and strong* functors on categories of Hilbert modules. We begin by establishing the Eilenberg-Watts map and its naturality.

Definition 3.2.1. If $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a strong* unital C^*-functor, E is the composition $E := F \circ \iota_{\mathcal{A}} : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$, and M is a right Hilbert $\mathcal{A}$-module, we define the *Eilenberg-Watts map*

$$\psi_M : M \otimes_{\mathcal{A}} E \to F(M)$$

on simple tensors $m \otimes e \in M(x) \otimes E(x)(z)$ by

$$\psi_M(m \otimes e) := F(\epsilon_m)(e) \in F(M)(z),$$

where $\epsilon_m \in \mathcal{K}(h_x, M)$ is defined as in Lemma 2.3.12.

Lemma 3.2.2. *The Eilenberg-Watts map as defined above is well-defined and extends to an map $\psi_M : M\bar{\otimes}_{\mathcal{A}}E \to F(M)$ on the completed module which is natural in M and in F, and which moreover preserves all inner products.*

Proof. We show first that ψ is well-defined on the uncompleted module $M\otimes_{\mathcal{A}}E$: recall that $m\otimes e \in M(x)\otimes E(x)(z)$ and $m'\otimes e' \in M(x')\otimes E(x')(z)$ are identified in $M \otimes_{\mathcal{A}} E(z)$ whenever there exists an element $a \in \mathcal{A}(x, x')$ such that we have $m' \cdot a = m$ and $a \cdot e = e'$. But then

$$\begin{aligned} \psi_x(m' \otimes e') &= F(\epsilon_{m'})(e') && = F(\epsilon_{m'})(a \cdot e) \\ &= F(\epsilon_{m'} \circ \iota(a))(e) && = F(\epsilon_{m' \cdot a})(e) \\ &= F(\epsilon_m)(e) && = \psi_x(m \otimes e). \end{aligned}$$

To see that ψ is a map of $\mathcal{B}$-modules, take $m \otimes e \in M(x) \otimes E(x)(z)$ and a morphism $b \in \mathcal{B}(z', z)$. Then note

$$\psi((m \otimes e) \cdot b) = \psi(m \otimes (e \cdot b)) = F(\epsilon_m)(e \cdot b) = F(\epsilon_m)(e) \cdot b = \psi(m \otimes e) \cdot b$$

as $F(\epsilon_m)$ is a $\mathcal{B}$-module map.

To extend ψ and show that it preserves inner products, take simple tensors $m \otimes e \in M(x) \otimes E(x)(z)$ and $m' \otimes e' \in M(y) \otimes E(y)(z)$, and note that from Lemma 2.3.12 we get that $\epsilon^*_{m'}\epsilon_m = \iota(\langle m', m\rangle)$. So we have

$$\begin{aligned} \langle \psi(m \otimes e), \psi(m' \otimes e')\rangle &= \langle F(\epsilon_m)(e), F(\epsilon_{m'})(e')\rangle && = \langle e, F(\epsilon^*_{m'}\epsilon_m)(e)\rangle \\ &= \langle e, E(\langle m, m'\rangle)(e)\rangle && = \langle m \otimes e, m' \otimes e'\rangle. \end{aligned}$$

Hence we see ψ preserves all inner products on simple tensors and finite sums thereof, so by Lemma 1.4.4 it extends to an isometry on $M\bar{\otimes}_{\mathcal{A}}E$ which must also preserve inner products.

Recall from Lemma 2.3.12 that for a bounded adjointable operator of right Hilbert $\mathcal{A}$-modules $\phi : M \to N$ we have $\phi \circ \epsilon_m = \epsilon_{\phi(m)}$. To prove naturality of ψ_M in M, we must show that $F(\phi) \circ \psi_M = \psi_N \circ (\phi \otimes \mathrm{id})$. But note for $m \otimes e \in M \otimes E$ that

$$\begin{aligned} (F(\phi) \circ \psi_M)(m \otimes e) &= F(\phi)F(\epsilon_m)(e) && = F(\phi\epsilon_m)(e) \\ &= F(\epsilon_{\phi(m)})(e) && = \psi_N(\phi(m) \otimes e) \\ & && = \psi_N((\phi \otimes \mathrm{id})(m \otimes e)) \end{aligned}$$

proving naturality on simple tensors, and hence again on the whole module. $\square$

Having defined the natural transformation ψ, we show in a few steps that it is a natural unitary isomorphism, beginning with the case for representable modules.

Proposition 3.2.3. *For $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $\psi : -\bar{\otimes}_{\mathcal{A}}(F \circ \iota) \Rightarrow F$ as above, ψ is a unitary isomorphism on the subcategory of representable right Hilbert $\mathcal{A}$-modules and their finite direct sums.*

Proof. We have shown so far that ψ preserves all inner products, so by Proposition 2.2.9 we only have left to show that for each $x \in \mathsf{Ob}\,\mathcal{A}$, the $\mathcal{B}$-module map $\psi_{h_x} : h_x \otimes_{\mathcal{A}} (F \circ \iota) \to F(h_x)$ is surjective at each $y \in \mathsf{Ob}\,\mathcal{B}$. But this is just an instance of Corollary 3.1.10 for the bimodule $F \circ \iota$.

Finally, F, being a unital C^*-functor, must preserve finite direct sums, and we easily deduce ψ is a unitary isomorphism on these, too. □

We are now in a position to prove the Eilenberg-Watts theorem using the finite free approximation result at the end of the previous chapter.

Theorem 3.2.4 (c.f. [Ble97, Theorem 5.4])**.** *If $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a strong* unital functor, the Eilenberg-Watts map $\psi_M : M \otimes (F \circ \iota_{\mathcal{A}}) \to F(M)$ is a unitary isomorphism on every right Hilbert $\mathcal{A}$-module M.*

Proof. As above, we only have left to show that ψ_M is surjective at every $y \in \mathsf{Ob}\,\mathcal{B}$. Recall from Theorem 2.5.6 that there exists a set of operators $\phi_\lambda : \bigoplus_{x \in S_\lambda} h_x \to M$, where each $S_\lambda : \lambda \in \Lambda$ is a finite list of $\mathcal{B}$-objects, such that the net of operators $\phi_\lambda \phi_\lambda^*$ converges strongly* to id_M. Writing S for S_λ we have a diagram

$$\begin{tikzcd}
M\bar{\otimes}_{\mathcal{A}}E \arrow[r, "\psi_M"] \arrow[d, shift right, "\phi_\lambda^*\otimes\mathrm{id}"'] & F(M) \arrow[d, shift right, "F(\phi_\lambda^*)"'] \\
(\oplus_S h_s)\bar{\otimes}_{\mathcal{A}}E \arrow[r, "\psi_{\oplus_S h_s}"] \arrow[u, shift right, "\phi_\lambda\otimes\mathrm{id}"'] & F(\oplus_S h_s) \arrow[u, shift right, "F(\phi_\lambda)"']
\end{tikzcd}$$

where the two squares with vertical arrows pointing in the same direction both commute by the naturality of ψ. We aim to show the top horizontal map is surjective at any $y \in \mathsf{Ob}\,\mathcal{B}$, so take an arbitrary $f \in F(M)(y)$. We know by Proposition 3.2.3 that $\psi_{\oplus_S h_s}$ is surjective at every object, so we have $F(\phi_\lambda^*)(f) = \psi_{\oplus_S h_s}(t)$ for some $t \in (\oplus_S h_s)\bar{\otimes}_{\mathcal{A}}E(y)$.

So we have

$$F(\phi_\lambda\phi_\lambda^*)(f) = F(\phi_\lambda)(\psi_{\oplus_S h_s}(t)) = \psi_M((\phi_\lambda \otimes \mathrm{id})(t)) \in \mathrm{im}\ \psi_M.$$

But F is strong* and unital and $\phi_\lambda\phi_\lambda^*$ is a bounded net, so $F(\phi_\lambda\phi_\lambda^*) \xrightarrow{\lambda} \mathrm{id}_{F(M)}$ strongly*, and we get $F(\phi_\lambda\phi_\lambda^*)(f) \xrightarrow{\lambda} f$. So ψ_M has dense image, so by Lemma 1.4.4 it is surjective. □

There is a related result about module categories that uses the approximate projectivity of Hilbert modules in a similar way:

Lemma 3.2.5. *Suppose $F, F' : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ are strong* unital C^*-functors, and let*

$$\eta, \zeta : F \Rightarrow F'$$

be two (bounded, adjointable) natural transformations. If η and ζ agree on representable modules, then $\eta = \zeta$.

Proof. Since $\eta - \zeta : F \Rightarrow F'$ is another bounded adjointable natural transformation, it suffices to show that if $\xi : F \Rightarrow F'$ is zero on representable modules, then $\xi = 0$.

Recall once more from Theorem 2.5.6 that there exists a set $\{\phi_\lambda : \lambda \in \Lambda\}$ of bounded adjointable operators $\phi_\lambda : \bigoplus_{x \in S_\lambda} h_x \to M$ (where S_λ is a finite list of $\mathcal{B}$-objects for each $\lambda \in \Lambda$) such that the net of operators $\phi_\lambda \phi_\lambda^*$ converges strongly* to id_M. We hence have for each $S := S_\lambda$ a diagram

$$\begin{tikzcd}
F(M) \arrow[r, "\xi_M"] \arrow[d, shift right, "F(\phi_\lambda^*)"'] & F'(M) \arrow[d, shift right, "F'(\phi_\lambda^*)"'] \\
F(\oplus_S h_s) \arrow[r, "\xi_{\oplus_S h_s}"] \arrow[u, shift right, "F(\phi_\lambda)"'] & F'(\oplus_S h_s) \arrow[u, shift right, "F'(\phi_\lambda)"']
\end{tikzcd}$$

We want to show for an arbitrary $f \in F(M)(x), x \in \mathsf{Ob}\,\mathcal{B}$ that $\xi_M(f) = 0$. Since $F(\phi_\lambda)F(\phi_\lambda^*) \xrightarrow{\lambda} \mathrm{id}_{F(M)}$ strongly*, we have $F(\phi_\lambda)F(\phi_\lambda^*)(f) \to f$. At the same time, since $\xi_{\oplus_S h_s} = 0$, by naturality of ξ we get

$$(\xi_M \circ F(\phi_\lambda) \circ F(\phi_\lambda^*))(f) = (F'(\phi_\lambda) \circ \xi_{\oplus_S h_s} \circ F(\phi_\lambda^*))(f) = 0.$$

Hence we have a net of elements converging in the norm to f which are in the kernel of the bounded map ξ_M, so $\xi_M(f) = 0$. $\square$

Another way of formulating this result is: if $\eta : -\bar{\otimes}_{\mathcal{A}} E \Rightarrow \bar{\otimes}_{\mathcal{A}} E'$ is any transformation of tensor product functors, it is in fact given on each module by tensoring with a fixed module map $\epsilon : E \to E'$, which can be obtained by whiskering η through $\iota_{\mathcal{A}}$.

3.3 Bi-Hilbert and imprimitivity modules

Up until now we have discussed bimodules which have inner products valued in the C^*-category acting on the right. In this section we discuss bimodules with products in both categories, and see that a subclass of these 'bi-Hilbert' bimodules are exactly those bimodules that give Morita equivalences, that is equivalences of module categories.

The thesis [Fer23] investigated the same question for small C^*-categories, with one key step being a result ([Fer23, Theorem 6.3.1]) saying that each small C^*-category is Morita equivalent to a C^*-algebra. We do not assume smallness so we take a somewhat different approach, adapting directly the results from [RW98, Chapter 3] and [Ech+06, Chapter 1] that characterize which bimodules give Morita equivalences in the C^*-algebra case. The aforementioned category-algebra equivalence becomes an example in this framework (see Proposition 3.4.9).

We begin by setting up the tools for 'switching' between two actions of non-unital categories.

For the following lemmas, recall from Definition 1.1.39 that for a C^*-category $\mathcal{A}$ we denote by $\mathcal{A}^+$ the *minimal unitization* of $\mathcal{A}$ and from Definition 1.1.41 that for any C^*-category $\mathcal{A}$ and unital complex category $\mathcal{C}$ we denote by $\mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathcal{C})$ the category of functors from $\mathcal{A}$ to $\mathcal{C}$ that restrict to unital maps on all unital endomorphism algebras in $\mathcal{A}$.

Lemma 3.3.1. *Let $\mathcal{A}$ be a C^*-category and $E : \mathcal{A}^{\mathrm{op}} \to \mathsf{Vect}_{\mathbb{C},0}$ be a right Hilbert $\mathcal{A}$-module. Then E is an object of* $\mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathsf{Vect}_{\mathbb{C},0})$.

Proof. We need to show that any existing units in $\mathcal{A}$ act unitally on the component spaces of E: but this follows immediately from Corollary 2.1.11. □

Lemma 3.3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories. There exist natural isomorphisms*

$$\begin{array}{rrl} & \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}^{\mathrm{op}}, \mathsf{Vect}_{\mathbb{C},0})) & \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}^+, \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}^{\mathrm{op}+}, \mathsf{Vect}_{\mathbb{C},0})) \\ \cong & \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}^+ \times \mathcal{B}^{\mathrm{op}+}, \mathsf{Vect}_{\mathbb{C},0}) & \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}^{\mathrm{op}+}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}^+, \mathsf{Vect}_{\mathbb{C},0})) \\ & & \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathsf{Vect}_{\mathbb{C},0})). \end{array}$$

Proof. The first and final isomorphisms follow by repeated application of Proposition 1.1.42, and the middle two are simply currying of unital functors. □

For the rest of this chapter we will identify functors with their images under these isomorphisms, for the sake of conceptual simplicity.

Finally, recall from Section 2.1 that we can define a left Hilbert $\mathcal{A}$-module as a right Hilbert $\mathcal{A}^{\mathrm{op}}$-module, or equivalently as a functor $E : \mathcal{A} \to \mathsf{Vect}_{\mathbb{C},0}$ with inner products $\langle -, - \rangle : E(x) \times E(y) \to \mathcal{A}(y, x)$ which are linear in the first variable and conjugate linear in the second, such that $\langle a \cdot e, f \rangle = a \circ \langle e, f \rangle$, and such that they satisfy the rest of the Hilbert module axioms verbatim.

Definition 3.3.3. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, a *bi-Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule* is a functor $E : \mathcal{A}^+ \times \mathcal{B}^{\mathrm{op}+} \to \mathsf{Vect}_{\mathbb{C},0}$ which is equipped for all $x \in \mathsf{Ob}\, \mathcal{A}, y \in \mathsf{Ob}\, \mathcal{B}$ with $\mathcal{A}$-valued products

$${}_{\mathcal{A}}\langle -, - \rangle : E(x')(y) \times E(x)(y) \to \mathcal{A}(x, x')$$

and $\mathcal{B}$-valued products

$$\langle -, - \rangle_{\mathcal{B}} : E(x)(y') \times E(x)(y) \to \mathcal{B}(y, y')$$

which give E (or rather, its images under Lemma 3.3.2) the structure of both an $\mathcal{A}$-$\mathcal{B}$ right Hilbert bimodule and a $\mathcal{B}^{\mathrm{op}}$-$\mathcal{A}^{\mathrm{op}}$ right Hilbert bimodule. Unpacking this a little, we require that:

- $E(x)(-)$ is a right Hilbert $\mathcal{B}$-module for each $x \in \mathsf{Ob}\,\mathcal{A}$, with product $\langle -, - \rangle_{\mathcal{B}}$ and right action by morphisms of the form (id_x, b).
- $E(-)(y)$ is a left Hilbert $\mathcal{A}$-module for each $y \in \mathsf{Ob}\,\mathcal{B}$, with product ${}_{\mathcal{A}}\langle -, - \rangle$ and left action by morphisms of the form (a, id_y)
- The action of $a \in \mathcal{A}$ on the $\mathcal{B}$-modules is bounded by $\|a\|$ and adjoint to the action of a^*; similarly for the action of $b \in \mathcal{B}$ on the $\mathcal{A}$-modules. Writing this out in equations, we get

$$\begin{array}{llll} \langle e, a \cdot f \rangle_{\mathcal{B}} & = \langle a^* \cdot e, f \rangle_{\mathcal{B}} \quad \text{and} & \|\langle a \cdot e, a \cdot e \rangle_{\mathcal{B}}\| & \leq \|a\|^2 \|\langle e, e \rangle_{\mathcal{B}}\| \\ {}_{\mathcal{A}}\langle e, f \cdot b \rangle & = {}_{\mathcal{A}}\langle e \cdot b^*, f \rangle \quad \text{and} & \|{}_{\mathcal{A}}\langle e \cdot b, e \cdot b \rangle\| & \leq \|b\|^2 \|{}_{\mathcal{A}}\langle e, e \rangle\|. \end{array}$$

We write

$$E^1 : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$$

and

$$E^2 : \mathcal{B}^{\mathrm{op}} \to \mathsf{Hilb}\text{-}\mathcal{A}^{\mathrm{op}}$$

for the two right Hilbert bimodule 'components' of E when it is necessary to consider them separately, but will generally speak simply of E. These two bimodules are automatically non-degenerate:

Lemma 3.3.4. *If E is a bi-Hilbert bimodule, the bimodules $E^1 : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $E^2 : \mathcal{B}^{\mathrm{op}} \to \mathsf{Hilb}\text{-}\mathcal{A}^{\mathrm{op}}$ are non-degenerate.*

Proof. Take any $x \in \mathsf{Ob}\,\mathcal{A}, y \in \mathsf{Ob}\,\mathcal{B}, e \in E(x)(y)$, and note that we can identify $E(x)(y) = E^1(x)(y) = E^2(y)(x)$. By Lemma 2.4.4 it suffices to show that for any approximate unit (u_λ) for $\mathcal{A}(x,x)$ we have $E^1(u_\lambda)(e) \xrightarrow{\lambda} e$. But note that $E^1(u_\lambda)(e) = e \cdot u_\lambda \in E^2(y)(x)$, and that (u_λ) is certainly an approximate unit of $\mathcal{A}^{\mathrm{op}}(x,x)$. So by Corollary 2.1.9 we have that $e \cdot u_\lambda \xrightarrow{\lambda} e$, proving the lemma. The non-degeneracy of E^2 follows similarly. $\square$

Definition 3.3.5. For any right Hilbert bimodule $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$, we let

$$\langle E, E \rangle$$

denote the subcategory of $\mathcal{B}$ given by the linear span of the inner products $\langle e, f \rangle \in \mathcal{B}(y, y')$ for all objects $x \in \mathsf{Ob}\,\mathcal{A}, y, y' \in \mathsf{Ob}\,\mathcal{B}$ and module elements $e \in E(x)(y'), f \in E(x)(y)$. Noting that this subcategory is closed under the involution and under multiplication by outside elements, we denote by

$$\overline{\langle E, E \rangle}$$

the ideal given by its norm closure. We say that E is *essential* if $\overline{\langle E, E \rangle}$ is an essential ideal and that E is *full* if $\overline{\langle E, E \rangle} = \mathcal{B}$.

Example 3.3.6. The Yoneda bimodule $\iota_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{A}$ is full: note that $\langle \iota_{\mathcal{A}}, \iota_{\mathcal{A}} \rangle$ consists of all morphisms of $\mathcal{A}$ which factorize through some object, but by Lemma 1.2.7 this is all of $\mathcal{A}$.

We are interested in bi-Hilbert bimodules that satisfy an additional axiom on the inner products:

Definition 3.3.7. A bi-Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule E is termed *a partial imprimitivity $\mathcal{A}$-$\mathcal{B}$ bimodule* when for all elements $e \in E(x)(y), f \in E(x')(y)$ and $g \in E(x')(y')$ the *imprimitivity equation*

$$_{\mathcal{A}}\langle e, f\rangle \cdot g = e \cdot \langle f, g\rangle_{\mathcal{B}},$$

holds in $E(x)(y')$. It is called *right essential* if the bimodule $E^1 : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is essential, *right full* if E^1 is full, *left-essential* if $E^2 : \mathcal{B}^{\mathrm{op}} \to \mathsf{Hilb}\text{-}\mathcal{A}^{\mathrm{op}}$ is essential and *left-full* if E^2 is full. A partial imprimitivity bimodule E is called an $\mathcal{A}$-$\mathcal{B}$ *imprimitivity bimodule* if both E^1 and E^2 are full.

Remark 3.3.8. In [Ech+06], which treats the C^*-algebra case, bimodules which are left full in our terminology are termed 'right partial', and vice versa. We modify their terminology here since strictly it implies that modules have full products on both sides when they are both left and right partial, a rather unintuitive statement.

Example 3.3.9. The representation bimodule $\iota_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{A}$ can be given the structure of a bi-Hilbert bimodule by setting for $a \in h_x(y), b \in h_z(y)$ the product $_{\mathcal{A}}\langle a, b\rangle := a \circ b^*$, and we see immediately that this is a partial imprimitivity bimodule since $_{\mathcal{A}}\langle a, b\rangle \cdot c = a \circ b^* \circ c = a \cdot \langle b, c\rangle_{\mathcal{A}}$. As explained in the previous example, it is evident that both products are full, so $\iota_{\mathcal{A}}$ is in fact an $\mathcal{A} - \mathcal{A}$ imprimitivity bimodule.

Lemma 3.3.10. *If E is a partial imprimitivity $\mathcal{A}$-$\mathcal{B}$ bimodule, the two norms on each space $E(x)(y)$ given by the $\mathcal{A}$- and $\mathcal{B}$-valued product are equal.*

Proof. Note that

$$\begin{aligned} \|\langle e, e\rangle_{\mathcal{B}}\|^2 &= \|\langle e, e\rangle_{\mathcal{B}}\langle e, e\rangle_{\mathcal{B}}\| && = \|\langle e, e \cdot \langle e, e\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\| \\ &= \|\langle e, {}_{\mathcal{A}}\langle e, e\rangle \cdot e\rangle_{\mathcal{B}}\| && \leq \|\langle e, e\rangle_{\mathcal{B}}\|\|_{\mathcal{A}}\langle e, e\rangle\| \end{aligned}$$

where the last inequality follows from Lemma 3.1.11.

So for $e \neq 0$ we can divide to get $\|\langle e, e\rangle_{\mathcal{B}}\| \leq \|_{\mathcal{A}}\langle e, e\rangle\|$, and it follows similarly that $\|_{\mathcal{A}}\langle e, e\rangle\| \leq \|\langle e, e\rangle_{\mathcal{B}}\|$, proving the lemma. $\square$

The imprimitivity equation gives us significant control over both actions:

Lemma 3.3.11. *If E is a partial imprimitivity $\mathcal{A}$-$\mathcal{B}$ bimodule, the images*

$$E^1({}_{\mathcal{A}}\langle E^2, E^2\rangle(x, x')) \subseteq \mathcal{L}(E^1(x), E^1(x'))$$

and

$$E^2(\langle E^1, E^1\rangle_{\mathcal{B}}(y, y')) \subseteq \mathcal{L}(E^2(y), E^2(y'))$$

both consist of exactly the finite-rank operators.

Proof. The imprimitivity equation states that $E^1({}_{\mathcal{A}}\langle e, f\rangle)(g) = \theta^{e,f}(g)$, so as E^1 is additive and continuous we get the first result.

The analogous result on E^2 follows similarly as the imprimitivity equation gives $E^2(\langle f, g\rangle_{\mathcal{B}})(e) = \theta^{g,f}(e)$. $\square$

Given additional information about the products, we can deduce that the actions of both categories are entirely compact:

Lemma 3.3.12. *If E is a left-essential partial imprimitivity $\mathcal{A}$-$\mathcal{B}$ bimodule, then in fact E^1 is an isometry on hom-spaces and satisfies*

$$E^1({}_{\mathcal{A}}\overline{\langle E^2, E^2\rangle}(x, x')) = \mathcal{K}(E^1(x), E^1(x')),$$

and if E is left full, E^1 gives a surjection of $\mathcal{A}(x, x')$ onto $\mathcal{K}(E^1(x), E^1(x'))$.

Similarly, if E is right-essential, E^2 acts isometrically and we have

$$E^2(\overline{\langle E^1, E^1\rangle_{\mathcal{B}}}(y', y)) = \mathcal{K}(E^1(y), E^1(y')),$$

and if E is right-full, E^2 in addition surjects $\mathcal{B}(y', y))$ onto $\mathcal{K}(E^1(y), E^1(y'))$.

Proof. We show that if E is left essential, E^1 acts isometrically on hom-spaces. Suppose $E^1(a) = 0$, then for all e, f we have $a \circ {}_{\mathcal{A}}\langle e, f\rangle = {}_{\mathcal{A}}\langle E^1(a)(e), f\rangle = 0$, so as $\langle E^2, E^2\rangle$ is essential, we in fact have $a = 0$. Hence E^1 is faithful and by Proposition 1.1.34 in fact acts isometrically, and in particular preserves closures of sets. So if E^2 is in fact full, clearly the final result on E^1 follows from Lemma 3.3.11.

The dual results follow similarly. $\square$

We'd like to build up to a converse to the above lemma, that specifies exactly when a right Hilbert bimodule is one 'half' of an imprimitivity bimodule.

Lemma 3.3.13. *If $\mathcal{D}$ is a subcategory of Hilb-$\mathcal{B}$ containing all the compact operators between its objects, then for every $y \in \mathsf{Ob}\,\mathcal{B}$ there is a large left Hilbert $\mathcal{D}$-module*

$$F_y : \mathcal{D} \to \mathsf{Vect}_{\mathbb{C},0}$$

given by

- *$F_y(D) := D(y)$ for all $D \in \mathsf{Ob}\,\mathcal{D}$,*
- *$F_y(T) := T_y : D(y) \to D'(y)$ for all $T \in \mathcal{L}(D, D')$, and*
- *${}_{\mathcal{D}}\langle d, d'\rangle := \theta_y^{d,d'} \in \mathcal{K}(D', D)$*

Proof. Note firstly that $\theta^{e,f} = (\theta^{f,e})^*$ and that $T \circ \theta^{e,f} = \theta^{T(e),f}$. To see for any $E \in \mathsf{Ob}\,\mathcal{D}, e \in E(y)$, that ${}_{\mathcal{D}}\langle e, e\rangle = \theta_y^{e,e} \in \mathcal{KD}(E, E)$ is positive, note that for all $f \in E(x)$ we have $\langle f, \theta_y^{e,e}(f)\rangle = \langle f, e\rangle\langle e, f\rangle \geq 0$, so by Lemma 2.5.1 we get positivity. Finally if $\theta_y^{e,e} = 0$, then $\langle e, e\rangle^2 = \langle e, \theta^{e,e}(e)\rangle = 0$ so $e = 0$.

Every space in the bimodule is complete in this norm as it is equivalent to the $\mathcal{B}$-norm; note that the proof of Lemma 3.3.10 uses only the imprimitivity equation and the compatibility of the actions. The action of $\mathcal{B}$ is bounded in the $\mathcal{D}$-norm since by Corollary 2.1.6 it is bounded in the $\mathcal{B}$-norm and the two norms are equal, and finally it is adjointable since $\theta^{e,f\cdot b} = \theta^{e\cdot b^*,f}$. This completes the proof. $\square$

Definition 3.3.14. A subcategory $\mathcal{D} \subseteq \mathsf{Hilb}\text{-}\mathcal{B}$ containing all compact operators between its objects is called *left small* if for all objects $y \in \operatorname{Ob}\mathcal{B}$, the large left Hilbert $\mathcal{D}$-module F_y is generated by a set of elements. A bimodule $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is called left small if its image is.

Example 3.3.15. If $\mathcal{D} \subseteq \mathsf{Hilb}\text{-}\mathcal{B}$ is small then $\mathcal{D}$ is left small by Lemma 2.1.15.

Lemma 3.3.16. *If $\mathcal{D} \subseteq \mathsf{Hilb}\text{-}\mathcal{B}$ contains every representable right Hilbert $\mathcal{B}$-module, then it is left small.*

Proof. Recall that by Corollary 2.1.11, for all $D \in \operatorname{Ob}\mathcal{D}, y \in \operatorname{Ob}\mathcal{B}$ and $d \in D(y)$ there exist $d' \in D(y)$ and $b \in \mathcal{B}(y,y)$ such that $d = d' \cdot b$. But then $d = \epsilon_{d'}(b)$, where $\epsilon_{d'} \in \mathcal{K}(h_y, D)$ is the map defined in Lemma 2.3.12. Therefore we see that for each $y \in \operatorname{Ob}\mathcal{B}$, the set $\mathcal{B}(y,y) \subseteq h_y(y)$ generates the $\mathcal{D}$-module F_y. $\square$

We can promote the above left-valued product to a left essential imprimitivity bimodule:

Proposition 3.3.17. *Suppose $\mathcal{D} \subseteq \mathsf{Hilb}\text{-}\mathcal{B}$ is a left small subcategory containing all compact operators between its objects. Then the inclusion*

$$\Delta_1 : \mathcal{D} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{B}$$

can be given the structure of a left-essential $\mathcal{D}$-$\mathcal{B}$ imprimitivity bimodule Δ with other half

$$\Delta_2 : \mathcal{B}^{\mathrm{op}} \to \mathsf{Hilb}\text{-}\mathcal{D}^{\mathrm{op}}$$

defined by setting $\Delta_2(y) = F_y$, where the latter is defined in Lemma 3.3.13. Furthermore, Δ is an imprimitivity bimodule if and only if $\mathcal{D}$ contains precisely the compact operators and the bimodule $\Delta_1 : \mathcal{D} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{B}$ is full.

Proof. The left action of $\mathcal{B}$ on the modules F_y is obvious, and the imprimitivity equation follows immediately from the definition of ${}_{\mathcal{D}}\langle -,-\rangle$.

Left essentiality follows *a fortiori* from the fact that $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$ is essential in $\mathsf{Hilb}\text{-}\mathcal{B}$, and left fullness is clearly equivalent to asking that $\mathcal{D}$ contains only compact operators. So Δ is an imprimitivity bimodule with described left product if it is also right full. $\square$

Corollary 3.3.18. *Every C^*-category $\mathcal{B}$ is linked to its own category of right Hilbert modules and compact operators $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$ by an imprimitivity bimodule.*

Proof. In the above lemma, simply let $\mathcal{D} = \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$: this is clearly full by considering for example the inner products on representable modules, and by Lemma 3.3.16 $\square$

Proposition 3.3.17 and Lemma 3.3.12 together allow us to give a complete characterization of which bimodules come from imprimitivity bimodules:

Proposition 3.3.19. *A bimodule $E^1 : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ can be given the structure of a left-full partial imprimitivity bimodule if and only if E^1 is left small and is a faithful functor that surjects onto the compact operators between the modules in its image. In this case the $\mathcal{A}$-valued product must equal ${}_{\mathcal{A}}\langle e, f\rangle = \theta^{e,f}$. This is an imprimitivity bimodule if and only if E^1 is full.*

Hence we see that for a given right Hilbert bimodule, being an imprimitivity bimodule or not is a *property* rather than a structure.

3.4 Morita equivalences of C^*-categories

The reason to study imprimitivity bimodules is that as in the C^*-algebra case, they turn out to be exactly the bimodules that induce strong* unitary equivalences of Hilbert module categories.

Definition 3.4.1. If $E : \mathcal{A}^+ \times \mathcal{B}^{\mathrm{op}+} \to \mathsf{Vect}_{\mathbb{C},0}$ is a bi-Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule, its *conjugate*

$$\widetilde{E} : \mathcal{B}^+ \times \mathcal{A}^{\mathrm{op}+} \to \mathsf{Vect}_{\mathbb{C},0}$$

is the bi-Hilbert $\mathcal{B}$-$\mathcal{A}$ bimodule defined as follows:

- For $x \in \mathsf{Ob}\,\mathcal{A}$ and $y \in \mathsf{Ob}\,\mathcal{B}$ we set $\widetilde{E}(y)(x) = E(x)(y)^*$, where * denotes the conjugate of a complex vector space.
- For any morphisms $a \in \mathcal{A}^+(x_1, x_2)$ and $b \in \mathcal{B}^+(y_1, y_2)$, and $e \in E(x_2, y_2)$, we define the action[1] by $\widetilde{E}(b, a)(\widetilde{e}) = \widetilde{E(a^*, b^*)(e)}$, where e.g. $\widetilde{e}$ signifies the element of $\widetilde{E}(x_2, y_2)$ corresponding to $e \in E(x_2, y_2)$.
- $\widetilde{E}$ has products ${}_{\mathcal{B}}\langle -, -\rangle$ and $\langle -, -\rangle_{\mathcal{A}}$ defined by setting ${}_{\mathcal{B}}\langle \widetilde{e}, \widetilde{f}\rangle = \langle e, f\rangle_{\mathcal{B}}$ and $\langle \widetilde{e}, \widetilde{f}\rangle_{\mathcal{A}} = {}_{\mathcal{A}}\langle e, f\rangle$ where e.g. $\widetilde{e} \in \widetilde{E}(y)(x)$ denotes the element of the conjugate space corresponding to $e \in E(x)(y)$.

Just like we write $E^1 : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $E^2 : \mathcal{B}^{\mathrm{op}} \to \mathsf{Hilb}\text{-}\mathcal{A}^{\mathrm{op}}$ for the 'components' of E, we denote by

$$\widetilde{E}^1 : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{A}$$

and

$$\widetilde{E}^2 : \mathcal{A}^{\mathrm{op}} \to \mathsf{Hilb}\text{-}\mathcal{B}^{\mathrm{op}}$$

the two right Hilbert bimodules associated to $\widetilde{E}$. It is immediate that many properties of $\widetilde{E}$ carry over from E, namely:

[1] Note that $\widetilde{E}$ is a $\mathbb{C}$-linear functor since $\widetilde{E}(\lambda(a, b))(\widetilde{e}) := (\widetilde{E(\overline{\lambda}a^*, \overline{\lambda}b^*)(e)}) = (\widetilde{\overline{\lambda}E(a^*, b^*)(e)}) = \lambda(\widetilde{E(a^*, b^*)(e)}) = \lambda\widetilde{E}(a, b)(\widetilde{e})$.

- $\widetilde{E}$ is a partial imprimitivity bimodule if and only if E is.
- $\widetilde{E}$ is left essential/full, respectively right essential/full if and only if E is right essential/full, respectively left essential/full.

Imprimitivity bimodules over small C^*-categories were termed 'equivalence bimodules' in [Fer23]: we stick here to imprimitivity bimodules to emphasize the analogy with the C^*-algebra case.

We begin by proving that an imprimitivity bimodule is invertible under the tensor product.

Proposition 3.4.2. *If E is a partial imprimitivity bi-Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule, there are isometric bimodule maps*

$$\phi : \widetilde{E}^1 \bar{\otimes}_{\mathcal{A}} E^1 \to \iota_{\mathcal{B}}$$

and

$$\psi : E^1 \bar{\otimes}_{\mathcal{B}} \widetilde{E}^1 \to \iota_{\mathcal{A}}.$$

If E is right-full, ϕ is a unitary isomorphism, and if E is left-full, ψ is a unitary isomorphism.

Proof. We construct ϕ first. For all $y, y' \in \mathsf{Ob}\,\mathcal{B}$ we define the bimodule map

$$\phi : (\widetilde{E}^1 \otimes_{\mathcal{B}} E^1)(y)(y') \to \iota_{\mathcal{B}}(y)(y') := \mathcal{B}(y', y)$$

on tensors $\widetilde{e} \otimes f \in \widetilde{E}^1(y)(x) \otimes E^1(x)(y')$ by

$$\phi(\widetilde{e} \otimes f) := \langle e, f \rangle_{\mathcal{B}}.$$

It is easily verified that this is a bilinear map which is natural with respect to the actions of $\mathcal{A}$ and $\mathcal{B}$, and that it respects the equivalence relation defining $\widetilde{E}^1 \otimes_{\mathcal{B}} E^2(y)(y')$. To show that it extends to an isometry on the completion, we show it respects the inner product on simple tensors:

$$\begin{aligned} \langle \phi(\widetilde{e} \otimes f), \phi(\widetilde{e'} \otimes f) \rangle_{\mathcal{B}} &= \langle e, f \rangle_{\mathcal{B}}^* \langle e', f' \rangle_{\mathcal{B}} &&= \langle f, e \rangle_{\mathcal{B}} \langle e', f' \rangle_{\mathcal{B}} \\ = \langle f, e \cdot \langle e', f' \rangle_{\mathcal{B}} \rangle_{\mathcal{B}} &= \langle f, {}_{\mathcal{A}}\langle e, e' \rangle \cdot f' \rangle_{\mathcal{B}} &&= \langle \widetilde{e} \otimes f, \widetilde{e'} \otimes f' \rangle_{\mathcal{B}}. \end{aligned}$$

So ϕ preserves inner products on finite sums of simple tensors, and by Lemma 1.4.4 it extends to an isometric bimodule map on the completed module $\widetilde{E}^1 \bar{\otimes}_{\mathcal{B}} E^1(y)(y')$. When E is right-full, this map is evidently surjective so by Proposition 3.1.1 it is a unitary equivalence.

The map ψ is defined on simple tensors by $\psi(e \otimes \widetilde{f}) = {}_{\mathcal{A}}\langle e, f \rangle$. The proof that ψ preserves inner products is completely analogous to the above, and clearly it is surjective (and hence a unitary equivalence) when E is left-full. $\square$

We follow with a converse of this lemma, which has the hardest proof in this chapter.

Lemma 3.4.3. *Let $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $F : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{A}$ be non-degenerate right Hilbert bimodules, and suppose that there exist bimodule isomorphisms*

$$\psi : E \bar{\otimes}_{\mathcal{B}} F \to \iota_{\mathcal{A}}$$

and

$$\phi : F \bar{\otimes}_{\mathcal{A}} E \to \iota_{\mathcal{B}}.$$

Then both E and F are full bimodules, and are faithful functors that surject onto the compact operators between the modules in their image.

Proof. By symmetry it is enough to prove the stated properties of E. Note that we can assume that all bimodule isomorphisms are unitary by Lemma 1.1.50.

Note also that $\langle \iota_{\mathcal{B}}, \iota_{\mathcal{B}} \rangle = \langle F \bar{\otimes}_{\mathcal{A}} E, F \bar{\otimes}_{\mathcal{A}} E \rangle \subseteq \langle E, E \rangle$ by definition of the inner product on $F \bar{\otimes}_{\mathcal{A}} E$, so E is full. Similarly, F is full since $\iota_{\mathcal{A}}$ is.

To show E is an isometry, note that we have unitary isomorphisms of functors

$$(-\bar{\otimes}_{\mathcal{B}} F) \bar{\otimes}_{\mathcal{A}} E \cong -\bar{\otimes}_{\mathcal{B}} (F \bar{\otimes}_{\mathcal{A}} E) \cong -\bar{\otimes}_{\mathcal{A}} \iota_{\mathcal{A}} \cong \mathrm{id}_{\mathsf{Hilb}\text{-}\mathcal{A}}.$$

Here we have used Lemma 3.1.8 for the first isomorphism, the hypothesis for the second and Lemma 3.1.9 for the third. We hence conclude that $-\bar{\otimes}_{\mathcal{A}} E$ must be faithful, but by Corollary 3.1.10, $-\bar{\otimes}_{\mathcal{A}} E$ is an extension of E from $\mathcal{A}$ to $\mathsf{Hilb}\text{-}\mathcal{A}$. So certainly E is faithful, and hence an isometry by Proposition 1.1.34.

The final claim we need to prove is that the image of $\mathcal{A}(x, x')$ under E is exactly $\mathcal{K}(E(x), E(x'))$. To this end, for every $x \in \mathsf{Ob}\,\mathcal{A}, y \in \mathsf{Ob}\,\mathcal{B}$ we define the map

$$\begin{aligned} \delta : \quad & F(y)(x) \to \mathcal{L}(E(x), h_y) \\ & \delta(f)(e) := \phi(f \otimes e) \end{aligned}$$

for each $y' \in \mathsf{Ob}\,\mathcal{B}$ and $e \in E(x)(y')$. It is clear that $\delta(f)$ is bounded by $\|f\|$, and since ϕ is a unitary isomorphism at every $y \in \mathsf{Ob}\,\mathcal{B}$ we see that $\delta(f)$ is adjointable if and only if the map $\phi_y^* \delta(f) : E(x) \to F \bar{\otimes}_{\mathcal{A}} E(y)$ is adjointable, where we have $\phi_y^* \delta(f)(e) := f \otimes e$. But $\phi_y^* \delta(f)$ has an obvious bounded adjoint sending $f' \otimes e'$ to $\langle f, f' \rangle_{\mathcal{B}} \cdot e'$. So δ has an adjoint δ^* found by $(\phi_y^* \delta)^* \phi_y$.

Note that if we fix y, not only does the domain of δ have an inner product

$$\langle -, - \rangle_F : F(y)(x') \times F(y)(x) \to \mathcal{A}(x, x'),$$

but its codomain has an inner product

$$\langle -, - \rangle_O : \mathcal{L}(E(x'), h_y) \times \mathcal{L}(E(x), h_y) \to \mathcal{L}(E(x), E(x'))$$

defined by $\langle S, T \rangle_O = S^* \circ T$. We are going to prove the following two claims about the map δ:

- For each $f \in F(y)(x), f' \in F(y)(x')$ we have $\langle \delta(f'), \delta(f) \rangle_O = E(\langle f', f \rangle_F)$.
- The image of δ at every $x \in \mathsf{Ob}\,\mathcal{A}, y \in \mathsf{Ob}\,\mathcal{B}$ is $\mathcal{K}(E(x), h_y)$.

Provided we can prove these claims, the desired result will then roll out as follows: note that as F is full, the span of $\bigcup_{y \in \mathsf{Ob}\,\mathcal{B}} \langle F(y)(x'), F(y)(x) \rangle_F$ is dense in $\mathcal{A}(x, x')$. Note also by Lemma 2.3.13 that the span of

$$\bigcup_{y \in \mathsf{Ob}\,\mathcal{B}} \langle \mathcal{K}(E(x'), h_y), \mathcal{K}(E(x), h_y) \rangle_O$$

is a dense subset of $\mathcal{K}(E(x), E(x'))$ (in the operator norm, which is the one induced by $\langle -, - \rangle_O$). But E is a faithful C^*-functor so in particular acts by linear isometries on hom-spaces, it preserves spans and closures, and the two claims together in fact give $E(\mathcal{A}(x, x')) = \mathcal{K}(E(x), E(x'))$.

To prove the first claim, let $f \in F(y)(x)$, $f' \in F(y)(x')$, and take elements $e \in E(x)(y')$, $e' \in E(x')(y'')$. Then note that we have

$$\begin{aligned}
\langle e', \delta(f')^* \delta(f)(e) \rangle &= \langle \delta(f') e', \delta(f)(e) \rangle \\
&= \langle \phi(f' \otimes e'), \phi(f \otimes e) \rangle \\
&= \langle f' \otimes e', f \otimes e \rangle \\
&= \langle e', E(\langle f', f \rangle)(e) \rangle.
\end{aligned}$$

Hence if we set $T = \delta(f')^* \delta(f) - E(\langle f', f \rangle)$ we see $\langle e', T(e) \rangle = 0$ for all e, e', and setting $e' = T(e)$ we conclude $T = 0$, proving the first claim.

To prove the second claim, note that we have for every $a \in \mathcal{A}(x', x)$ and $f \in F(y)(x)$ that $\delta(f \cdot a) = \delta(f) \circ E(a)$, and for every $b \in \mathcal{B}(y, y')$ that $\delta(F(b)(f)) = \iota_{\mathcal{B}}(b) \circ \delta(f)$. From now on, we will denote all these actions as simply $- \cdot a$ and $b \cdot -$, which δ preserves. Now note that

$$\begin{aligned}
\delta(F(y)(x)) &\overset{(1)}{=} \delta(\overline{\operatorname{span}_{y' \in \mathsf{Ob}\,\mathcal{B}} \mathcal{B}(y, y') \cdot F(y')(x)}) \\
&\overset{(2)}{=} \overline{\operatorname{span}_{y' \in \mathsf{Ob}\,\mathcal{B}} \iota_{\mathcal{B}}(y')(y) \circ \delta(F(y')(x))} \\
&\overset{(3)}{=} \overline{\operatorname{span}_{y' \in \mathsf{Ob}\,\mathcal{B}} (\iota_{\mathcal{B}}(y)(y'))^* \circ \delta(F(y')(x))} \\
&\overset{(4)}{=} \overline{\operatorname{span}_{y' \in \mathsf{Ob}\,\mathcal{B}} (\phi(F \bar{\otimes}_{\mathcal{A}} E)(y)(y'))^* \circ \delta(F(y')(x))} \\
&\overset{(5)}{=} \overline{\operatorname{span}_{y' \in \mathsf{Ob}\,\mathcal{B}, x' \in \mathsf{Ob}\,\mathcal{A}} \mathcal{K}(E(x'), h_y) \circ \delta(F(y')(x'))^* \circ \delta(F(y')(x))} \\
&\overset{(6)}{=} \overline{\operatorname{span}_{y' \in \mathsf{Ob}\,\mathcal{B}, x' \in \mathsf{Ob}\,\mathcal{A}} \mathcal{K}(E(x'), h_y) \circ E(\langle F(y')(x'), F(y')(x) \rangle)} \\
&\overset{(7)}{=} \overline{\operatorname{span}_{x' \in \mathsf{Ob}\,\mathcal{A}} \mathcal{K}(E(x'), h_y) \circ E(\mathcal{A}(x', x))} \\
&\overset{(8)}{=} \mathcal{K}(E(x)(h_y)).
\end{aligned}$$

Where we have used the symbol * to take adjoints of sets of operators. We justify each of these equalities as follows:

1. By non-degeneracy of F.
2. Since E acts isometrically, the first claim we proved shows that δ is a norm isometry, so it preserves closures, and we also know δ is natural with respect to the left action of $\mathcal{B}$.
3. Since the involution of operators is an isometry.

4. By the surjectivity of ϕ.

5. Notice that if $\epsilon_e : h_y \to E(x')$ is the compact operator corresponding to $e \in E(x')(y)$, then by definition of δ, for all $f \in F(y')(x')$ we have

$$\phi(f \otimes e) = \iota_{\mathcal{B}}(\delta(f)(e)) = \delta(f) \circ \epsilon_e : h_y \to h_{y'},$$

since for all $b \in h_y$, we have

$$(\delta(f) \circ \epsilon_e)(b) = \phi(f \otimes e \cdot b) = \phi(f \otimes e) \circ b.$$

Hence $\phi(F \bar{\otimes}_{\mathcal{A}} E)(y')(y) = \overline{\operatorname{span}_{x' \in \mathsf{Ob}\,\mathcal{A}} \delta(F(y')(x')) \circ \mathcal{K}(h_y, E(x'))}$, and applying the involution we deduce this equality.

6. By the first of the two claims.

7. By the fullness of F.

8. By the non-degeneracy of E.

This concludes the proof. $\square$

Corollary 3.4.4 (c.f. [Fer23, Proposition 8.1.3])**.** *Let $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ and $F : \mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{A}$ be non-degenerate right Hilbert bimodules, and suppose there exist bimodule isomorphisms $\psi : E \bar{\otimes}_{\mathcal{B}} F \to \iota_{\mathcal{A}}$ and $\phi : F \bar{\otimes}_{\mathcal{A}} E \to \iota_{\mathcal{B}}$. Then E and F are imprimitivity bimodules and we in fact have $F \cong \widetilde{E}$.*

Proof. Notice that at any $x \in \mathsf{Ob}\,\mathcal{A}, y \in \mathsf{Ob}\,\mathcal{B}$, the map δ in the proof of Lemma 3.4.3 gives an isometry from $F(y)(x)$ to $\mathcal{K}(E(x), h_y)$, which is of course conjugate isomorphic to $\mathcal{K}(h_y, E(x)) \cong E(x)(y)$. Hence we see that E is left small since F lands in small $\mathcal{A}$-modules and vice versa.

The imprimitivity structure is then deduced by Proposition 3.3.19, and the isomorphism $F \cong \widetilde{E}$ follows from the above characterization of δ. $\square$

Notice that this directly generalizes the case of C^*-algebras, which is explored in e.g. [RW98, Chapter 3]. We are now ready to summarize the results of this section into one theorem:

Theorem 3.4.5. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories. A strong* unital functor $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is an equivalence if and only if its restriction $E := F \circ \iota_{\mathcal{A}}$ is left small, a full bimodule, a faithful functor, and surjects onto the compact operators between the modules in its image.*

Proof. Suppose F is an equivalence. Firstly, the Eilenberg-Watts theorem guarantees that $F \cong -\bar{\otimes}_{\mathcal{A}} E$. But then by Corollary 3.4.4 we see that E is an imprimitivity $\mathcal{A}$-$\mathcal{B}$ module. So by Proposition 3.3.19, we see that E is an isometry onto the compacts between modules in its image.

The converse is even simpler; a bimodule with the specified properties is always an imprimitivity bimodule and by Proposition 3.4.2, we see that $-\bar{\otimes}_{\mathcal{A}} E$ is a strong* unital unitary equivalence. $\square$

We will hereafter describe such an equivalence F as a *Morita equivalence* between $\mathcal{A}$ and $\mathcal{B}$.

An interesting corollary to this states that such an equivalence must necessarily preserve compact operators. Recall from Definition 1.4.10 that a multiplier equivalence between $\mathcal{A}$ and $\mathcal{B}$ is a pair of non-degenerate functors $F : \mathcal{A} \to \mathcal{M}\mathcal{B}$ and $G : \mathcal{B} \to \mathcal{M}\mathcal{A}$ whose lifts $\bar{F}, \bar{G}$ to the respective multiplier categories give an equivalence.

Corollary 3.4.6. *If a unital functor $G : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a strong* equivalence, then it restriction $G|_{\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}} : \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ lands in $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$ and gives a multiplier equivalence.*

Conversely, any multiplier equivalence $\Gamma : \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} \to \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B}$ extends to a strong* unital equivalence $G : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$.

Proof. For the rightward implication, note that Theorem 3.4.5 tells us that G is given by tensoring with an $\mathcal{A}$-$\mathcal{B}$ bimodule E which acts entirely by compact operators. So by Corollary 3.1.18 we see that G preserves compact operators.

For the converse, note that the extension $\bar{\Gamma} : \mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}) \to \mathcal{M}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{B})$ provided by Theorem 1.4.5 is necessarily unital and strictly continuous on bounded subsets. So then by Proposition 2.3.16, $\bar{\Gamma}$ is strong* and unital. □

We are long overdue some examples of Morita equivalences of C^*-categories, other than the Yoneda bimodules: it follows from Lemma 3.1.9 that these are self-Morita equivalences. Luckily, the results in this section set us up well to provide these. Specifically, if we can provide an example of a faithful, left small, full right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule whose action consists of all compact operators between the $\mathcal{B}$-modules in its image, then by Corollary 3.4.4 we know there is a unital, strong*, unitary equivalence between $\mathsf{Hilb}\text{-}\mathcal{A}$ and $\mathsf{Hilb}\text{-}\mathcal{B}$. We will begin with an example that already played a role in Chapter 1:

Lemma 3.4.7. *If $\mathcal{A}$ is a C^*-category, there is a Morita equivalence between $\mathcal{A}_\oplus$ and $\mathcal{A}$ given by tensoring with the right Hilbert $\mathcal{A}_\oplus - \mathcal{A}$ bimodule that sends $x_1 \oplus \cdots \oplus x_n$ to $h_{x_1} \oplus \cdots \oplus h_{x_n}$.*

Proof. It follows from Proposition 2.4.11 and Lemma 2.3.9 that there is an isomorphism

$$\mathcal{A}_\oplus(x_1 \oplus \cdots \oplus x_n, y_1 \oplus \cdots \oplus y_k) \cong \mathcal{K}(h_{x_1} \oplus \cdots \oplus h_{x_n}, h_{y_1} \oplus \cdots \oplus h_{y_k})$$

which is natural in the objects, so the described assignment is functorial and isometric onto the compact operators. This bimodule is left small by Lemma 3.3.16. Hence as this bimodule is certainly full, by Theorem 3.4.5 we see the described functor is a Morita equivalence. □

We show next that two unital C^*-categories are Morita equivalent if and only if they are Morita equivalent as additive categories:

Proposition 3.4.8. *Two unital C^*-categories $\mathcal{A}$, $\mathcal{B}$ are Morita equivalent if and only if there is a unitary equivalence*

$$\mathcal{A}_\oplus^\natural \cong \mathcal{B}_\oplus^\natural .$$

Proof. Recall from Corollary 2.3.10 that for a unital C^*-category $\mathcal{A}$ there is an equivalence between $\mathcal{A}_\oplus^\natural$ (the closure of $\mathcal{A}$ under idempotents and direct sums) and the category $\mathsf{fgProj}\text{-}\mathcal{A}$ of finitely generated projective $\mathcal{A}$-modules.

Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital and E is an $\mathcal{A}$-$\mathcal{B}$ imprimitivity bimodule. Then $E^1 : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is non-degenerate by Lemma 3.3.4, so it must in fact be unital, but by Lemma 3.3.12 it must also have image in compact operators, so by Proposition 2.5.8 we see that for each $x \in \mathsf{Ob}\,\mathcal{A}$, the module $E^1(x)$ is finitely generated and projective. Hence the functor $-\bar{\otimes}_{\mathcal{A}} E^1$ sends representable modules to finitely generated projectives, and hence sends all finitely generated projectives to finitely generated projectives. We can make a similar argument on the inverse functor $-\bar{\otimes}_{\mathcal{B}} E^2$ and then we obtain by the equivalence $\mathcal{A}_\oplus^\natural \cong \mathcal{B}_\oplus^\natural$.

For the converse assume there is an equivalence $F : \mathsf{fgProj}\text{-}\mathcal{A} \xrightarrow{\cong} \mathsf{fgProj}\text{-}\mathcal{B}$: let E^1 be the right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodule given by postcomposing F with the inclusion $\mathsf{fgProj}\text{-}\mathcal{B} \hookrightarrow \mathsf{Hilb}\text{-}\mathcal{B}$ and precomposing with the Yoneda embedding $\mathcal{A} \hookrightarrow \mathsf{fgProj}\text{-}\mathcal{A}$. Let E^2 be the $\mathcal{B}$-$\mathcal{A}$ bimodule similarly obtained from the inverse of F. Then it is evident that $E^1 \bar{\otimes}_{\mathcal{A}} E^2 \cong \iota_{\mathcal{A}}$ and $E^2 \bar{\otimes}_{\mathcal{B}} E^1 \cong \iota_{\mathcal{B}}$. □

This means our terminology coincides with that in [DT14], where Morita equivalence of (unital) C^*-categories was *defined* by the second criterion above.

Recall from Definition 1.2.5 that Mat-$\mathcal{A}$ is the direct limit of the endomorphism algebras of *non-repeating* direct sums in $\mathcal{A}_\oplus$. Using the results we have at our disposal it is now very easy to prove using these algebras that many C^*-categories are Morita equivalent to C^*-algebras.

Proposition 3.4.9 (c.f. [Fer23, Theorem 6.3.1])**.** *If $\mathcal{A}$ is a small C^*-category, there is a Morita equivalence between* (Mat-$\mathcal{A}$) *and $\mathcal{A}$ given by tensoring with the right Hilbert* (Mat-$\mathcal{A}$)-$\mathcal{A}$ *bimodule $\bigoplus_{x \in \mathsf{Ob}\,\mathcal{A}} h_x$.*

More generally, if $\mathcal{A}$ is a locally small C^*-category with a small full subcategory $\mathcal{B}$ such that every morphism in $\mathcal{A}$ is a norm-limit of finite sums of morphisms factoring through some object of $\mathcal{B}$, then there is a Morita equivalence between (Mat-$\mathcal{B}$) *and $\mathcal{A}$ given by tensoring with the right Hilbert* (Mat-$\mathcal{B}$)-$\mathcal{A}$ *bimodule $\bigoplus_{x \in \mathsf{Ob}\,\mathcal{B}} h_x$.*

Proof. Note firstly that both bimodules are left small since they have a single-object C^*-category acting on the left.

For the first case Mat-$\mathcal{A} \cong \bigoplus_{x,y \in \mathsf{Ob}\,\mathcal{A}} \mathcal{A}(x,y) \cong \bigoplus_{x,y \in \mathsf{Ob}\,\mathcal{A}} \mathcal{K}(h_x, h_y)$ (with the obvious multiplication on the latter two algebras), which by Proposition 2.4.11 is isomorphic to $\mathcal{K}(\bigoplus_{x \in \mathsf{Ob}\,\mathcal{A}} h_x)$. The module $\bigoplus_{x \in \mathsf{Ob}\,\mathcal{A}} h_x$ is clearly full, so tensoring with this bimodule gives a Morita equivalence by Theorem 3.4.5.

The second result follows similarly as the requirement on $\mathcal{B}$ equates to asking that $\bigoplus_{x \in \mathsf{Ob}\,\mathcal{B}} h_x$ is full. □

These results are a significant generalization of those obtained in [Joa03], where it was proved that (translated to our terminology) Mat-$\mathcal{A}$ and $\mathcal{A}$ are Morita equivalent in the case where $\mathcal{A}$ is unital and has a countable set of objects.

Chapter 4

(Bi)categories of C^*-categories and C^*-algebras

In this chapter, we use the results in this thesis to prove results on a variety of categories whose objects are C^*-categories and C^*-algebras. First we use the Eilenberg-Watts theorem to show the equivalence of several bicategories of C^*-algebras and C^*-categories. Second, we exhibit a *reflective localization* of a category of C^*-categories at the Yoneda embeddings, resulting in a 'Morita homotopy category' of C^*-categories where two C^*-categories are isomorphic if and only if they are Morita equivalent.

4.1 The bicategories C^*-HCat and C^*-Bimod

We begin by considering several bicategories. We will assume basic results on bicategories and refer the reader to [JY21] for a primer.

Our first bicategory has evidently strictly associative and unital compositions, so is in fact a 2-category:

Definition 4.1.1. We let C^*-HCat be the 2-category defined as follows:

- Its objects are locally small C^*-categories.
- The 1-morphisms from $\mathcal{A}$ to $\mathcal{B}$ are strong* unital C^*-functors from Hilb-$\mathcal{A}$ to Hilb-$\mathcal{B}$.
- The 2-morphisms are bounded adjointable natural transformations between these functors.
- C^*-HCat has 1- and 2-composition laws are defined in the obvious way.

Our second bicategory is an honest bicategory:

Proposition 4.1.2. *There is a bicategory* C^*-Bimod *defined as follows:*

- *Its objects are locally small C^*-categories.*

- *The 1-morphisms from $\mathcal{A}$ to $\mathcal{B}$ are non-degenerate right Hilbert $\mathcal{A}$-$\mathcal{B}$ bimodules, and horizontal identities are the Yoneda bimodules.*
- *The 2-morphisms are bounded adjointable natural transformations.*
- *Composition of 1-morphisms is given by tensor products of bimodules (recall from Lemma 3.1.16 that a tensor product where the left bimodule is non-degenerate is again non-degenerate).*
- *Composition of 2-morphisms is given by composition of transformations.*
- *Associators provided by Lemma 3.1.8 and unitors by Lemma 3.1.9.*

The equivalences in this bicategory are given by imprimitivity bimodules.

Proof. It is easily verified that the associators satisfy the pentagon identity and that the unitors satisfy the triangle identity: the proof proceeds exactly as in [Bro03, Proposition 2.3.1].

The final statement follows directly from Proposition 3.4.2 and Lemma 3.4.3. □

This bicategory is analogous to similar ones defined over rings and operator algebras, see [Bro03].

The results in Chapters 2 and 3 let us show without much effort that these two bicategories are equivalent:

Theorem 4.1.3. *There is a biequivalence $\Psi : \mathsf{C^*\text{-}Bimod} \to \mathsf{C^*\text{-}HCat}$ given by:*

- *The identity on objects.*
- *Sending every bimodule $E : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ to the functor*
  $-\bar{\otimes}_{\mathcal{A}} E : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$.
- *Sending a bounded adjointable transformation $T : E \to E'$ of right Hilbert $\mathcal{A}$-$\mathcal{B}$-bimodules to the bounded adjointable natural transformation*
  $\Psi(T) : -\bar{\otimes}_{\mathcal{A}} E \to -\bar{\otimes}_{\mathcal{A}} E'$ *whose existence is given by Lemma 3.1.15.*
- *Natural isomorphisms $\gamma : \Psi(E\bar{\otimes}_{\mathcal{B}} F) \xrightarrow{\cong} \Psi(F) \circ \Psi(E)$ given for every right Hilbert $\mathcal{A}$-module M, $\mathcal{A}$-$\mathcal{B}$ bimodule E, and $\mathcal{B}$-$\mathcal{C}$ bimodule F simply by the associator isomorphism $M\bar{\otimes}_{\mathcal{A}}(E\bar{\otimes}_{\mathcal{B}} F) \cong (M\bar{\otimes}_{\mathcal{A}} E)\bar{\otimes}_{\mathcal{B}} F$ from Lemma 3.1.8.*
- *Natural isomorphisms $I : \Psi(\iota_{\mathcal{A}}) \cong \mathrm{id}_{\mathcal{A}}$ given for every right Hilbert $\mathcal{A}$-module M by the unitor isomorphism $M\bar{\otimes}_{\mathcal{A}}(\iota_{\mathcal{A}}) \cong M$ from Lemma 3.1.9.*

Proof. It is straightforward to verify that this data satisfies the commutation diagrams for a pseudofunctor.

To show that Ψ is a biequivalence, we employ the criterion in [JY21, Theorem 7.4.1], which says Ψ is a biequivalence if and only if it is essentially surjective on objects, and gives for any $\mathcal{A}$ and $\mathcal{B}$ an equivalence of 1-categories $\Psi : \mathsf{C^*\text{-}Bimod}(\mathcal{A}, \mathcal{B}) \to \mathsf{C^*\text{-}HCat}(\mathcal{A}, \mathcal{B})$. The essential surjectivity is immediate.

To show that $\Psi : \mathsf{C^*\text{-}Bimod}(\mathcal{A}, \mathcal{B}) \to \mathsf{C^*\text{-}HCat}(\mathcal{A}, \mathcal{B})$ is an equivalence of 1-categories, we show that it is essentially surjective and fully faithful.

For the first claim, suppose $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ is a strong* unital functor; then by Theorem 3.2.4 we know that there is a natural isomorphism of functors $F \cong -\bar{\otimes}_{\mathcal{A}}(F \circ \iota_{\mathcal{A}})$, hence F is in the essential image under Ψ of $\mathsf{C^*\text{-}Bimod}(\mathcal{A}, \mathcal{B})$.

To show the faithfulness, we note simply that $\Psi(T) : -\bar{\otimes}_{\mathcal{A}}E \to -\bar{\otimes}_{\mathcal{A}}E'$ whiskers through the inclusion $\iota_{\mathcal{A}} : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{A}$ to $T : E \to E'$, so if two transformations $T, T' : E \to E'$ have the same image under Ψ, they must already be the same, showing Ψ is faithful.

To show fullness, let $\tau : -\bar{\otimes}_{\mathcal{A}}E \to -\bar{\otimes}_{\mathcal{A}}E'$ be a bounded adjointable natural transformation. Let $T : E \to E'$ be the whiskering of τ through $\iota_{\mathcal{A}}$. It is clear by definition that $\Psi(T)$ and τ agree on representable $\mathcal{A}$-modules, but then by Lemma 3.2.5 they must agree everywhere. Hence Ψ is full. □

This theorem tells us that the 2-category $\mathsf{C^*\text{-}HCat}$ is a model for the *strictification* of the bicategory $\mathsf{C^*\text{-}Bimod}$. We also record that the Morita equivalences in Proposition 3.4.9 provide us with further biequivalences.

Proposition 4.1.4. *The following bicategories are biequivalent:*

- *The full sub-bicategory of* $\mathsf{C^*\text{-}Bimod}$ *whose objects are small* C^**-categories.*
- *The full sub-2-category of* $\mathsf{C^*\text{-}HCat}$ *whose objects are small* C^**-categories.*
- *The full sub-bicategory of* $\mathsf{C^*\text{-}Bimod}$ *whose objects are the one-object categories, i.e.* C^**-algebras.*
- *The full sub-2-category of* $\mathsf{C^*\text{-}HCat}$ *whose objects are the* C^**-algebras.*

Proof. The equivalence of the first two bicategories follows from the preceding theorem: note that Ψ is the identity on objects.

The inclusion of the third into the first category is an equivalence since by Proposition 3.4.9 it is essentially surjective on objects, as is the inclusion from the fourth into the second; hence again by [JY21, Theorem 7.4.1] these inclusions are biequivalences. □

Variants of several categories in this proposition were studied in other works: the third category, often labelled the 'correspondence bicategory', was first defined in [Lan01] and further studied in [BMZ13]. The equivalence between the first and third bicategory was proven in the [Fer23, Section 7].

We characterize briefly an interesting sub-2-category of $\mathsf{C^*\text{-}HCat}$ that relates to other works and will become relevant in the next section:

Definition 4.1.5. For any locally small C^*-category $\mathcal{A}$, let $\mathsf{Hilb_{rep}}\text{-}\mathcal{A}$ be the category of representable right Hilbert $\mathcal{A}$-modules, i.e. the essential image of the Yoneda embedding $\iota_{\mathcal{A}} : \mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{A}$.

Let $\mathsf{C}^*\text{-}\mathsf{HCat}_{\mathsf{rep}}$ be the subcategory of $\mathsf{C}^*\text{-}\mathsf{HCat}$ with the same objects but whose 1-morphisms are strong* unital functors $\mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ that preserve representable modules, i.e. they send $\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}$ to $\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{B}$.

Finally, let $\mathsf{C}^*\text{-}\mathsf{Cat}_{\mathsf{ndg}}$ be the 2-category whose objects are locally small categories, whose 1-morphisms are non-degenerate functors $\mathcal{A} \to \mathcal{MB}$ and whose 2-morphisms are natural transformations of functors: identities are given by the embeddings $\kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{MA}$.

Note that by Corollary 2.4.8, the Yoneda embedding $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ extends to an equivalence $\overline{\iota_{\mathcal{A}}} : \mathcal{MA} \to \mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}$ of unital C^*-categories.

Proposition 4.1.6. *There is a 2-equivalence* $\Xi : \mathsf{C}^*\text{-}\mathsf{HCat}_{\mathsf{rep}} \to \mathsf{C}^*\text{-}\mathsf{Cat}_{\mathsf{ndg}}$ *given by*

- *The identity on objects*
- *Sending a strict unital functor* $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ *to the functor*

  $$\Xi(F) := \overline{\iota_{\mathcal{B}}}^{-1} \circ F|_{\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}} \circ \iota_{\mathcal{A}},$$

  where $\overline{\iota_{\mathcal{B}}}^{-1}$ *is a chosen inverse functor to the 'multiplier Yoneda embedding'* $\overline{\iota_{\mathcal{B}}} : \mathcal{MB} \to \mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{B}$*: this composition is defined since by assumption* $F|_{\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}}$ *lands in* $\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{B}$.
- *Sending a natural transformation* $\eta : F \Rightarrow F'$ *of functors to the natural transformation* $\Xi(\eta) : \overline{\iota_{\mathcal{B}}}^{-1} \circ F|_{\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}} \circ \iota_{\mathcal{A}} \Rightarrow \overline{\iota_{\mathcal{B}}}^{-1} \circ F'|_{\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}} \circ \iota_{\mathcal{A}}$ *given by whiskering through the two functors* $\iota_{\mathcal{A}}$ *and* $\overline{\iota_{\mathcal{B}}}^{-1}$.
- *Natural isomorphisms* $I : \Xi(\mathrm{id}_{\mathsf{Hilb}\text{-}\mathcal{A}}) \xrightarrow{\cong} \mathrm{id}_{\mathcal{A}}$ *given by the isomorphism* $\epsilon_{\mathcal{A}} : \iota_{\mathcal{A}} \circ \overline{\iota_{\mathcal{A}}}^{-1} \Rightarrow \kappa_{\mathcal{A}}$ *associated to the chosen inverse* $\overline{\iota_{\mathcal{A}}}^{-1}$.
- *Natural isomorphisms* $\gamma : \Xi(G \circ F) \xrightarrow{\cong} \Xi(G) \circ \Xi(F)$ *given for all strong* unital* $F : \mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$ *and* $G : \mathsf{Hilb}\text{-}\mathcal{B} \to \mathsf{Hilb}\text{-}\mathcal{C}$ *by a whiskering* $\overline{\iota_{\mathcal{C}}}^{-1} \circ G|_{\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{B}} \circ \epsilon_{\mathcal{B}} \circ F|_{\mathsf{Hilb}_{\mathsf{rep}}\text{-}\mathcal{A}} \circ \iota_{\mathcal{A}}$ *where* $\epsilon_{\mathcal{B}} : \iota_{\mathcal{B}} \circ \overline{\iota_{\mathcal{B}}}^{-1} \Rightarrow \kappa_{\mathcal{B}}$ *is the chosen transformation as in the point above.*

Proof. As before it is elementary to show that Ξ is a pseudofunctor, and as it is the identity on objects, we just need to show that its components

$$\Xi(\mathcal{A}, \mathcal{B}) : \mathsf{C}^*\text{-}\mathsf{HCat}_{\mathsf{rep}}(\mathcal{A}, \mathcal{B}) \to \mathsf{C}^*\text{-}\mathsf{Cat}_{\mathsf{ndg}}(\mathcal{A}, \mathcal{B})$$

are equivalences.

As in the proof of Theorem 4.1.3, we obtain faithfulness by Lemma 3.2.5: a natural transformation of functors on Hilbert module categories is determined by its values on the representables.

To prove fullness, whisker a natural transformation $\eta : F \Rightarrow F'$ of functors $\mathcal{A} \to \mathcal{MB}$ through the inclusion $\mathcal{MB} \to \mathsf{Hilb}\text{-}\mathcal{B}$ to get a transformation $\overline{\eta}$ of bimodules $\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$. Then by Lemma 3.1.15 this gives a natural transformation $\mathrm{id}\,\bar{\otimes}_{\mathcal{B}}\overline{\eta}$ of tensor functors $\mathsf{Hilb}\text{-}\mathcal{A} \to \mathsf{Hilb}\text{-}\mathcal{B}$, and one easily sees that $\Xi(\mathrm{id}\,\bar{\otimes}_{\mathcal{B}}\overline{\eta}) = \eta$.

Essential surjectivity follows simply by noting that by Corollary 2.4.9, a non-degenerate functor $\mathcal{A} \to \mathcal{MB}$ gives a non-degenerate bimodule by post-composing with the embedding $\mathcal{MB} \to \mathsf{Hilb}\text{-}\mathcal{B}$, and tensoring a representable $\mathcal{A}$-module with this bimodule then gives a representable $\mathcal{B}$-module by Corollary 3.1.10. □

Several full subcategories of the bicategory $\mathsf{C}^*\text{-}\mathsf{Cat}_{\mathsf{ndg}}$ were studied in [AV20]: in particular, they prove that the full sub-bicategory of $\mathsf{C}^*\text{-}\mathsf{Cat}_{\mathsf{ndg}}$ whose objects are small C^*-categories containing all countable direct sums, which they call $\mathrm{C}^*\mathrm{Lin}$, is closed under bicolimits.

4.2 Localizing a 1-category of C^*-categories at the Morita equivalences

In this subsection we construct a category of C^*-categories where Morita equivalences have been inverted, i.e. made into isomorphisms: we will moreover do this in a *universal* way, so that the end result constitutes a localization of 1-categories. To be specific, we will invert the Yoneda embeddings, so that isomorphisms between $\mathsf{Hilb}\text{-}\mathcal{A}$ and $\mathsf{Hilb}\text{-}\mathcal{B}$ give isomorphisms between $\mathcal{A}$ and $\mathcal{B}$. We establish a *reflective localization*, which gives excellent control over the hom-spaces of a localization, as done in recent works on C^*-categories and C^*-algebras such as [Bun23].

Recall first the definition of a localization.

Definition 4.2.1. Let $\mathcal{C}$ be a unital category and W be a class of morphisms in $\mathcal{C}$. The *localization of $\mathcal{C}$ at W*, if it exists, is a category $\mathcal{C}[W^{-1}]$ admitting a functor

$$Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$$

such that for all categories $\mathcal{D}$, the precomposition functor

$$- \circ Q : \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and has essential image consisting of those functors $F : \mathcal{C} \to \mathcal{D}$ that send all morphisms in W to isomorphisms.

Our localizations will be of a special type, namely reflective localizations. These are localizations which are at once adjoints:

Definition 4.2.2. The localization $Q : \mathcal{C} \to \mathcal{C}[W^{-1}]$ of $\mathcal{C}$ at W is called a *reflective localization* if it has a fully faithful right adjoint $R : \mathcal{C}[W^{-1}] \to \mathcal{C}$

Reflective localizations have several convenient equivalent characterizations:

Proposition 4.2.3. *Suppose $\mathcal{B}$ and $\mathcal{C}$ are unital categories and $R : \mathcal{B} \to \mathcal{C}$ is a functor with a left adjoint $Q : \mathcal{C} \to \mathcal{B}$. The following are equivalent:*

1. *$R : \mathcal{B} \to \mathcal{C}$ is a fully faithful functor.*

2. *$Q : \mathcal{C} \to \mathcal{B}$ exhibits $\mathcal{B}$ as the reflective localization of $\mathcal{C}$ at the class of morphisms which get sent to isomorphisms by Q.*

3. *$Q : \mathcal{C} \to \mathcal{B}$ exhibits $\mathcal{B}$ as the reflective localization of $\mathcal{C}$ at the class of units $\eta_c : c \to RQ(c)$.*

4. *The counit $\epsilon : QR \Rightarrow \mathrm{id}_{\mathcal{B}}$ is an isomorphism of functors.*

Proof. The equivalence of (1), (2), and (4) is standard, see for example [nLa23b, Proposition 3.1]. To see how (3) follows from the other statements, we show that in the presence of statements (1) and (4), the localizations in (2) and (3) are defined by the same universal property.

Note first that one of the adjunction axioms gives the commutation of the diagram
$$\begin{tikzcd} Q(c) \arrow[r, "Q(\eta_c)"] \arrow[rr, bend right, "\mathrm{id}"'] & QRQ(c) \arrow[r, "\epsilon_{Q(c)}"] & Q(c) \end{tikzcd}$$
so as the right morphism is an isomorphism by (4), the left one must be too. Hence if a functor sends all those morphisms to isomorphisms which are sent to isomorphisms under Q, it must certainly send all units η_c to isomorphisms.

Suppose conversely that a functor $F : \mathcal{C} \to \mathcal{D}$ sends all units η_c to isomorphisms, and let $g \in \mathcal{C}(c, c')$ be a morphism sent to an isomorphism by Q. Note that we can whisker the natural transformation η through F and obtain a commutative diagram

$$\begin{tikzcd} F(c) \arrow[r, "F(g)"] \arrow[d, "F(\eta_c)"] & F(c') \arrow[d, "F(\eta_{c'})"] \\ FRQ(c) \arrow[r, "FRQ(g)"] & FRQ(c') \end{tikzcd}$$

where the vertical arrows are isomorphisms by hypothesis, and the bottom arrow is an isomorphism since $Q(g)$ is. Hence $F(g)$ is an isomorphism. $\square$

We now apply this theory to the following category of C^*-categories:

Definition 4.2.4. Let
$$\mathbf{C^*Cat}$$
be the unital category whose objects are (locally small, not necessarily unital) C^*-categories, and whose morphisms from $\mathcal{A}$ to $\mathcal{B}$ are natural (unitary) isomorphism classes of non-degenerate functors $\mathcal{A} \to \mathcal{MB}$.

This is of course a '1-truncation' of the bicategory $\mathsf{C^*\text{-}Cat_{ndg}}$ from the previous subsection.

Definition 4.2.5. Let
$$\mathcal{K}\mathsf{Hilb}\text{-} : \mathbf{C^*Cat} \to \mathbf{C^*Cat}$$

be the endofunctor that sends every C^*-category to its C^*-category of Hilbert modules and compact operators, and sends every non-degenerate functor $F : \mathcal{A} \to \mathcal{MB}$ to the tensor product with the non-degenerate bimodule given by composing with the Yoneda embedding $\overline{\iota_{\mathcal{B}}} : \mathcal{MB} \to \mathsf{Hilb}\text{-}\mathcal{B}$. Let

$$\mathbf{HMod}$$

be the full subcategory of $\mathbf{C^*Cat}$ with objects those in the image of $\mathcal{K}\mathsf{Hilb}\text{-}$.

Proposition 4.2.6. *The functor $\mathcal{K}\mathsf{Hilb}\text{-}$ is left adjoint to the inclusion functor*

$$\mathrm{incl} : \mathbf{HMod} \to \mathbf{C^*Cat}.$$

Proof. We need to show that for any two C^*-categories $\mathcal{A} \in \mathsf{Ob}\,\mathbf{C^*Cat}$ and $\mathcal{B} \in \mathsf{Ob}\,\mathbf{HMod}$, there is a natural isomorphism of hom-sets

$$\mathbf{C^*Cat}(\mathcal{A}, \mathcal{B}) \cong \mathbf{C^*Cat}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}, \mathcal{B}).$$

Since $\mathcal{B}$ is in the image of $\mathcal{K}\mathsf{Hilb}\text{-}$ we can write $\mathcal{B} = \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{C}$. But then the first hom-space refers to non-degenerate $\mathcal{A}$-$\mathcal{C}$ bimodules and the second corresponds to strong* unital functors from $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ to $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{C}$, and these are in correspondence by the Eilenberg-Watts theorem. This correspondence is easily seen to be natural in both $\mathcal{A}$ and $\mathcal{C}$. $\square$

We hence derive the main result of this subsection:

Theorem 4.2.7. *The functor $\mathcal{K}\mathsf{Hilb}\text{-}$ is the (reflective) localization of the category $\mathbf{C^*Cat}$ at the class of Yoneda inclusions $\iota_{\mathcal{A}}$, as well as at the class of non-degenerate functors $\mathcal{A} \to \mathcal{MB}$ inducing Morita equivalence.*

Proof. The functor incl from Proposition 4.2.6 is clearly full and faithful, so the adjunction satisfies statement 1 in Proposition 4.2.3. The Yoneda bimodules $\iota_{\mathcal{A}}$ give the identity on $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ upon tensoring, so give us the units of the adjunction. Hence statement 3 tells us that $\mathcal{K}\mathsf{Hilb}\text{-}$ is the localization at the class of Yoneda embeddings. Statement 2 gives us the final part of the theorem. $\square$

Hence, **HMod** is the first model for the *Morita homotopy theory* of C^*-categories, akin to similar results on dg-categories ([Toë07][Tab05]), unital C^*-categories ([DT14]), and $(\infty, 1)$-categories ([CG19]). This construction may well be of interest in noncommutative geometry since invariants such as KK-theory have long been modelled as functors from a category of C^*-algebras that invert Morita equivalences.

We also derive easily the idempotence of the functor $\mathcal{K}\mathsf{Hilb}\text{-}$, which can alternatively be deduced from Corollary 3.3.18.

Proposition 4.2.8. *For any locally small C^*-category $\mathcal{A}$, the map*

$$\iota_{\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}} : \mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A} \to \mathcal{K}\mathsf{Hilb}\text{-}(\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A})$$

is an isomorphism.

Proof. We prove that in the set-up of Proposition 4.2.3, for all $b \in \mathsf{Ob}\,\mathcal{B}$ the unit $\eta_{R(b)} : R(b) \to RQR(b)$ is an isomorphism.

Note that one of the adjunction axioms states that the diagram

$$\begin{tikzcd} R(b) \arrow[r, "\eta_{R(b)}"] \arrow[rr, bend right, "\mathrm{id}"'] & RQR(b) \arrow[r, "R(\epsilon_b)"] & R(b) \end{tikzcd}$$

commutes. But by statement 3 we know ϵ_b is an isomorphism. So $R(\epsilon_b)$ must be an isomorphism, and hence $\eta_{R(b)}$ is also an isomorphism.

Now note that in the setting of Proposition 4.2.6, $\iota_{\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}}$ is exactly the morphism $\eta_{R(b)}$ for $b = \mathcal{A}$. $\square$

It follows that C^*-categories which are equivalent to one of the form $\mathcal{K}\mathsf{Hilb}\text{-}\mathcal{A}$ for some locally small C^*-category $\mathcal{A}$ are *exactly* those C^*-categories whose Yoneda embedding is an equivalence. This gives us a partial answer to the questions raised about 'pre-completeness' in [Hen15].

Appendix A

The non-degenerate tensor-hom adjunction

joint work with Benjamin Dünzinger

In this appendix we prove a tensor-hom adjunction for non-degenerate functors of C^*-categories.

As this will involve iterated functor categories, we assume in this appendix that all categories are *small*, although this assumption is not strictly necessary if one pays sufficient care to size issues.

Recall from Proposition 1.1.46 that for any two small C^*-categories $\mathcal{B}$ and $\mathcal{C}$ there is a small C^*-category $\mathrm{Fun}(\mathcal{B}, \mathcal{C})$ of C^*-functors. The objects of these functor categories are, of course, given by (not necessarily unital) C^*-functors. The morphisms are uniformly bounded natural transformations and the involution is given pointwise. We begin by defining a few variants of the functor category that will come in handy, some of which we saw already in Chapter 1.

Definition A.1. If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-categories we denote by

$$\mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathcal{B})$$

the C^*-category of unital C^*-functors from $\mathcal{A}$ to $\mathcal{B}$. For any two C^*-categories $\mathcal{A}$ and $\mathcal{B}$ we denote by

$$\mathrm{Fun}^{\mathrm{strict}}(\mathcal{M}\mathcal{A}, \mathcal{M}\mathcal{B})$$

the full subcategory of $\mathrm{Fun}^{\mathrm{u}}(\mathcal{M}\mathcal{A}, \mathcal{M}\mathcal{B})$ spanned by those functors which are strictly continuous on norm-bounded subsets. We denote by

$$\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, \mathcal{M}\mathcal{B})$$

the subcategory of $\mathrm{Fun}(\mathcal{A}, \mathcal{M}\mathcal{B})$ spanned by the non-degenerate functors. Finally we denote by

$$\mathrm{Fun}^{\mathrm{ndg,prop}}(\mathcal{A}, \mathcal{M}\mathcal{B})$$

the subcategory of $\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, \mathcal{M}\mathcal{B})$ whose objects are the non-degenerate functors landing in the ideal $\mathcal{B} \subseteq \mathcal{M}\mathcal{B}$, though still with transformations in $\mathcal{M}\mathcal{B}$.

Recall from Proposition 1.4.12 that if $\kappa = \kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}\mathcal{A}$ is the standard embedding, then the precomposition functor

$$- \circ \kappa : \mathrm{Fun}^{\mathrm{strict}}(\mathcal{M}\mathcal{A}, \mathcal{M}\mathcal{B}) \to \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}, \mathcal{M}\mathcal{B})$$

is an isomorphism of C^*-categories.

Dell'Ambrogio defines the maximal tensor product $\mathcal{A} \otimes_{\max} \mathcal{B}$ of two unital C^*-categories $\mathcal{A}$ and $\mathcal{B}$ ([Del12, Section 3.2]) and proves a tensor-hom adjunction

$$\mathrm{Fun}^{\mathrm{u}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{C}) \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}, \mathcal{C})).$$

In [BEL23], the maximal tensor product of non-unital C^*-categories is defined. Upon inspection one realizes it is unlikely that for all non-unital C^*-categories we have

$$\mathrm{Fun}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{C}) \cong \mathrm{Fun}(\mathcal{A}, \mathrm{Fun}(\mathcal{B}, \mathcal{C}))$$

since such an isomorphism usually works through a 'currying' procedure that uses the units in $\mathcal{A}$ to determine a functor from $\mathcal{B}$ to $\mathcal{C}$ for each object in $\mathcal{A}$.

In this appendix, we combine the results in Section 1.4 of this thesis with Dell'Ambrogio's tensor-hom adjunction to obtain a 'non-degenerate tensor-hom adjunction'. More precisely, we will obtain an isomorphism between

$$\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{M}\mathcal{C})$$

and a subcategory of $\mathrm{Fun}(\mathcal{A}, \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{B}, \mathcal{M}\mathcal{C}))$.

We briefly recapitulate the work of Bunke on maximal tensor products of non-unital C^*-categories. We begin by defining the maximal norm, which can in fact be defined on any complex linear $*$-category:

Definition A.2. If $\mathcal{A}$ is a complex linear $*$-category, the maximal norm on $\mathcal{A}$ is the seminorm given by

$$\|f\|_{max} := \sup_{\rho : \mathcal{A} \to B} \|\rho(f)\|$$

where we vary over all $*$-functors $\rho : \mathcal{A} \to B$, where B is a C^*-algebra.

It is shown in [Bun19, Remark 2.15] that a small complex linear $*$-category is a C^*-category if and only if the maximal norm is positive-definite and complete (note that the smallness assumption can easily be dropped by allowing B to live in an enlargement of the enriching category of $\mathcal{A}$). The following lemma immediately follows:

Lemma A.3. *If $\mathcal{A}$ is a complex linear $*$-category, the normed complex linear $*$-category Compl$(\mathcal{A})$ obtained by quotienting out elements with zero maximal norm and completing is a C^*-category.*

We can now define the maximal tensor product, which will form a completion of the following complex linear $*$-category.

Definition A.4. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, then the *algebraic tensor product* of $\mathcal{A}$ and $\mathcal{B}$, also labelled $\mathcal{A} \odot \mathcal{B}$, is the complex linear $*$-category with objects $\mathsf{Ob}\,\mathcal{A} \times \mathsf{Ob}\,\mathcal{B}$ and hom-spaces

$$\mathcal{A} \odot \mathcal{B}((x,y),(x',y')) := \mathcal{A}(x,x') \otimes_{\mathbb{C}} \mathcal{B}(y,y').$$

The composition and involution is defined in the obvious way.

Definition A.5. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, the *maximal tensor product* of $\mathcal{A}$ and $\mathcal{B}$ is the C^*-category

$$\mathcal{A} \otimes_{\max} \mathcal{B} := \mathrm{Compl}(\mathcal{A} \odot \mathcal{B}).$$

It turns out that there are no elements with maximal norm zero in $\mathcal{A} \odot \mathcal{B}$:

Lemma A.6 ([BEL23, Lemma 7.3])**.** *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, then for any morphism c in $\mathcal{A} \odot \mathcal{B}$ we have $\|c\|_{max} = 0$ only if $c = 0$.*

Hence there is a faithful functor $\mathcal{A} \odot \mathcal{B} \to \mathcal{A} \otimes_{\max} \mathcal{B}$.

It is possible to tensor together two approximate units and get another one:

Lemma A.7. *If $(u_\lambda)_{\lambda \in \Lambda}$ and $(v_\mu)_{\mu \in M}$ are approximate units for the C^*-algebras $\mathcal{A}(x,x)$ and $\mathcal{B}(y,y)$, then the net $(u_\lambda \otimes v_\mu)_{(\lambda,\mu) \in \Lambda \times M}$ is an approximate unit for $(\mathcal{A} \otimes_{\max} \mathcal{B})((x,y),(x,y))$.*

Proof. We define two commuting, uniformly bounded nets U_λ, V_μ of multipliers in $\mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B})((x,y)(x,y))$ on simple tensors by

$$U_\lambda(a \otimes b) = (u_\lambda a \otimes b) \text{ and } (a \otimes b)U_\lambda = (a u_\lambda \otimes b),$$

and similarly for V_μ. Note that $U_\lambda V_\mu = \kappa(u_\lambda \otimes v_\mu)$. Note also that since $\|a \otimes b\|_{\max} \leq \|a\|\|b\|$, the tensor product is continuous in each variable and hence both nets U_λ and V_μ converge strictly to the identity. But then by Lemma 1.1.45 we see that their product $U_\lambda V_\mu = \kappa(u_\lambda \otimes v_\mu)$ converges strictly to the identity, and by Lemma 1.1.19 it is positive, meaning it is an approximate unit. □

The maximal tensor product has the following universal property:

Lemma A.8. *If $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are C^*-categories and $F : \mathcal{A} \odot \mathcal{B} \to \mathcal{C}$ is a linear $*$-functor, F extends to a C^*-functor*

$$\overline{F} : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$$

Proof. This is [BEL23, Definition 7.2]: the statement that this characterization is equivalent to Definition A.5 is in the proof of [BEL23, Proposition 7.5]. □

We easily deduce that the maximal tensor product is functorial in both arguments.

Corollary A.9. *For any two C^*-functors $F : \mathcal{A} \to \mathcal{A}', G : \mathcal{B} \to \mathcal{B}'$, there is a C^*-functor*

$$F \otimes G : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A}' \otimes_{\max} \mathcal{B}'$$

given on simple tensors by $(F \otimes G)(a \otimes b) = F(a) \otimes G(b)$.

Proof. Simply extend the obvious linear $*$-functor $F \odot G : \mathcal{A} \odot \mathcal{B} \to \mathcal{A}' \otimes_{\max} \mathcal{B}'$. □

One such tensored functor is of interest here:

Definition A.10. For two non-unital C^*-categories $\mathcal{A}, \mathcal{B}$ we denote by

$$J_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}$$

the C^*-functor given by tensoring the two inclusions $\kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}\mathcal{A}$ and $\kappa_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}\mathcal{B}$. We denote by

$$I_{\mathcal{A},\mathcal{B}} : \mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B} \to \mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B})$$

the C^*-functor extending the obvious functor $\mathcal{M}\mathcal{A} \odot \mathcal{M}\mathcal{B} \to \mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B})$.

Lemma A.11. *We have*

$$I_{\mathcal{A},\mathcal{B}} \circ J_{\mathcal{A},\mathcal{B}} = \kappa_{\mathcal{A} \otimes_{\max} \mathcal{B}}.$$

Furthermore, the functor $J_{\mathcal{A},\mathcal{B}}$ is faithful.

Proof. For any $a \in \mathcal{A}(x, x'), b \in \mathcal{B}(y, y')$ we clearly have

$$I_{\mathcal{A},\mathcal{B}} \circ J_{\mathcal{A},\mathcal{B}}(a \otimes b) = \kappa_{\mathcal{A} \otimes_{\max} \mathcal{B}}(a \otimes b).$$

But then we see that $I_{\mathcal{A},\mathcal{B}} \circ J_{\mathcal{A},\mathcal{B}} = \kappa_{\mathcal{A} \otimes_{\max} \mathcal{B}}$ on the subspace

$$(\mathcal{A} \odot \mathcal{B})((x, y), (x', y')) \subseteq (\mathcal{A} \otimes_{\max} \mathcal{B})((x, y), (x', y'))$$

and as this subspace is by definition norm-dense, we see $I_{\mathcal{A},\mathcal{B}} \circ J_{\mathcal{A},\mathcal{B}} = \kappa_{\mathcal{A} \otimes_{\max} \mathcal{B}}$ everywhere. It follows immediately that $J_{\mathcal{A},\mathcal{B}}$ is faithful since $\kappa_{\mathcal{A} \otimes_{\max} \mathcal{B}}$ is. □

We caution that the functor $I_{\mathcal{A},\mathcal{B}}$ is not necessarily faithful: already in the algebra case this fails, see the comments in [KLQ21, p.19].

We now have the tools to prove our promised tensor-hom adjunction for non-degenerate functors. We recall again Dell'Ambrogio's result:

Proposition A.12 ([Del12, Lemma 3.1.4])**.** *For all unital C^*-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ there is a trinatural bijection between the objects of the C^*-categories*

$$\phi : \mathrm{Fun}^{\mathrm{u}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{C}) \xrightarrow{\cong} \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}, \mathcal{C})).$$

We describe explicitly ϕ and its inverse. For a functor $F : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$, the map of $\phi(F)$ on objects is simply obtained by currying that of F. For an object $x \in \mathsf{Ob}\,\mathcal{A}$ we obtain $\phi(F)(x) : \mathcal{B} \to \mathcal{C}$ by the action of F on morphisms of the form $\mathrm{id}_x \otimes b \in \mathcal{A} \otimes_{\max} \mathcal{B}((x, y), (x, y'))$ and similarly we obtain the action of $\mathcal{A}$ by natural transformations of functors using morphisms of the form $a \otimes \mathrm{id}_y$.

Conversely, for a functor $G : \mathcal{A} \to \mathrm{Fun}(\mathcal{B}, \mathcal{C})$, the functor $\phi^{-1}(G)$ is obtained on objects by un-currying and on simple tensors of morphisms

$$a \otimes b \in \mathcal{A} \otimes_{\max} \mathcal{B}((x, y), (x', y'))$$

by

$$\phi^{-1}(G)(a \otimes b) := G(a)_y \circ G(x)(b) = G(y)(b) \circ G(a)_{y'},$$

where the second equality holds since $G(a)$ is a natural transformation.

Strictly speaking, Dell'Ambrogio only proves an isomorphism between the objects of these two functor C^*-categories. It is easy to promote this to an isomorphism of C^*-categories, however:

Lemma A.13. *For any two unital C^*-functors $F, F' : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{C}$ there is a bijection between the spaces of natural transformations $Nat(F, F')$ and $Nat(\phi(F), \phi(F'))$.*

Proof. A transformation $\eta : F \Rightarrow F'$ is given by a $\mathcal{C}$-morphism

$$\eta_{(x,y)} : F(x, y) \to F'(x, y)$$

for each $(x, y) \in \mathsf{Ob}\,\mathcal{A} \times \mathsf{Ob}\,\mathcal{B}$ satisfying naturality under all morphisms in $\mathcal{A} \otimes_{\max} \mathcal{B}$. Now for $\phi(F), \phi(F') : \mathcal{A} \to \mathrm{Fun}^{\mathrm{u}}(\mathcal{B}, \mathcal{C})$ as above, define the transformation $\phi(\eta) : \phi(F) \Rightarrow \phi(F')$ to have components $\phi(\eta)_x : \phi(F)(x) \Rightarrow \phi(F')(x)$ where

$$(\phi(\eta)_x)_y := \eta_{(x,y)} : \phi(F)(x)(y) \Rightarrow \phi(F')(x)(y)$$

for all $x \in \mathsf{Ob}\,\mathcal{A}, y \in \mathsf{Ob}\,\mathcal{B}$. Naturality under both $\mathcal{A}$ and $\mathcal{B}$ is clearly implied by considering morphisms of the form $(\mathrm{id} \otimes b)$ and $(a \otimes \mathrm{id})$. It is clear that ϕ is injective since $\phi(\eta)$ consists of the same array of maps as η, and it is surjective since naturality under $\mathcal{A} \otimes_{\max} \mathcal{B}$ is equivalent to naturality under morphisms of the form $(\mathrm{id} \otimes b)$ or $(a \otimes \mathrm{id})$. $\square$

Hence in particular for all non-unital C^*-categories there is a unitary equivalence

$$\mathrm{Fun}^{\mathrm{u}}(\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}, \mathcal{M}\mathcal{C}) \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{M}\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{M}\mathcal{B}, \mathcal{M}\mathcal{C})).$$

We now describe the subcategories of these two C^*-categories that we would like to identify:

Definition A.14. We say that a unital C^*-functor $F : \mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B} \to \mathcal{M}\mathcal{C}$ is *strict* when the composition

$$\mathcal{A} \otimes_{\max} \mathcal{B} \xrightarrow{J_{\mathcal{A},\mathcal{B}}} \mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B} \xrightarrow{F} \mathcal{M}\mathcal{C}$$

is *non-degenerate*. We denote by $\mathrm{Fun}^{\mathrm{strict}}(\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}, \mathcal{M}\mathcal{C})$ the full subcategory of $\mathrm{Fun}^{\mathrm{u}}(\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}, \mathcal{M}\mathcal{C})$ spanned by these functors.

Proposition A.15. *The functor*

$$\mathrm{Fun}^{\mathrm{strict}}(\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}, \mathcal{M}\mathcal{C}) \xrightarrow{-\circ J_{\mathcal{A},\mathcal{B}}} \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{M}\mathcal{C})$$

is an isomorphism.

Proof. Write κ for the canonical functor $\kappa_{\mathcal{A}\otimes_{\max}\mathcal{B}} : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B})$ and recall that precomposition with κ induces an isomorphism

$$(-\circ\kappa) : \mathrm{Fun}^{\mathrm{strict}}(\mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B}), \mathcal{M}\,\mathcal{C}) \to \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{M}\mathcal{C}),$$

where the former denotes those functors from $\mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B})$ to $\mathcal{M}\mathcal{C}$ which are strictly continuous on unit balls. Let Ψ be the inverse of the above isomorphism. We claim that

$$(-\circ I_{\mathcal{A},\mathcal{B}}) \circ \Psi : \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{M}\mathcal{C}) \to \mathrm{Fun}^{\mathrm{strict}}(\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}, \mathcal{M}\mathcal{C})$$

is an inverse of $-\circ J_{\mathcal{A},\mathcal{B}}$. The equality

$$(-\circ J_{\mathcal{A},\mathcal{B}}) \circ (-\circ I_{\mathcal{A},\mathcal{B}}) \circ \Psi = \mathrm{id}_{\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A}\otimes_{\max}\mathcal{B},\mathcal{M}\mathcal{C})}$$

is true since by Lemma A.11 we have $-\circ\kappa = -\circ I_{\mathcal{A},\mathcal{B}} J_{\mathcal{A},\mathcal{B}}$.

For the other direction, take a strict functor $F : \mathcal{M}\,\mathcal{A} \otimes_{\max} \mathcal{M}\,\mathcal{B} \to \mathcal{M}\,\mathcal{C}$, and consider the diagram

$$\begin{tikzcd}
 & \mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B}) \arrow[dr, "\Psi(FJ_{\mathcal{A},\mathcal{B}})"] & \\
\mathcal{A} \otimes_{\max} \mathcal{B} \arrow[ur, "\kappa"] \arrow[r, "J_{\mathcal{A},\mathcal{B}}"] & \mathcal{M}\,\mathcal{A} \otimes_{\max} \mathcal{M}\,\mathcal{B} \arrow[u, "I_{\mathcal{A},\mathcal{B}}"] \arrow[r, "F"] & \mathcal{M}\,\mathcal{C}
\end{tikzcd}$$

observing that commutation of the right-hand triangle is equivalent to the equation $F = (-\circ I_{\mathcal{A},\mathcal{B}})\Psi(-\circ J_{\mathcal{A},\mathcal{B}})(F)$, giving the other inverse equation. Note also that the right-hand triangle commutes on the image of $J_{\mathcal{A},\mathcal{B}}$ by the other inverse condition.

To show that the right triangle commutes on all morphisms in $\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}$, it suffices to check that

$$F(T \otimes T') = \Psi(FJ_{\mathcal{A},\mathcal{B}}) I_{\mathcal{A},\mathcal{B}}(T \otimes T')$$

for T a morphism in $\mathcal{M}\,\mathcal{A}$ and T' a morphism in $\mathcal{M}\,\mathcal{B}$, since sums of such simple tensors are norm-dense in $\mathcal{M}\mathcal{A} \otimes_{\max} \mathcal{M}\mathcal{B}$. Let (u_λ) be an approximate unit of the domain of T and (v_μ) be an approximate unit of the domain of T'.

Recall from Lemma A.7 that $(u_\lambda \otimes v_\mu)$ is an approximate unit for the domain of $T \otimes T'$. Since $FJ_{\mathcal{A},\mathcal{B}}$ is non-degenerate, we know that

$$\begin{aligned} F(T \otimes T') &= \lim_{\lambda,\mu} F(T \otimes T') \circ FJ_{\mathcal{A},\mathcal{B}}(u_\lambda \otimes v_\mu) \\ &= \lim_{\lambda,\mu} FJ_{\mathcal{A},\mathcal{B}}(T(u_\lambda) \otimes T'(v_\mu)) \\ &= \lim_{\lambda,\mu} \Psi(FJ_{\mathcal{A},\mathcal{B}})(\kappa(T(u_\lambda) \otimes T'(v_\mu))) \\ &= \lim_{\lambda,\mu} \Psi(FJ_{\mathcal{A},\mathcal{B}})(I_{\mathcal{A},\mathcal{B}}(T \otimes T') \circ \kappa(u_\lambda \otimes v_\mu)) \\ &= \lim_{\lambda,\mu} \Psi(FJ_{\mathcal{A},\mathcal{B}})(I_{\mathcal{A},\mathcal{B}}(T \otimes T')) \circ \Psi(FJ_{\mathcal{A},\mathcal{B}})(\kappa(u_\lambda \otimes v_\mu)) \\ &= \Psi(FJ_{\mathcal{A},\mathcal{B}})(I_{\mathcal{A},\mathcal{B}}(T \otimes T')) \end{aligned}$$

Where the final equality follows from the strict continuity of $\Psi(FJ_{\mathcal{A},\mathcal{B}})$. $\square$

We move on to describe the relevant objects in $\mathrm{Fun}^{\mathrm{u}}(\mathcal{MA}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC}))$ that correspond to those in the above category.

Definition A.16. For all C^*-categories $\mathcal{B}$ and $\mathcal{C}$, the *objectwise strict topology* on $\mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})$ is defined as follows: for $F, G \in \mathsf{Ob}\,\mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})$, a net of natural transformations

$$(T^\lambda) = \{T^\lambda_x : F(x) \to G(x) : x \in \mathsf{Ob}\,\mathcal{B}\, \lambda \in \Lambda\}$$

converges objectwise strictly[1] to the natural transformation $T : F \Rightarrow G$ if for each $x \in \mathsf{Ob}\,\mathcal{B}$, the net T^λ_x of multipliers in $M\,\mathcal{C}(F(x), G(x))$ converges strictly to T_x.

We also have an objectwise strict topology on $\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{B}, \mathcal{MC})$ since by Proposition 1.4.12 it is isomorphic to a subcategory of $\mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})$.

Note that the objectwise strict topology is *not* the topology relative to any obvious class of natural transformations, as far as we can see.

Definition A.17. For any three C^*-categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, we denote by

$$\mathrm{Fun}^{\mathrm{strict}}(\mathcal{MA}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC}))$$

the subcategory of $\mathrm{Fun}^{\mathrm{u}}(\mathcal{MA}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC}))$ spanned by those functors which are continuous on unit balls when one gives $\mathcal{MA}$ the strict topology and $\mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})$ the objectwise strict topology. We denote by

$$\mathrm{Fun}^{\mathrm{strict}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC}))$$

the subcategory of $\mathrm{Fun}(\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC}))$ spanned by those functors satisfying the same continuity requirement.

Lemma A.18. *If $\kappa_\mathcal{A} : \mathcal{A} \to M\,\mathcal{A}$ is the standard inclusion functor, then the precomposition functor*

$$(- \circ \kappa_\mathcal{A}) : \mathrm{Fun}^{\mathrm{strict}}(\mathcal{MA}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})) \to \mathrm{Fun}^{\mathrm{strict}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC}))$$

is an isomorphism.

[1] This is a legitimate definition for a topology since it is given by a family of seminorms $\{T \mapsto \|T_x(a)\|\}$ where x varies over objects of $\mathcal{B}$ and a varies over morphisms with domain $F(x)$.

Proof. Since $\mathcal{A}$ is strictly dense in $\mathcal{MA}$ (Lemma 1.3.8) we immediately see the functor $(- \circ \kappa_{\mathcal{A}})$ is injective on objects.

Now take a strict functor $F : \mathcal{A} \to \mathrm{Fun}(\mathcal{MB}, \mathcal{MC})$. To show $(- \circ \kappa_{\mathcal{A}})$ is surjective on objects, we must show that every such F lifts to a strict functor $\overline{F} : \mathcal{MA} \to \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})$. To do this, note that for any given $y \in \mathsf{Ob}\,\mathcal{B}$ there is a functor $F_y : \mathcal{A} \to \mathcal{MC}$ given for each $x \in \mathsf{Ob}\,\mathcal{A}$ by $F_y(x) = F(x)(y)$ and for each $a \in \mathcal{A}(x, x')$ by $F_y(a) = F(a)_x$. Each functor F_y is non-degenerate by the strict continuity requirement so extends to a unital functor $\overline{F_y} : \mathcal{MA} \to \mathcal{MC}$ which is strictly continuous on bounded subsets. As in the proof of Proposition 1.4.12, for any $T \in \mathcal{MA}(x, x')$ the assemblage $\{\overline{F_y}(T)\}$ defines a natural transformation from $F(x)$ to $F(x')$, since we can strictly approximate it by a net of natural transformations $\{F_y(Tu_\lambda)\}$ where u_λ is an approximate unit for $\mathcal{A}(x, x)$.

Hence we can lift F to a unital functor $\overline{F} : \mathcal{MA} \to \mathrm{Fun}^{\mathrm{u}}(\mathcal{MB}, \mathcal{MC})$, which is strictly continuous on bounded subsets since every component $\overline{F_y}$ is.

The proof that $(- \circ \kappa_{\mathcal{A}})$ is fully faithful proceeds exactly as in Proposition 1.4.12. □

We now show that the two forms of 'double strictness' we've defined correspond to each other under the unital tensor-hom adjunction.

Lemma A.19. *A unital C^*-functor*

$$F : \mathcal{MA} \otimes_{\max} \mathcal{MB} \to \mathcal{MC}$$

is strict if and only if its image under currying

$$\phi(F) : \mathcal{MA} \to \mathrm{Fun}(\mathcal{MB}, \mathcal{MC})$$

both lands in $\mathrm{Fun}^{\mathrm{strict}}(\mathcal{MB}, \mathcal{MC})$ and *is strict in the sense defined in Definition A.17.*

Proof. Suppose $\phi(F)$ satisfies the requirements above, and take any $x \in \mathsf{Ob}\,\mathcal{A}$ and $y \in \mathsf{Ob}\,\mathcal{B}$. Let (u_λ) be an approximate unit for $\mathcal{A}(x, x)$ and (v_μ) be an approximate unit for $\mathcal{B}(y, y)$. Then since $\phi(F)$ lands in non-degenerate functors we can deduce that $F(\mathrm{id} \otimes v_\mu) \xrightarrow{\mu} \mathrm{id}$ strictly in $\mathcal{MC}(F(x)(y), F(x)(y))$. Furthermore since $\phi(F)$ is strict as defined in Definition A.17 we deduce that $F(u_\lambda \otimes \mathrm{id}) \xrightarrow{\lambda} \mathrm{id}$ strictly in $\mathcal{MC}(F(x)(y), F(x)(y))$. But then by Lemma 1.1.45 we have that

$$F(u_\lambda \otimes v_\mu) = F(\mathrm{id} \otimes v_\mu) F(u_\lambda \otimes \mathrm{id}) \xrightarrow{\lambda,\mu} \mathrm{id}_{F(x,y)}$$

strictly.

But then as $(u_\lambda \otimes v_\mu)$ is an approximate unit for $(\mathcal{A} \otimes_{\max} \mathcal{B})((x, y)(x, y))$ by Lemma A.7, we easily see that $F \circ J_{\mathcal{A},\mathcal{B}}$ is non-degenerate: note that in Theorem 1.4.5 it is enough for criterion 3 to hold of just one approximate unit in each endomorphism algebra since this implies criterion 1.

Suppose conversely that F is strict. Then by Proposition A.15 we know F factors as $F = \overline{F} \circ I_{\mathcal{A},\mathcal{B}}$, where $\overline{F} = \Psi(F \circ J_{\mathcal{A},\mathcal{B}}) : \mathcal{M}(\mathcal{A} \otimes_{\max} \mathcal{B}) \to \mathcal{MC}$ is unital and strictly continuous on unit balls and $I_{\mathcal{A},\mathcal{B}}$ and $J_{\mathcal{A},\mathcal{B}}$ are the canonical functors defined in Definition A.10. Then since the net $I_{\mathcal{A},\mathcal{B}}(\mathrm{id} \otimes v_\mu)$ converges strictly to the identity in $(\mathcal{MA} \otimes_{\max} \mathcal{MB})((x,y),(x,y))$, the elements $\overline{F} I_{\mathcal{A},\mathcal{B}}(\mathrm{id} \otimes v_\mu)$ converge strictly to the identity in $\mathcal{MC}(F(x)(y), F(x)(y))$, and hence we see that $\phi(F)(x) : \mathcal{B} \to \mathcal{MC}$ is a non-degenerate functor for each x.

We prove finally that $\phi(F)$ is strict in the sense defined in Definition A.17. Take a norm-bounded net (T_λ) of multipliers in $\mathcal{MA}(x,x')$ converging strictly to 0. Then certainly the net $(I_{\mathcal{A},\mathcal{B}}(T_\lambda \otimes \mathrm{id}))$ in $\mathcal{M}(A \otimes_{\max} \mathcal{B})((x,y),(x',y)$ is also norm-bounded and strictly converging to zero, and hence we have $\overline{F} I_{\mathcal{A},\mathcal{B}}(T_\lambda \otimes \mathrm{id}) \xrightarrow{\lambda} 0$ strictly. But then we see the net $(\phi(F)(T_\lambda))$ in the space $\mathcal{MC}(\phi(F)(x)(y), \phi(F)(x')(y))$ converges strictly to zero for all $y \in \mathsf{Ob}\,\mathcal{B}$, proving our claim. □

We can now prove the non-degenerate tensor-hom adjunction.

Proposition A.20. *If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are C^*-categories, there is a tensor-hom equivalence*

$$\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{MC}) \cong \mathrm{Fun}^{\mathrm{strict}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{B}, \mathcal{MC})).$$

Proof. Combining Lemma A.19 with the unital tensor-hom adjunction we obtain an isomorphism of C^*-categories

$$\mathrm{Fun}^{\mathrm{strict}}(\mathcal{MA} \otimes_{\max} \mathcal{MB}, \mathcal{MC}) \cong \mathrm{Fun}^{\mathrm{strict}}(\mathcal{MA}, \mathrm{Fun}^{\mathrm{strict}}(\mathcal{MB}, \mathcal{MC})).$$

The left category is isomorphic to $\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{MC})$ by Proposition A.15 The right category is isomorphic to $\mathrm{Fun}^{\mathrm{strict}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{ndg}}(\mathcal{B}, \mathcal{MB}))$ by Proposition 1.4.12 and Lemma A.18. □

Corollary A.21. *If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are C^*-categories where $\mathcal{A}$ is unital, there is a tensor-hom adjunction*

$$\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{MC}) \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, Fun^{ndg}(\mathcal{B}, \mathcal{MC}))$$

which is natural in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$.

Proof. As noted before, the strict topology on a unital C^*-category is simply the norm topology, and since C^*-functors are norm-decreasing and norm convergence in $\mathrm{Fun}^{\mathrm{ndg}}(\mathcal{B}, \mathcal{MC})$ certainly implies objectwise strict convergence, Proposition A.20 specializes to the above. □

We end by specializing the non-degenerate tensor-hom adjunction to the proper case, recalling that a functor to $\mathcal{MC}$ is labelled proper if it lands in $\mathcal{C} \subseteq \mathcal{MC}$.

Proposition A.0.1. *If $\mathcal{A}$ is a unital C^*-category, then a non-degenerate C^*-functor*

$$F : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{M}\mathcal{C}$$

is proper if and only if its image

$$\phi(F) : \mathcal{A} \to \mathrm{Fun}(\mathcal{B}, \mathcal{M}\mathcal{C})$$

under the isomorphism in Proposition A.20 lands in $\mathrm{Fun}^{\mathrm{ndg,prop}}(\mathcal{B}, \mathcal{M}\mathcal{C})$.

Proof. If F is proper, then certainly we have $F(\mathrm{id} \otimes b) \in \mathcal{C}$ for all simple tensors $\mathrm{id} \otimes b \in \mathcal{A} \otimes_{\max} \mathcal{B}((x, y), (x, y'))$. Hence we see $\phi(F)(x)$ is proper for all objects $x \in \mathsf{Ob}\, \mathcal{A}$.

On the other hand, if $\phi(F)(x) : \mathcal{B} \to \mathcal{M}\mathcal{C}$ is proper for all $x \in \mathsf{Ob}\, \mathcal{A}$, then for all $a \otimes b \in \mathcal{A} \otimes_{\max} \mathcal{B}((x, y), (x', y'))$ we have

$$F(a \otimes b) = \phi(F)(a)_y \circ \phi(F)(x)(b) \in \mathcal{C}$$

as $\phi(F)(x)(b) \in \mathcal{C}$ and $\mathcal{C}$ is an ideal in $\mathcal{M}\mathcal{C}$. It follows that F is proper. $\square$

Corollary A.22. *If $\mathcal{A}$ is a unital C^*-category, then for all C^*-categories $\mathcal{B}$ and $\mathcal{C}$ there is an isomorphism of functor categories*

$$\mathrm{Fun}^{\mathrm{ndg,prop}}(\mathcal{A} \otimes_{\max} \mathcal{B}, \mathcal{C}) \cong \mathrm{Fun}^{\mathrm{u}}(\mathcal{A}, \mathrm{Fun}^{\mathrm{ndg,prop}}(\mathcal{B}, \mathcal{C})).$$

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