dear Michael,

I have found an answer to Neeman's riddle. Perhaps you can find a shorter proof. Anyway, in case you haven't found your own yet, here is mine. Recall the situation: we are working in a "pre-triangulated category" which is not assumed to have biproducts, but only to have a zero object (which is also necessary to formulate axiom (TR0)), and to be enriched on abelian groups: the Hom sets are abelian groups and composition is additive in each variable.

If a, b are two arbitrary objects, we must find a biproduct for them, that is, an object c together with maps $i_a : a \to c$, $i_b : b \to c$, $p_a : c \to a$ and $p_b : c \to b$ such that

$$p_a i_a = 1_a, \quad p_b i_b = 1_b \quad \text{and} \quad i_a p_a + i_b p_b = 1_c.$$
 (1)

It is then a general fact that the category is additive; in particular (c, i_a, i_b) is a coproduct for a, b and (c, p_a, p_b) a product (see e.g. Mac Lane's book).

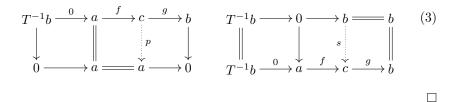
As we noticed last time, by axiom (TR1) there exists a distinguished triangle

$$T^{-1}b \xrightarrow{0} a \xrightarrow{f} c \xrightarrow{g} b.$$
 (2)

containing the zero map $T^{-1}b \to a$. I claim that the data $(c, i_a := f, p_b := g)$ can be completed to a biproduct for a and b. First, the two remaining maps are provided by:

Lemma 1. Let $T^{-1}b \xrightarrow{0} a \xrightarrow{f} c \xrightarrow{g} b$ be a distinguished triangle with vanishing connecting map. Then f is a split mono and g is a split epi, namely, there exist $p: c \rightarrow a$ with $pf = 1_a$ and $s: b \rightarrow c$ with $gs = 1_b$.

Proof. Use axioms (TR0), (TR2) and (TR3) to fill in the following morphisms of distinguished triangles:



Now it is tempting to set $p_a := p$ and $i_b := s$ as in the lemma, so that the first two equations in (1) are satisfied. But there's a catch: try as I may, I couldn't prove that (fp)(sg) = 0 (which is needed below), i.e., that fp and sg are *orthogonal* idempotents of c.

Fortunately, this is easily corrected. Namely set $p_a := p$ as above but

$$i_b := (1_c - fp)s : b \to c.$$

With these definitions we still have

$$p_a i_a = pf = 1_a$$

and

$$p_b i_b = g(1 - fp)s = gs - \underbrace{gf}_{=0} ps = gs = 1_b,$$

but now we have gained orthogonality:

$$(i_b p_b)(i_a p_a) = (1 - fp)sg(fp) = (1 - fp)s\underbrace{gf}_{=0} p = 0$$
 (4)

$$(i_a p_a)(i_b p_b) = (fp)(1-fp)sg = fpsg - f \underbrace{pf}_{-1} psg = 0.$$
 (5)

In order to prove the third crucial equation, I also need the following two standard results (neither uses the existence of biproducts):

Lemma 2. In every distinguished triangle $T^{-1}z \to x \to y \to z$, the map $x \to y$ is a weak kernel of $y \to z$ and $y \to z$ is a weak cohernel of $x \to y$ (that is, the property of a (co)kernel is satisfied up to the uniqueness of the induced factorization, which may fail).

Proof. This must be somewhere in Neeman's book. Anyway, it is an easy application of the axioms (TR1), (TR2) and (TR3), similarly to lemma 1. \Box

Lemma 3. Consider an automorphism of a distinguished triangle:

$$\begin{array}{ccc} T^{-1}z \longrightarrow x \longrightarrow y \longrightarrow z \\ T^{-1}h_3 \downarrow & h_1 \downarrow & h_2 \downarrow & h_3 \downarrow \\ T^{-1}z \longrightarrow x \longrightarrow y \longrightarrow z \end{array}$$
(6)

If two of the three components (h_1, h_2, h_3) are zero, then the third is nilpotent with zero square.

Proof. See e.g. my thesis, lemma 1.1.12 (it uses lemma 2).

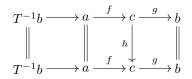
Now consider the endomorphism

$$h := i_a p_a + i_b p_b = fp + (1 - fp)sg : c \to c.$$

Since

$$hf = fpf + (1-fp)sgf = f \quad \text{ and } \quad gh = gfp + g(1-fp)sg = g$$

(using that $gs = 1_b$, $pf = 1_a$ and gf = 0) we have the following automorphism of the distinguished triangle (2):



Now subtract this automorphism from the identical one, and apply lemma 3 to conclude that $(1_c - h)^2 = 0$. Finally, we compute

$$0 = (1_c - h)^2 = (1_c - i_a p_a - i_b p_b)^2$$

= $1_c - i_a p_a - i_b p_b$

because $i_a p_a$ and $i_b p_b$ are orthogonal idempotent endomorphisms of c. Hence $i_a p_a + i_b p_b = 1_c$, completing the proof.

tschüss, Ivo