SERRE DUALITY AND GEOMETRIC MORPHISMS

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Abstract. For a rigidly-compactly generated tensor triangulated category, a k-linear structure over a field k is the same as to give a geometric morphism from the derived category of k. Moreover, if this is the case, the subcategory of rigid-compact objects enjoys Serre duality precisely when the geometric morphism is Gorenstein, in the sense that its dualising object is invertible.

[This note should be seen as a complement to Section 6 of [BDS16], in particular Proposition 7 offers a converse of Corollary 6.12 of *loc. cit.*]

Let $\mathbb{k}$ be a field. A *Serre functor* on a $\mathbb{k}$-linear triangulated category $\mathcal{C}$ is a $\mathbb{k}$-linear equivalence $S\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ together with a natural isomorphism $\sigma_{x,y}\colon \mathcal{C}(x,y)^\star \cong \mathcal{C}(y, Sx)$ for all objects $x, y \in \mathcal{C}$, where $(-)^\star$ denotes the $\mathbb{k}$-linear dual; see [BK89] and [BO01]. By the Yoneda lemma, if such a pair (S, σ) exists then it is uniquely determined up to a unique isomorphism, so its existence is really a property of the category and we may want to say that $\mathcal{C}$ *has Serre duality.*

If $\mathcal{C}$ happens to also be endowed with a tensor product $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, then it is natural to ask whether the Serre functor is given by a twist, *i.e.* if it is given by tensoring with a tensor-invertible object of $\mathcal{C}$. We claim that this is necessarily the case, and we begin by proving this general fact.

1. **Lemma.** *Let $\mathcal{C}$ be $\mathbb{k}$-linear tensor category with Serre duality (S, σ). Let $F\colon \mathcal{C} \to \mathcal{C}$ be any $\mathbb{k}$-linear functor admitting a two-sided adjoint G. Then S and F commute up to an isomorphism, $S \circ F \cong F \circ S$, which is natural in F.*

Proof. (We learned this argument from Bernhard Keller.) Just compute as follows

$$\begin{aligned}
\mathcal{C}(x, S(Fy)) &\cong \mathcal{C}(Fy, x)^\star && \text{by Serre duality}\\
&\cong \mathcal{C}(y, Gx)^\star && \text{by } F \dashv G\\
&\cong \mathcal{C}(Gx, Sy) && \text{by Serre duality}\\
&\cong \mathcal{C}(x, F(Sy)) && \text{by } G \dashv F
\end{aligned}$$

and conclude with the Yoneda lemma. □

By taking $F := x \otimes -$ in the lemma and applying the functors to the tensor unit object, we obtain the above claim:

2. **Corollary.** *Let $\mathcal{C}$ be a rigid $\mathbb{k}$-linear tensor category with tensor unit $\mathbb{1}$. If $\mathcal{C}$ has Serre duality, the Serre functor S must be given by the twist $S \cong (-) \otimes S(\mathbb{1})$.* □

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Incidentally, another special case of the lemma is the following well-known property of Serre functors:

3. **Corollary.** *Let $\mathcal{C}$ be a $\Bbbk$-linear category with Serre duality (S, σ). Then the Serre functor commutes, up to isomorphism, with every self-equivalence of $\mathcal{C}$.* □

4. *Remark.* In general not all (exact) self-equivalences of a tensor-triangulated category arise as twists. In algebraic geometry, numerous examples (with Serre duality) occur as bounded derived categories of smooth varieties ([Muk81] [Orl02] [ST01]).

From now on, we focus our attention to the case of *rigidly-compactly generated* tensor triangulated categories in the sense of [BDS16, Def. 2.7] (a.k.a. unital algebraic stable homotopy categories in the sense of [HPS97]). The question now is whether the tensor subcategory of rigid-compact objects has Serre duality.

The next lemma links k-linear structures with the theory of [BDS16]:

5. **Lemma.** *Let $\mathcal{T}$ be a rigidly-compactly generated tensor-triangulated category in the sense of* [BDS16] *(a.k.a. a unital algebraic stable homotopy category in the sense of* [HPS97]*). Then the following two structures on $\mathcal{T}$ are equivalent:*

(a) *a $\Bbbk$-linear structure, i.e. an enrichment of $\mathcal{T}$ over $\Bbbk$-modules (as in* [Kel05]*), compatible with the triangulation in the sense that the suspension functor Σ is $\Bbbk$-linear: $\Sigma(\lambda \cdot \alpha) = \lambda \cdot \Sigma(\alpha)$ for all morphisms α of $\mathcal{T}$ and all $\lambda \in \Bbbk$.*

(b) *a tensor-product preserving (= symmetric monoidal) and coproduct-preserving exact functor $f^* \colon \mathrm{D}(\Bbbk) \to \mathcal{T}$ (a.k.a. a geometric functor in the sense of* [HPS97]*), where $\mathrm{D}(\Bbbk)$ denotes the unbounded derived category of $\Bbbk$.*

Proof. Assume $\mathcal{T}$ is $\Bbbk$-linear. The cohomology functor H^* yields a $\Bbbk$-linear tensor equivalence of $\mathrm{D}(\Bbbk)$ with the category of $\mathbb{Z}$-graded $\Bbbk$-vector spaces and degree-preserving $\Bbbk$-linear maps. Exploiting this equivalence, we can easily extend the assignment $\mathbb{1}_{\mathrm{D}(\Bbbk)} = \Bbbk[0] \mapsto \mathbb{1}_{\mathcal{T}}$, in an essentially unique way, to a coproduct-preserving, shift-preserving and $\Bbbk$-linear functor $f^* \colon \mathrm{D}(\Bbbk) \to \mathcal{T}$. Being additive and commuting with the suspensions, this functor is readily seen to be exact and symmetric monoidal.

Conversely, such a functor $f^* \colon \mathrm{D}(\Bbbk) \to \mathcal{T}$ satisfies the basic hypotheses of [BDS16]; in particular it admits a right adjoint f_* (by Brown representability) and the tensor structure of $\mathcal{T}$ admits an internal Hom functor $\mathsf{hom}_{\mathcal{T}}(-,-)$ (by the hyptheses on $\mathcal{T}$). As in [BDS16, Rem. 6.8], it is straightforward to check that $\underline{\mathcal{T}}(x,y) := f_* \mathsf{hom}_{\mathcal{T}}(x,y)$ extends (by exploiting the tensor-Hom and (f^*, f_*) adjunctions) to an enrichment of $\mathcal{T}$ over the tensor category $\mathrm{D}(\Bbbk)$. By applying the 0th cohomology functor H^0, we then obtain an enrichment of $\mathcal{T}$ in $\Bbbk$-modules. □

6. *Remark.* Alternatively for (b)⇒(a) in Lemma 5, note that $f^* \colon \mathrm{D}(\Bbbk) \to \mathcal{T}$ restricts to a ring map $\Bbbk = \mathrm{End}_{\mathrm{D}(\Bbbk)}(\Bbbk) \to \mathrm{End}_{\mathcal{T}}(\mathbb{1})$ which can be combined with the standard action of the target on $\mathcal{T}$ coming from the tensor structure.

* * *

The tensor-exact functor $f^* \colon \mathrm{D}(\Bbbk) \to \mathcal{T}$ satisfies the basic hypotheses of [BDS16, Hyp. 1.2]. It follows that f^* has a right adjoint f_* (as already mentioned) which itself has a right adjoint $f^{(1)} \colon \mathrm{D}(\Bbbk) \to \mathcal{T}$ (see [BDS16, Cor. 2.14]). We recall that in this situation the *relative dualizing object* $\omega_f := f^{(1)}(\Bbbk[0])$ is of particular interest.

7. **Proposition.** *Let $\mathcal{T}$ be a $\Bbbk$-linear rigidly-compactly generated category, with associated geometric functor $f^*\colon \mathrm{D}(k) \to \mathcal{T}$ as in Lemma 5. Then the following are equivalent:*

(a) *The subcategory of rigid-compact objects $\mathcal{C} := \mathcal{T}^c$ has Serre duality.*
(b) *The relative dualizing object ω_f is tensor-invertible.*

Moreover in this case $S(\mathbb{1}) \cong \omega_f$, i.e. the Serre functor is given by $S \cong (-) \otimes \omega_f$.

8. *Remark.* The proposition provides a source of *invertible* relative dualizing objects ω_f, even in the absence of Grothendieck-Neeman duality (this is what may be called the 'fourth case' of the Trichotomy Theorem [BDS16, Cor. 1.13]).

9. *Remark.* The functor f^* constructed in Lemma 5 may look trivial, but its adjoint f_*, and the further right adjoint $f^{(1)}$, typically are anything but. In algebraic geometry, for instance, f_* specializes to sheaf cohomology and $f^{(1)}$ to the exceptional pullback functor whose computation motivates much of Grothendieck duality theory (see [BDS16, Ex. 3.22 and 6.14]).

Proof of Proposition 7. Let us write $\omega_\Bbbk := \omega_f$ for the dualising object of the functor $f^*\colon \mathrm{D}(\Bbbk) \to \mathcal{T}$, as we are thinking in terms of the $\Bbbk$-enrichment. For such a functor f^* the 'relative Serre duality theorem' [BDS16, Thm. 6.9] always gives us an isomorphism

$$\mathsf{hom}_{\mathrm{D}(\Bbbk)}(\underline{\mathcal{T}}(x,y), \Bbbk[0]) \xrightarrow{\sim} \underline{\mathcal{T}}(y, x \otimes \omega_\Bbbk) \tag{10}$$

natural in $x \in \mathcal{T}^c$ and $y \in \mathcal{T}$, where as above $\underline{\mathcal{T}}$ denotes the $\mathrm{D}(\Bbbk)$-enrichment of $\mathcal{T}$ and hom the internal Hom. (In general, however, $\omega_\Bbbk$ is not necessarily invertible.)

By applying H^0 to (10), we deduce a natural isomorphism

$$\mathcal{T}(x,y)^\star \xrightarrow{\sim} \mathcal{T}(y, x \otimes \omega_\Bbbk) \tag{11}$$

of $\Bbbk$-vector spaces. Clearly if $\omega_\Bbbk$ is tensor-invertible then $S := (-) \otimes \omega_\Bbbk \colon \mathcal{T}^c \to \mathcal{T}^c$ is a Serre functor by (11). Hence (b) implies (a).

Conversely, assuming now that $\mathcal{C} = \mathcal{T}^c$ has Serre duality (S, σ). We claim that the twisting object $S(\mathbb{1})$ is our $\omega_\Bbbk$ (*cf.* Lemma 2). For $x, y \in \mathcal{T}^c$ we obtain a composite natural isomorphism

$$\mathcal{T}(y, S(x)) = \mathcal{T}^c(y, S(x)) \overset{\sigma}{\cong} \mathcal{T}^c(x,y)^\star \overset{(11)}{\cong} \mathcal{T}(y, x \otimes \omega_\Bbbk) \tag{12}$$

and therefore by setting $x := \mathbb{1}$ a natural isomorphism

$$\phi_y \colon \mathcal{T}(y, S(\mathbb{1})) \xrightarrow{\sim} \mathcal{T}(y, \omega_\Bbbk) \tag{13}$$

for all $y \in \mathcal{T}^c$. (We could be tempted to apply the Yoneda lemma at this point, but it would be premature as we still don't know whether $\omega_\Bbbk$ belongs to $\mathcal{T}^c$!) In particular, by choosing $y := S(\mathbb{1}) \in \mathcal{T}^c$ in (13) we find in $\mathcal{T}$ a morphism $\alpha := \phi_{S(\mathbb{1})}(\mathrm{id}_{S(\mathbb{1})})\colon S(\mathbb{1}) \to \omega_\Bbbk$. Consider now the induced natural transformation

$$\alpha_* \colon \mathcal{T}(-, S(\mathbb{1})) \longrightarrow \mathcal{T}(-, \omega_\Bbbk)$$

of cohomological functors $\mathcal{T}^{\mathrm{op}} \to \mathrm{Mod}(\Bbbk)$. By naturality, for every map $\beta\colon y \to S(\mathbb{1})$ with y compact we have a commutative square

$$\begin{array}{ccc} \mathcal{T}(S(\mathbb{1}), S(\mathbb{1})) & \xrightarrow[\simeq]{\phi_{S(\mathbb{1})}} & \mathcal{T}(S(\mathbb{1}), \omega_{\Bbbk}) \\ {\scriptstyle \beta^*}\downarrow & & \downarrow{\scriptstyle \beta^*} \\ \mathcal{T}(y, S(\mathbb{1})) & \xrightarrow[\simeq]{\phi_y} & \mathcal{T}(y, \omega_{\Bbbk}) \end{array}$$

showing that $\phi_y(\beta) = \phi_y(\beta^*(\mathrm{id})) = \beta^*(\alpha) = \alpha_*(\beta)$. Thus the restriction of α_* on $\mathcal{T}^c$ coincides with ϕ, and in particular it is an isomorphism. As $\mathcal{T}$ is compactly generated this implies that α_* is an isomorphism on all $\mathcal{T}$, hence by Yoneda the map α is invertible. It follows that the dualizing object $\omega_{\Bbbk} \cong S(\mathbb{1})$ is compact. But then, the isomorphism (12) takes place inside $\mathcal{T}^c$, hence by Yoneda it defines a natural isomorphism $(-) \otimes \omega_{\Bbbk} \cong S$ of functors $\mathcal{T}^c \to \mathcal{T}^c$.

For the true skeptic, let us still verify that $\omega_{\Bbbk}$ is tensor-invertible: as $(-) \otimes \omega_{\Bbbk}$ is an equivalence $\mathcal{T}^c \xrightarrow{\sim} \mathcal{T}^c$ it has a pseudo-inverse S^{-1}, from which it follows that $S^{-1}(\mathbb{1}) \otimes \omega_{\Bbbk} \cong S(S^{-1}(\mathbb{1})) \cong \mathbb{1}$. Thus (b) implies (a).

This concludes the proof of Proposition 7. $\square$

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