ERRATUM TO "AFFINE WEAKLY REGULAR TENSOR TRIANGULATED CATEGORIES"

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Many thanks to Eike Lau for raising this issue.

The proof of [DS16, Lemma 2.2] has a gap, and it is unclear whether its conclusion holds in the claimed generality. However, as we explain below, the lemma's conclusion *does* hold under the running hypotheses of the article, so that none of the other results are affected by this. In particular, all claims made in the introduction remain true as stated.

In details, Lemma 2.2 claims that the comparison map $\rho\colon \operatorname{Spc}\mathcal{K} \to \operatorname{Spec} R$ of [Bal10], between the Balmer spectrum of a tensor triangulated category $\mathcal{K}$ and the Zariski spectrum of its graded central ring $R = \operatorname{End}^*_{\mathcal{K}}(\mathbf{1})$, is a homeomorphism as soon as it is bijective. Since ρ is a spectral map in the sense of Hochster, the claim boils down to the equivalence

$$\mathcal{P} \subseteq \mathcal{Q} \;\Leftrightarrow\; \rho(\mathcal{Q}) \subseteq \rho(\mathcal{P}) \qquad (\text{for } \mathcal{P}, \mathcal{Q} \in \operatorname{Spc}\mathcal{K})$$

but, while $\Rightarrow$ is immediate, it is unclear why the implication $\Leftarrow$ should hold in general. Although we do not know any counterexamples, this leaves a gap in the proof.

Fortunately, supposing $\mathcal{K}$ satisfies properties (a) and (b) of [DS16, Theorem 1.1] as we do throughout the article, we can still obtain the same result. Indeed, without using Lemma 2.2 everything still works up an including the proof of Theorem 1.1 (p. 102), which now only tells us that ρ is bijective and continuous. The argument for Lemma 3.10 also still works verbatim, showing that $\rho(\operatorname{supp} X) = \operatorname{Supp}_R H^*X$ for all $X \in \mathcal{K}$. Since the Zariski support $\operatorname{Supp}_R H^*X$ is closed in $\operatorname{Spec} R$ and since the subsets of the form $\operatorname{supp} X$ form a closed basis for the topology of $\operatorname{Spc}\mathcal{K}$, this shows that the bijection ρ preserves closed subsets and thus is a homeomorphism.

Alternatively, the conclusion of Lemma 2.2 also holds by [Lau21, Corollary 2.5]. Indeed, it is easily checked that our hypotheses (a) and (b) ensure that $\mathcal{K}$ is rigid and End-finite (in the terminology of *loc. cit.*), indeed in (b) we do not even need the graded ring R to be regular, only noetherian.

References

[Bal10] Paul Balmer. Spectra, spectra, spectra – tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.

[DS16] Ivo Dell'Ambrogio and Donald Stanley. Affine weakly regular tensor triangulated categories. *Pac. J. Math.*, 285(1):93–109, 2016.

[Lau21] Eike Lau. The Balmer spectrum of certain Deligne-Mumford stacks. Preprint, 31 pages. `arXiv:2101.01446`, 2021.

Date: June 11, 2022.