dear Michael,

I have found an answer to Neeman's riddle. Perhaps you can find a shorter proof. Anyway, in case you haven't found your own yet, here is mine. Recall the situation: we are working in a "pre-triangulated category" which is not assumed to have biproducts, but only to have a zero object (which is also necessary to formulate axiom (TR0)), and to be enriched on abelian groups: the Hom sets are abelian groups and composition is additive in each variable.

If a, b are two arbitrary objects, we must find a biproduct for them, that is, an object c together with maps $i_a : a \to c$, $i_b : b \to c$, $p_a : c \to a$ and $p_b : c \to b$ such that

$$p_a i_a = 1_a, \quad p_b i_b = 1_b \quad \text{and} \quad i_a p_a + i_b p_b = 1_c. \tag{1}$$

It is then a general fact that the category is additive; in particular (c, i_a, i_b) is a coproduct for a, b and (c, p_a, p_b) a product (see e.g. Mac Lane's book).

As we noticed last time, by axiom (TR1) there exists a distinguished triangle

$$T^{-1}b \xrightarrow{\;0\;} a \xrightarrow{\;f\;} c \xrightarrow{\;g\;} b. \tag{2}$$

containing the zero map $T^{-1}b \to a$. I claim that the data $(c, i_a := f, p_b := g)$ can be completed to a biproduct for a and b. First, the two remaining maps are provided by:

Lemma 1. *Let $T^{-1}b \xrightarrow{0} a \xrightarrow{f} c \xrightarrow{g} b$ be a distinguished triangle with vanishing connecting map. Then f is a split mono and g is a split epi, namely, there exist $p : c \to a$ with $pf = 1_a$ and $s : b \to c$ with $gs = 1_b$.*

Proof. Use axioms (TR0), (TR2) and (TR3) to fill in the following morphisms of distinguished triangles:

$$\begin{array}{ccccccc} T^{-1}b & \xrightarrow{0} & a & \xrightarrow{f} & c & \xrightarrow{g} & b \\ \downarrow & & \| & & \vdots\, p & & \downarrow \\ 0 & \longrightarrow & a & = & a & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccc} T^{-1}b & \longrightarrow & 0 & \longrightarrow & b & = & b \\ \| & & \downarrow & & \vdots\, s & & \| \\ T^{-1}b & \xrightarrow{0} & a & \xrightarrow{f} & c & \xrightarrow{g} & b \end{array} \tag{3}$$

□

Now it is tempting to set $p_a := p$ and $i_b := s$ as in the lemma, so that the first two equations in (1) are satisfied. But there's a catch: try as I may, I couldn't prove that $(fp)(sg) = 0$ (which is needed below), i.e., that fp and sg are *orthogonal* idempotents of c.

Fortunately, this is easily corrected. Namely set $p_a := p$ as above but

$$i_b := (1_c - fp)s : b \to c.$$

With these definitions we still have

$$p_a i_a = pf = 1_a$$

and

$$p_b i_b = g(1 - fp)s = gs - \underbrace{gf}_{=0} ps = gs = 1_b,$$

but now we have gained orthogonality:

$$(i_b p_b)(i_a p_a) = (1 - fp)sg(fp) = (1 - fp)s \underbrace{gf}_{=0} p = 0 \tag{4}$$

$$(i_a p_a)(i_b p_b) = (fp)(1 - fp)sg = fpsg - f \underbrace{pf}_{=1} psg = 0. \tag{5}$$

In order to prove the third crucial equation, I also need the following two standard results (neither uses the existence of biproducts):

Lemma 2. *In every distinguished triangle $T^{-1}z \to x \to y \to z$, the map $x \to y$ is a weak kernel of $y \to z$ and $y \to z$ is a weak cokernel of $x \to y$ (that is, the property of a (co)kernel is satisfied up to the uniqueness of the induced factorization, which may fail).*

Proof. This must be somewhere in Neeman's book. Anyway, it is an easy application of the axioms (TR1), (TR2) and (TR3), similarly to lemma 1. ☐

Lemma 3. *Consider an automorphism of a distinguished triangle:*

$$\begin{array}{ccccccc} T^{-1}z & \longrightarrow & x & \longrightarrow & y & \longrightarrow & z \\ \downarrow{\scriptstyle T^{-1}h_3} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} \\ T^{-1}z & \longrightarrow & x & \longrightarrow & y & \longrightarrow & z \end{array} \tag{6}$$

If two of the three components (h_1, h_2, h_3) are zero, then the third is nilpotent with zero square.

Proof. See e.g. my thesis, lemma 1.1.12 (it uses lemma 2). ☐

Now consider the endomorphism

$$h := i_a p_a + i_b p_b = fp + (1 - fp)sg : c \to c.$$

Since

$$hf = fpf + (1 - fp)sgf = f \quad \text{and} \quad gh = gfp + g(1 - fp)sg = g$$

(using that $gs = 1_b$, $pf = 1_a$ and $gf = 0$) we have the following automorphism of the distinguished triangle (2):

$$\begin{array}{ccccccc} T^{-1}b & \longrightarrow & a & \xrightarrow{f} & c & \xrightarrow{g} & b \\ \| & & \| & & \downarrow{\scriptstyle h} & & \| \\ T^{-1}b & \longrightarrow & a & \xrightarrow{f} & c & \xrightarrow{g} & b \end{array}$$

Now subtract this automorphism from the identical one, and apply lemma 3 to conclude that $(1_c - h)^2 = 0$. Finally, we compute

$$\begin{aligned} 0 = (1_c - h)^2 &= (1_c - i_a p_a - i_b p_b)^2 \\ &= 1_c - i_a p_a - i_b p_b \end{aligned}$$

because $i_a p_a$ and $i_b p_b$ are orthogonal idempotent endomorphisms of c. Hence $i_a p_a + i_b p_b = 1_c$, completing the proof.

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