The Spanier-Whitehead category is always triangulated

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1 Introduction

Our goal is to prove the following theorem:

Theorem 1.1. *The Spanier-Whitehead category* $SW(\mathrm{Ho}\mathcal{M}, \Sigma)$ *of the homotopy category of a pointed model category* $\mathcal{M}$ *obtained by inverting its suspension endofunctor* Σ *is always a (classical) triangulated category.*

We derive the triangulation on $SW(\mathrm{Ho}\mathcal{M}, \Sigma)$ in a natural way from the collection of cofiber sequences in $\mathrm{Ho}\mathcal{M}$ (see Def. (6.3)). In order to prove the theorem, we shall first isolate a small set of properties (denoted **(H0),...,(H6)**, see Section 3) of the homotopy category $\mathrm{Ho}\mathcal{M}$ which suffice to prove Theorem (1.1) in conditional form. That is, given a category $\mathcal{H}$ and a functor $\Sigma : \mathcal{H} \to \mathcal{H}$ satisfying conditions **(H0)-(H6)**, we shall prove that $SW(\mathcal{H}, \Sigma)$ is triangulated (Thm. (6.5)).

This we do in Part I. In Part II we shall refer to the literature, basically just Daniel Quillen's seminal work on model categories *Homotopical algebra* [4] and Mark Hovey's hi-tech monography *Model categories* [2], in order to prove that the model category of an arbitrary pointed model category does indeed satisfy our conditions **(H0)-(H6)**. As recalled at the beginning of Section 8, the assumption that the model category be pointed can always be satisfied by a very easy construction; thus the above theorem provides a simple way of deriving a triangulated category out of *any* model category. For this second part we need some of the rather technical language of model category theory, so we have provided a section (Section 7) in which we recall the basic notions that will be needed. This section is really a sorry excuse for an introduction to algebraic homotopy theory: the neophyte is warmly encouraged to read the introductory paper by Dwyer and Spalinski [1].

The reader who wants to know how triangulated categories and model categories arose is invited to read the enlightening and pleasant historical paper [7] by Charles A. Weibel.

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Part I
General formalism

2 (Co)groups and (co)actions in a category

In this section we make some preliminary definitions which shall be needed later. The confident reader may surely skip this section and come back if need should arise.

Definition 2.1. Let $\mathcal{C}$ be a category with finite products and a final object 1. A *group object* (or simply *group*) in $\mathcal{C}$ is a quadruple (G, m, e, i) consisting of an object G and three morphisms

$$m : G \times G \longrightarrow G \qquad \text{(the multiplication)}$$

$$e : 1 \longrightarrow G \qquad \text{(the unit)}$$

$$i : G \longrightarrow G \qquad \text{(the inverse)}$$

such that the three diagrams below commute.

$$\begin{tikzcd}
G \times (G \times G) \arrow[d, "1\times m"'] & \simeq & (G \times G) \times G \arrow[r, "m\times 1"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[rrr, "m"] & & & G
\end{tikzcd}$$

$$\begin{tikzcd}
1 \times G \arrow[r, "e\times 1"] \arrow[dr, "\simeq"'] & G \times G \arrow[d, "m"] & G \times 1 \arrow[l, "1\times e"'] \arrow[dl, "\simeq"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "(1,i)"] \arrow[d, "!"'] & G \times G \arrow[d, "m"] \\
1 \arrow[r, "e"] & G
\end{tikzcd}$$

These are the associativity axiom, the unit axiom and the inverse axiom respectively (writing them down as equations gives the usual group axioms). The three isomorphisms in the diagrams are the canonical ones. One should note here that the unit e and the inverse i of a group are completely determined by the axioms, once the multiplication m is known (this is incidentally proved in Remark (3.1)); it is perhaps better though to give them explicitly. The same remark holds for cogroups, defined below.

A group is *commutative* or *abelian* if this diagram also commutes:

$$\begin{tikzcd}
G \times G \arrow[r, "T"] \arrow[dr, "m"'] & G \times G \arrow[d, "m"] \\
& G
\end{tikzcd}$$

where T is the “interchange map” with $p_1 T = p_2$ and $p_2 T = p_1$, and p_1, p_2 being the canonical maps of the product $G \times G$. Choosing $\mathcal{C} = \mathcal{S}\text{et}$, these definition is equivalent to the usual definition of an (abelian) group; the same is also true for the definition of morphism of groups and for the definition of action below.

A *morphism of groups* is a map $f : G \to G'$ between groups (G, m, e, i), (G', m', e', i') which respects the structure maps, i.e., such that the three diagrams below commute.

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\ G \times G' \arrow[r, "m'"] & G' \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[dr, "e'"'] & G \arrow[d, "f"] \\ & G' \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "i"] \arrow[d, "f"'] & G \arrow[d, "f"] \\ G' \arrow[r, "i'"] & G' \end{tikzcd}$$

One should note here that the commutativity of the second and third diagrams follows from that of the first one. But, again, we allow some redundancy in our definitions.

Definition 2.2. Given a group object $G = (G, m, e, i)$ and some other object X in $\mathcal{C}$, a *(right) action* of G on X is a map $\nu : X \times G \to X$ such that this diagram commutes:

$$\begin{tikzcd} (X \times G) \times G \arrow[r, phantom, "\simeq"] \arrow[d, "\nu\times 1"] & X \times (G \times G) \arrow[r, "1\times m"] & X \times G \arrow[d, "\nu"'] & X \times 1 \arrow[l, "1\times e"'] \arrow[dl, "\simeq"] \\ X \times G \arrow[rr, "\nu"] & & X & \end{tikzcd}$$

(As equations, the commuting square on the left and the triangle on the right are recognizable as the usual axioms for the action of a group.)

In a category $\mathcal{C}$ with finite coproducts and an initial object 0, one has the following dual definitions.

Definition 2.3. A *cogroup object* (or simply *cogroup*) in $\mathcal{C}$ is a quadruple (A, m, e, i) with $m : A \to A \vee A$, $e : A \to 0$ and $i : A \to A$ making the following diagrams commute.

$$\begin{tikzcd} A \vee (A \vee A) \arrow[r, phantom, "\simeq"] & (A \vee A) \vee A & A \vee A \arrow[l, "m\vee 1"'] \\ A \vee A \arrow[u, "1\vee m"] & & A \arrow[ll, "m"] \arrow[u, "m"'] \end{tikzcd}$$

$$\begin{tikzcd} 0 \vee A & A \vee A \arrow[l, "e\vee 1"'] \arrow[r, "1\vee e"] & A \vee 0 \\ & A \arrow[ul, "\simeq"] \arrow[u, "m"] \arrow[ur, "\simeq"'] & \end{tikzcd} \qquad \begin{tikzcd} A & A \vee A \arrow[l, "(1,i)"'] \\ 0 \arrow[u, "!"] & A \arrow[l, "e"] \arrow[u, "m"'] \end{tikzcd} \tag{1}$$

A *morphism of cogroups* $(A, m, e, i), (A', m', e', i')$ is then a map $f : A \to A'$ such that the following three diagrams commute.

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f"'] & A \vee A \arrow[d, "f\vee f"] \\ A' \arrow[r, "m'"] & A' \vee A' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & 0 \\ A' \arrow[ur, "e'"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A' \arrow[r, "i'"] & A' \end{tikzcd}$$

Definition 2.4. Given a functor $F : \mathcal{C} \to \mathcal{C}$, we shall say that FX *is a natural cogroup object* if

(i) for all objects X the object FX is a cogroup object, and

(ii) for all maps $f : X \to Y$ the map Ff is a morphism of cogroups.

One then has the obvious dual definition of a *natural* group object.

Definition 2.5. A *(right) coaction* of the cogroup object (A, m, e, i) on X is a map $\nu : A \to X \vee A$ such that

$$\begin{tikzcd}
(X \vee A) \vee A \arrow[r, phantom, "\simeq"] & X \vee (A \vee A) & X \vee A \arrow[l, "1\vee m"'] \arrow[r, "1\vee e"] & X \vee 0 \\
X \vee A \arrow[u, "\nu\vee 1"] & & X \arrow[ll, "\nu"] \arrow[u, "\nu"'] \arrow[ur, "\simeq"']
\end{tikzcd} \tag{2}$$

commutes.

We shall also need the following definition. Given a coaction $\nu : X \to X \vee A$ of the cogroup (A, m, e, i) on X, a coaction $\nu' : X' \to X' \vee A'$ of (A', m', e', i') on X, and a morphism of cogroups $f : A \to A'$, one defines a map $x : X \to X'$ to be *f-equivariant* if the square below commutes. There is obviously also a dual definition for actions.

$$\begin{tikzcd}
X \arrow[r, "\nu"] \arrow[d, "x"'] & X \vee A \arrow[d, "x\vee f"] \\
X' \arrow[r, "\nu'"] & X' \vee A'
\end{tikzcd}$$

Equivalently, one can look at all the above-defined concepts also on another "plane", as the following proposition suggests.

Proposition 2.6. *Let $\mathcal{C}$ have finite products. Let G, X and A be objects of $\mathcal{C}$. Then one has the following bijections.*

(i) The group structures (G, m, e, i) on G in $\mathcal{C}$ are in one-to-one correspondence to the group structures $(\mathcal{C}(-, G), \overline{m}, \overline{e}, \overline{i})$ on $\mathcal{C}(-, G)$ in the functor category $\mathcal{S}\mathrm{et}^{\mathcal{C}^{op}}$.

(ii) The actions $\nu : X \times G \to X$ of the group (G, m, e, i) on X in $\mathcal{C}$ are in one-to-one correspondence with the actions $\nu : \mathcal{C}(-, X) \times \mathcal{C}(-, G) \to \mathcal{C}(-, X)$ in $\mathcal{S}\mathrm{et}^{\mathcal{C}^{op}}$ of the group $(\mathcal{C}(-, G), \overline{m}, \overline{e}, \overline{i})$ on $\mathcal{C}(-, X)$ in $\mathcal{S}\mathrm{et}^{\mathcal{C}^{op}}$.

If $\mathcal{C}$ is a category with finite coproducts, the dual statements are true:

(iii) The cogroup structures (A, m, e, i) on A in $\mathcal{C}$ are in one-to-one correspondence to the group (!) structures $(\mathcal{C}(A, -), \overline{m}, \overline{e}, \overline{i})$ on $\mathcal{C}(A, -)$ in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$.

(iv) The coactions $\nu : X \to X \vee A$ of the cogroup (A, m, e, i) in $\mathcal{C}$ are in one-to-one correspondence with the actions (!) $\nu : \mathcal{C}(X, -) \times \mathcal{C}(A, -) \to \mathcal{C}(X, -)$ of the group (!) $(\mathcal{C}(A, -), \overline{m}, \overline{e}, \overline{i})$ in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$.

Furthermore, a group or a cogroup is abelian if and only if its "brother" in the functor category is so. Also the concept of equivariance can be shifted between "planes".

Proof: The proof is an exercise application of the Yoneda lemma, see [3] Prop. 1 p. 75 for a statement and partial proof of (i). Instead of a detailed proof, we give the bijections that will concern us most, namely those in (iii) and (iv). Given a cogroup (A, m, e, i) in $\mathcal{C}$, the comultiplication $m : A \to A \vee A$ determines a unique natural transformation

$$\overline{m} : \ \mathcal{C}(A,-) \times \mathcal{C}(A,-) \xrightarrow{\simeq} \mathcal{C}(A \vee A,-) \xrightarrow{m^*} \mathcal{C}(A,-)$$

where the natural isomorphism on the left is the one given by the definition of the coproduct $A \vee A$, and $m^* = \mathcal{C}(m,-)$ is precomposition with m. On the other hand, by the Yoneda lemma, for any natural transformation $\mu : \mathcal{C}(A \vee A,-) \to \mathcal{C}(A,-)$ there is a unique $m : A \to A \vee A$ such that $\mu = m^*$. Similarly, i and e correspond to the natural maps $\overline{i} = i^* : \mathcal{C}(A,-) \to \mathcal{C}(A,-)$ and $\overline{e} = e^* : \mathcal{C}(0,-) \to \mathcal{C}(A,-)$ (Notice that the functor $\mathcal{C}(0,-)$ is final in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$ as required, because for all Y the set $\mathcal{C}(0,Y)$ contains exactly one element). Then one checks that $(\mathcal{C}(A,-), \overline{m}, \overline{e}, \overline{i})$ makes the diagrams for a group commute in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$ if and only if (A, m, e, i) makes the diagrams for a cogroup commute in $\mathcal{C}$. A coaction $\nu : A \to X \vee A$ gives rise to the action

$$\overline{\nu} : \ \mathcal{C}(X,-) \times \mathcal{C}(A,-) \xrightarrow{\simeq} \mathcal{C}(X \vee A,-) \xrightarrow{\nu^*} \mathcal{C}(X,-) \ .$$

□

Remark 2.7. It is easy to check that the two following statements are equivalent:
(a) $(\mathcal{C}(A,-), \overline{m}, \overline{e}, \overline{i})$, as above, is a group in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$.
(b) For each $X \in \mathrm{Ob}\mathcal{C}$, the set $\mathcal{C}(C,X)$ is a group in the usual sense, and for each $f : X \to Y$ in $\mathcal{C}$, $f_* = \mathcal{C}(A, f)$ (composition with f) is a group homomorphism.

Therefore, instead of saying that $(\mathcal{C}(A,-), \overline{m}, \overline{e}, \overline{i})$ is a group in $\mathcal{S}\mathrm{et}^{\mathcal{C}}$, one can equivalently say that the functor $\mathcal{C}(A,-) : \mathcal{C} \to \mathcal{S}\mathrm{et}$ takes values into the category $\mathcal{G}\mathrm{rp}$ of (usual) groups and homomorphisms (or that it lifts to $\mathcal{G}\mathrm{rp}$ along the forgetful functor $\mathcal{G}\mathrm{rp} \to \mathcal{S}\mathrm{et}$). A similar remark applies to groups $\mathcal{C}(-,G)$ in $\mathcal{S}\mathrm{et}^{\mathcal{C}^{op}}$.

3 The set of assumptions

Let $(\mathcal{H}, \Sigma)$ be a pair consisting of a category $\mathcal{H}$ and a functor $\Sigma : \mathcal{H} \to \mathcal{H}$, which we shall call *suspension*. This choice of notation, of course, wants to be reminiscent of the $\mathcal{H}$omotopy category of a pointed model category and its suspension Σ. In this section we carefully isolate all the properties that $(\mathcal{H}, \Sigma)$ must enjoy, in order for $SW(\mathcal{H}, \Sigma)$ to be a triangulated category (Thm. (6.5)).

(H0) (a) $\mathcal{H}$ admits finite coproducts and is a pointed category. Denote the zero object (the final and initial object) by $*$. As usual one defines the zero maps $0 : X \to * \to Y$ for all objects X, Y. Moreover, Σ preserves the coproducts and the zero object.

(b) $\Sigma^n X$ is a natural cogroup object for all $n \geq 1$, abelian as soon as $n \geq 2$. (See Def. (2.4).)

(c) Σ preserves the structure maps of all cogroups ΣX (that is, the three structure maps of the cogroup $\Sigma^2 X$ are the images under Σ of the structure maps of ΣX).

(d) There is a collection of diagrams in $\mathcal{H}$ called *cofiber sequences*. Cofiber sequences must satisfy the following seven conditions.

(H1) Cofiber sequences are in particular diagrams of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad C \xrightarrow{\nu} C \vee \Sigma A \tag{3}$$

where ν is a right coaction of the cogroup ΣA on C. To every cofiber sequence there belongs a *boundary map* ∂, which is defined to be the following composition:

$$\partial : \; C \xrightarrow{\nu} C \vee \Sigma A \xrightarrow{(0,1)} \Sigma A \,.$$

Define a *morphism of cofiber sequences* to be a commuting diagram as the one below. Call it an *isomorphism* if the vertical arrows are all isomorphisms.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"'] & C \arrow[r, "\nu"] \arrow[d, "c"'] & C \vee \Sigma A \arrow[d, "c \vee \Sigma a"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & C' \arrow[r, "\nu'"] & C' \vee \Sigma A' \end{tikzcd}$$

The meaning of requiring the square on the right to commute is that we want the third map c to be Σa-equivariant. (Note that Σa is a morphism of cogroups, because of **(H0)(b)**, cf. Def. (2.4).)

(H2) For any object X, the diagram $* \longrightarrow X \xrightarrow{1_X} X$ is a cofiber sequence, with the only possible coaction $X \to X \vee \Sigma *$. (Condition **(H0)(a)** says that $\Sigma * = *$, and then it is easy to see that the only possible such coaction is the canonical isomorphism.)

(H3) Every map $f : A \to B$ is part of some cofiber sequence (3).

(H4) Cofiber sequences can be shifted to the right. More precisely, given a cofiber sequence (3), the following diagram is also a cofiber sequence:

$$B \xrightarrow{g} C \xrightarrow{\partial} \Sigma A \qquad \Sigma A \xrightarrow{\nu_f} \Sigma A \vee \Sigma B$$

where ∂ is the boundary map of (3) and the coaction ν_f is given by the composition

$$\nu_f : \; \Sigma A \xrightarrow{m} \Sigma A \vee \Sigma A \xrightarrow{1 \vee (i \circ \Sigma f)} \Sigma A \vee \Sigma B$$

(here m is the comultiplication of the cogroup ΣA and i is the coinverse of the cogroup ΣB).

(H5) "Fill-in maps exist". That is, given a commutative (solid) diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, dotted, "c"] & C \arrow[r, "\nu"] \arrow[d, dotted, "c"'] & C \vee \Sigma A \arrow[d, dotted, "c \vee \Sigma a"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & C' \arrow[r, "\nu'"'] & C' \vee \Sigma A' \end{tikzcd} \tag{4}$$

where the two rows are cofiber sequences, there exists a (nonunique) map $c : C \to C'$ which completes the above diagram to a morphism of cofiber sequences (In other words, c is such that $cg = g'b$ and it is Σa-equivariant).

(H6) Given a composition of maps $h := g \circ f : X \xrightarrow{f} Y \xrightarrow{g} Z$, there exist cofiber sequences containing f, g and h

$$X \xrightarrow{f} Y \xrightarrow{f'} U \qquad U \longrightarrow U \vee \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{g'} W \qquad W \xrightarrow{\nu_g} W \vee \Sigma Y$$

$$X \xrightarrow{h} Z \xrightarrow{h'} V \qquad V \longrightarrow V \vee \Sigma X$$

and also a cofiber sequence

$$U \xrightarrow{s} V \xrightarrow{s'} W \qquad W \xrightarrow{\nu_s} W \vee \Sigma U$$

which enjoy the following good properties. The coaction ν_s is the composite

$$\nu_s : W \xrightarrow{\nu_g} W \vee \Sigma Y \xrightarrow{1 \vee \Sigma f'} W \vee \Sigma U \ .$$

Moreover s is $\Sigma 1_X$-equivariant, s' is Σf-equivariant and

$$s'h' = g', \qquad h'g = sf'.$$

Remark 3.1. In the presence of **(H0)(a,b)**, condition **(H0)(c)** is equivalent to the following.

(H0)(c') For all $n \geq 1$ and objects X, Y, the map of sets $\Sigma : \mathcal{H}(\Sigma^n X, \Sigma^n Y) \to \mathcal{H}(\Sigma^{n+1} X, \Sigma^{n+1} Y)$ is a group homomorphism.

Proof: Write $A := \Sigma^n X$ and $B := \Sigma^n Y$. In the light of Proposition (2.6) and Remark (2.7), the comultiplications $m_{\Sigma A} : \Sigma A \to \Sigma A \vee \Sigma A$ and $m_{\Sigma^2 A} : \Sigma^2 A \to \Sigma^2 A \vee \Sigma^2 A$ provided by **(H0)(b)** give rise to (usual) group operations $\overline{m_{\Sigma A}}$ and $\overline{m_{\Sigma^2 A}}$, depicted as the two rows in the following diagram.

$$\begin{tikzcd}
\mathcal{H}(\Sigma A, B) \times \mathcal{H}(\Sigma A, B) \arrow[r, "\simeq"] \arrow[d, "\Sigma \times \Sigma"'] & \mathcal{H}(\Sigma A \vee \Sigma A, B) \arrow[r, "(m_{\Sigma A})^*"] \arrow[d, "\Sigma"'] & \mathcal{H}(\Sigma A, B) \arrow[d, "\Sigma"] \\
\mathcal{H}(\Sigma^2 A, \Sigma B) \times \mathcal{H}(\Sigma^2 A, \Sigma B) \arrow[r, "\simeq"] & \mathcal{H}(\Sigma^2 A \vee \Sigma^2 A, \Sigma B) \arrow[r, "(m_{\Sigma^2 A})^*"] & \mathcal{H}(\Sigma^2 A, \Sigma B)
\end{tikzcd}$$

The square on the left commutes because Σ preserves coproducts by **(b)**. If **(c)** holds, $\Sigma(m_{\Sigma A}) = m_{\Sigma^2 A}$ and the right square commutes also (because Σ is a functor), and the commutativity of the outer square is **(c')**. Conversely, if **(c')** holds the outer square commutes for all B, which implies that the right square commutes, i.e. for all B and $f \in \mathcal{H}(\Sigma A \vee \Sigma A, B)$ we have $\Sigma(f \circ m_{\Sigma A}) = \Sigma f \circ m_{\Sigma^2 A}$, hence by choosing $B = \Sigma A \vee \Sigma A$ and $f = 1_{\Sigma A \vee \Sigma A}$ we obtain $\Sigma(m_{\Sigma A}) = m_{\Sigma^2 A}$. Then to prove that Σ preserves also the counit and coinvers maps of the cogroups ΣX, it suffices to notice that these are always uniquely determined by the comultiplication. In fact, given a cogroup (G, m, e, i), this corresponds to a usual group $\mathcal{H}(G, B)$ natural in B with inverse and unit natural maps $\bar{i}$, $\bar{e}$. It is trivial to show that the inverse and unit of a usual group are determined by the operation, hence for all B the morphisms $\bar{i}$ and $\bar{e}$ are uniquely determined, hence they are also as natural transformations. The Yoneda Lemma shows the unicity of the maps i and e by which they were induced. □

Remark 3.2. In **(H5)**, the fact that c is Σa-equivariant (i.e. that in (4) the square on the right commutes) implies immediately that the following square commutes:

$$\begin{tikzcd} C \arrow[r, "\partial"] \arrow[d, "c"'] & \Sigma A \arrow[d, "\Sigma a"] \\ C' \arrow[r, "\partial'"] & \Sigma A' \end{tikzcd}$$

4 Inverting an endofunctor

We now want to define and study the category obtained from $\mathcal{H}$ by formally inverting the suspension Σ. This construction though is so simple that it applies to *any* category equipped with *any* endofunctor. Accordingly, throughout this section we shall investigate the Spanier-Whitehead category in its full generality.

4.1 The Spanier-Whitehead category

Definition 4.1. Let $\mathcal{C}$ be any category equipped with an endofunctor $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$. The *Spanier-Whitehead category $SW = SW(\mathcal{C}, \Sigma)$ (obtained by inverting Σ)* consists of the following data.

- The objects of SW are pairs (X, i), where X is an object of $\mathcal{C}$ and $i \in \mathbb{Z}$ is an integer.
- Given two objects (X, i) and (Y, j), their hom set $SW((X, i), (Y, j))$ is defined by the following colimit (of sets):

$$\operatorname*{colim}_{n \geq -i, -j} \left(\cdots \xrightarrow{\Sigma} \mathcal{C}(\Sigma^{n+i}X, \Sigma^{n+j}Y) \xrightarrow{\Sigma} \mathcal{C}(\Sigma^{n+1+i}X, \Sigma^{n+1+j}Y) \xrightarrow{\Sigma} \cdots \right)$$

  That is, a morphism $\alpha : (X, i) \longrightarrow (Y, j)$ is an equivalence class $[n, f]$ of some map $f : \Sigma^{n+i}X \longrightarrow \Sigma^{n+j}Y$ in $\mathcal{C}$ for some $n \geq -i, -j$, with respect to the equivalence relation generated by:

$$(n, f) \sim (n + l, \Sigma^l f), \quad \text{for all} \quad l \geq 0.$$

  Thus, $(n, f) \sim (m, g)$ if and only if there is a (big enough) N such that $\Sigma^{N-n} f = \Sigma^{N-m} g$.
- Given two composable arrows $\alpha : (X, i) \longrightarrow (Y, j)$ and $\beta : (Y, j) \longrightarrow (Z, k)$, where $\alpha = [n, f]$ and $\beta = [m, g]$, define their composition to be $\beta \circ \alpha := [n + m, \Sigma^n g \circ \Sigma^m f]$.

- For every object (X,i) in SW, define its identity map to be $1_{(X,i)} := [-i, 1_X]$.

One checks easily that these data define a category.

There is a *canonical functor* $\iota : \mathcal{C} \longrightarrow SW(\mathcal{C},\Sigma)$ which comes with the category and is defined by $\iota(X) := (X,0)$ and $\iota(f) := [0,f]$. We can also recover Σ in the new category as

$$\Sigma' : SW(\mathcal{C},\Sigma) \longrightarrow SW(\mathcal{C},\Sigma)$$

$$\Sigma'(X,i) := (X,i+1), \quad \Sigma'[n,f] := [n,\Sigma f] = [n-1,f]\ .$$

This functor is an automorphism of $SW(\mathcal{C},\Sigma)$ having the obvious inverse

$$\Sigma'^{-1}(X,i) = (X,i-1), \quad \Sigma'^{-1}[n,f] = [n+1,f]\ .$$

Since $1_{\Sigma X} \in \mathcal{C}(\Sigma X, \Sigma X) = \mathcal{C}(\Sigma^{0+1}X, \Sigma^{0+0}\Sigma X)$, the identity map $1_{\Sigma X}$ represents also the class of an isomorphism $[0,1_{\Sigma X}]$ in SW from $(X,1)$ to $(\Sigma X, 0)$. Because the diagram below trivially commutes for all $f : X \to Y$ in $\mathcal{C}$, this is a natural isomorphism $[0,1_{\Sigma X}] : \Sigma' \circ \iota \simeq \iota \circ \Sigma$.

$$\begin{tikzcd}
(X,1) \arrow[r, "{[0,\Sigma f]}"] \arrow[d, "{[0,1_{\Sigma X}]}"'] & (Y,1) \arrow[d, "{[0,1_{\Sigma Y}]}"] \\
(\Sigma X,0) \arrow[r, "{[0,\Sigma f]}"] & (\Sigma Y,0)
\end{tikzcd}$$

In the following, especially in the calculations of Section 6, we will often write (and think) $\Sigma\iota = \iota\Sigma$ for the above isomorphism $\Sigma'\iota \simeq \iota\Sigma$. In particular, we will drop the $'$ and denote the new functor Σ' also by Σ.

Of course, the canonical functor ι is in general not an equivalence, nor is it in general full, faithful or essentially surjective. Notice though that it is "essentially essentially surjective", that is to say, every object of SW is isomorphic to the canonical image ιX of some X in $\mathcal{C}$, but only up to some iteration of Σ or Σ^{-1}.

On many occasions we shall want to study a given commutative diagram in SW by tracing it back to some commutative diagram in $\mathcal{C}$. This can be done by applying Σ enough times. For example, if we have a commutative diagram

$$\begin{tikzcd}
(X,i) \arrow[rr, "{[l,h]}"] \arrow[dr, "{[n,f]}"'] & & (Z,k) \\
& (Y,j) \arrow[ur, "{[m,g]}"'] &
\end{tikzcd}$$

in SW, then for $N \geq n,m,l$ (which by definition (4.1) implies also $N \geq -i,-j,-k$) we get the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
\Sigma^{N+i}X \arrow[rr, "\Sigma^{N-l}h"] \arrow[dr, "\Sigma^{N-n}f"'] & & \Sigma^{N+k}Z \\
& \Sigma^{N+j}Y \arrow[ur, "\Sigma^{N-m}g"'] &
\end{tikzcd}$$

If we take an even bigger N, then this diagram commutes. (The existence of this bigger N means exactly that the equation $[m,g][n,f]=[l,h]$ holds.) Clearly, this same operation is possible with any finite diagram in SW.

The Spanier-Whitehead category is characterized up to isomorphism by the following universal property. An immediate consequence of this is that whenever $\Sigma:\mathcal{C}\to\mathcal{C}$ is already invertible, then $SW(\mathcal{C},\Sigma)$ and $\mathcal{C}$ are isomorphic categories.

Proposition 4.2 (Universal property of $(SW(\mathcal{C},\Sigma),\iota,\Sigma')$)**.**

(i) *$\Sigma' : SW \to SW$ is an isomorphism, and there is a natural isomorphism $\Sigma'\iota \simeq \iota\Sigma$.*

(ii) *For any such triple $(\mathcal{T},\tau,S)$, i.e. a category $\mathcal{T}$ and functors*

$$\mathcal{C} \xrightarrow{\tau} \mathcal{T} \xrightarrow{S} \mathcal{T}$$

such that S is an isomorphism and such that there is a natural isomorphism $S\tau \simeq \tau\Sigma$, there is a unique functor $\bar{\tau} : SW(\mathcal{C},\Sigma) \to \mathcal{T}$ such that $S\bar{\tau} = \bar{\tau}\Sigma'$.

In a diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\tau"] \arrow[dr, "\iota"'] \arrow[dd, "\Sigma"'] & & \mathcal{T} \arrow[dd, "S"] \\
& SW(\mathcal{C},\Sigma) \arrow[ur, dashed, "\bar{\tau}"'] \arrow[dd, "\Sigma'"] & \\
\mathcal{C} \arrow[rr] \arrow[dr, "\iota"'] & & \mathcal{T} \\
& SW(\mathcal{C},\Sigma) \arrow[ur, dashed, "\bar{\tau}"'] &
\end{tikzcd}$$

Proof: We've seen that $SW(\mathcal{C},\Sigma)$ has property (i). Let's now begin with the unicity of $\bar{\tau}$ in (ii). Consider two functors τ_1 and τ_2 such that

$$\tau_1\iota = \tau = \tau_2\iota \quad \text{and} \quad S\tau_1 = \tau_1\Sigma',\ S\tau_2 = \tau_2\Sigma'.$$

We know thus from the first equation that for any map f in $\mathcal{C}$ we have $\tau_1([0,f]) = \tau_2([0,f]) = \tau(f)$. Using the two other equations, one also calculates

$$\tau_1[n,f] = \tau_1\Sigma'^{-n}[0,f] = S^{-n}\tau_1[0,f] = S^{-n}\tau_2[0,f] = \tau_2\Sigma'^{-n}[0,f] = \tau_2[n,f]$$

for *any* map in $SW(\mathcal{C})$.

Let's now prove the existence of $\bar{\tau}$. Denote by μ the natural isomorphism $S\tau \to \tau\Sigma$. For any map $f : \Sigma^{n+i}X \to \Sigma^{n+j}Y$ ($n \geq -i,-j$) in $\mathcal{C}$ look at the following diagram, where the dotted arrow is the composition of all others.

$$\begin{tikzcd}
S^{n+i}\tau X \arrow[r, "S^{n+i-1}\mu"] \arrow[d, dotted] & \cdots \arrow[r, "S\mu"] & S\tau\Sigma^{n+i-1}X \arrow[r, "\mu"] & \tau\Sigma^{n+i}X \arrow[d, "\tau f"] \\
S^{n+j}\tau Y \arrow[r, "S^{n+j-i}\mu"'] & \cdots \arrow[r, "S\mu"'] & S\tau\Sigma^{n+j-1} \arrow[r, "\mu"'] & \tau\Sigma^{n+j}Y
\end{tikzcd}$$

Applying S^{-n} to it provides a map

$$S^{-n}((\underbrace{S^{n+j-1}\mu^{-1}S^{n+j-2}\mu^{-1}\cdots S\mu^{-1}\mu^{-1}}_{\mu^{-1}_{n+j}})(\tau f)(\underbrace{\mu\, S\mu\cdots S^{n+i-1}\mu}_{\mu_{n+i}}))$$

in $\mathcal{T}(S^iX, S^jY)$. Denote it by $\bar{\tau}_n^{(X,i)(Y,j)}$, and abbreviate the left composition of isomorphisms by μ^{-1}_{n+j} and the right one by μ_{n+i}. Then $\bar{\tau}_{n+1}^{(X,i)(Y,j)}$ sends Σf to

$$\begin{aligned} S^{-(n+1)}(\mu^{-1}_{n+1+j}\,\tau(\Sigma f)\,\mu_{n+1+i}) &= S^{-n}S^{-1}(S^{n+j}\mu^{-1}\cdots S\mu^{-1}\mu^{-1}(\tau\Sigma f)\mu\, S\mu\cdots S^{n+i}\mu) \\ &= S^{-n}S^{-1}(S^{n+j}\mu^{-1}\cdots S\mu^{-1}(S\tau f)S\mu\cdots S^{n+i}\mu) \\ &= S^{-n}(S^{n-1+j}\mu^{-1}\cdots S\mu^{-1}\mu^{-1}(S\tau f)\mu\, S\mu\cdots S^{n-1+i}\mu) \\ &= \bar{\tau}_n^{(X,i)(Y,j)}(f) \end{aligned}$$

where the second equality is the naturality of μ. Thus we have constructed a cone on the sequential diagram whose colimit is $SW((X,i),(Y,j))$. By the definition of colimit there is a well defined map of sets

$$\bar{\tau}^{(Y,j)}_{(X,i)} : SW((X,i),(Y,j)) \longrightarrow \mathcal{T}(S^i\tau X, S^j\tau Y)$$

$$[n,f] \longmapsto S^{-n}(\mu^{-1}_{n+j}\,\tau f\,\mu_{n+i})$$

The collection of these maps for all pairs of objects $(X,i),(Y,j)$ of SW make up a functor $\bar{\tau}$. Let's check this carefully.
Identity axiom. Use that $\mu_0 = \mathrm{Id}$:

$$\bar{\tau}(1_{(X,i)}) = \bar{\tau}^{(X,i)}_{(X,i)}([-i,1_X]) = S^i(\mu^{-1}_{-i+i}\circ\tau 1_X\circ\mu_{-i+i}) = S^i\tau 1_X = 1_{S^i\tau X}$$

Composition axiom. Note that for all $l \geq 0$ the map we abbreviated by μ_l^{-1} *is* the inverse of μ_l. Hence for two composable maps $(X,i)\xrightarrow{[n,f]}(Y,j)\xrightarrow{[m,g]}(Z,k)$ in SW one has:

$$\bar{\tau}([m,g]\circ[n,f]) = \bar{\tau}([m{+}n,\Sigma^n g\circ\Sigma^m f]) = S^{-m-n}(\mu^{-1}_{m+n+k}\circ\tau(\Sigma^n g\circ\Sigma^m f)\circ\mu_{m+n+i})$$

$$= S^{-m-n}(\mu^{-1}_{m+n+k}\circ\tau(\Sigma^n g)\circ\mu_{m+n+i}\circ\mu^{-1}_{m+n+i}\circ\tau(\Sigma^m f)\circ\mu_{m+n+i})$$

$$= S^{-m-n}(\mu^{-1}_{m+n+k}\circ\tau(\Sigma^n g)\circ\mu_{m+n+i})\circ S^{-m-n}(\mu^{-1}_{m+n+i}\circ\tau(\Sigma^m f)\circ\mu_{m+n+i})$$

$$= \bar{\tau}([m+n,\Sigma^n g])\circ\bar{\tau}([n+m,\sigma^m f]) = \bar{\tau}([m,g])\circ\bar{\tau}([n,f])$$

Thus $\bar{\tau}$ is a functor. Now, again because $\mu_0 = \mathrm{Id}$, we have trivially that $\bar{\tau}\iota = \tau$. Moreover,

$$S\bar{\tau}(X,i) = SS^i\tau X = S^{i+1}\tau X = \bar{\tau}(X,i+1) = \bar{\tau}\Sigma'(X,i)$$

and

$$S\bar{\tau}[n,f] = SS^{-n}(\mu^{-1}_{n+j}\tau(f)\mu_{n+1}) = S^{-(n-1)}(\mu^{-1}_{(n-1)+(j+1)}\tau(f)\mu_{(n-1)+(j+1)})$$

$$= \bar{\tau}[n-1,f] = \bar{\tau}\Sigma'[n,f]$$

show that $S\bar{\tau} = \bar{\tau}\Sigma'$. $\square$

4.2 Limits and colimits

Now we prove a couple of results about limits and colimits that will be useful later.

Lemma 4.3. *Let $\mathcal{I}$ be a finite category (i.e. with a finite number of morphisms). If $\Sigma : \mathcal{C} \to \mathcal{C}$ preserves limits (colimits) of $\mathcal{I}$-shaped diagrams in $\mathcal{C}$, then also the canonical functor $\iota : \mathcal{C} \to SW$ preserves such limits (colimits).*

Proof: We consider colimits (the proof for limits is similar). Let $F : \mathcal{I} \to \mathcal{C}$ be a finite diagram in $\mathcal{C}$ with colimit $(\mathrm{colim}F, (\eta_i)_{i \in \mathrm{Ob}\mathcal{I}})$. In SW this becomes

$$(\iota\, \mathrm{colim}F, (\iota\eta_i)_i) = ((\mathrm{colim}F, 0), ([0, \eta_i])_i) \ .$$

Consider a test cone $((T, n), ([n_i, t_i])_i)$ on ιF. (Recall that then in particular $[n_j, t_j][0, F(f)] = [n_i, t_i]$ for all $f : i \to j$ in $\mathcal{I}$.)
We now have to find a unique τ which makes the following diagram in SW commute for all i in $\mathcal{I}$.

$$\begin{tikzcd}
(F_i, 0) \arrow[rr, "{[n_i,t_i]}"] \arrow[dr, "{[0,\eta_i]}"'] & & (T, n) \\
& (\mathrm{colim}F, 0) \arrow[ur, dotted, "\tau"'] &
\end{tikzcd}$$

Existence: Since the diagram ιF and the test cone on it are composed by a finite number of commuting diagrams

$$\begin{tikzcd}
(F_i, 0) \arrow[rr, "{[n_i,t_i]}"] \arrow[dr, "{[0,f]}"'] & & (T, n) \\
& (F_j, 0) \arrow[ur, "{[n_j,t_j]}"'] &
\end{tikzcd}$$

we can recover them in $\mathcal{C}$ as the collection of the following commuting diagrams

$$\begin{tikzcd}
\Sigma^N F_i \arrow[rr, "\Sigma^{N-n_i} t_i"] \arrow[dr, "\Sigma^N f"'] & & \Sigma^{N+n} T \\
& \Sigma^N F_j \arrow[ur, "\Sigma^{N-n_j} t_j"'] &
\end{tikzcd}$$

for some big $N \in \mathbb{Z}$. When glued together, these diagrams form the image under Σ^N of $F(\mathcal{I})$ plus a test cone. Since Σ (and so Σ^N) preserves finite colimits by hypothesis, there exists a unique morphism $t : \Sigma^N \mathrm{colim}F \to \Sigma^{N+n}T$ in $\mathcal{C}$ such that for all $i \in \mathrm{Ob}\mathcal{I}$ the following diagram commutes:

$$\begin{tikzcd}
\Sigma^N F_i \arrow[rr, "\Sigma^{N-n_i} t_i"] \arrow[dr, "\Sigma^N \eta_i"'] & & \Sigma^{N+n} T \\
& \Sigma^N \mathrm{colim}F \arrow[ur, dotted, "t"'] &
\end{tikzcd}$$

Hence by applying ι and then Σ^{-N}we get in SW a map $\Sigma^{-N}[0,t] = [N,t]$ (not necessarily unique anymore) such that the diagram

$$\begin{tikzcd}
(F_i,0) \arrow[rr, "{[n_i,t_i]}"] \arrow[dr, "{[0,\eta_i]}"'] & & (T,n) \\
& (\mathrm{colim} F, 0) \arrow[ur, dotted, "{[N,t]}"'] &
\end{tikzcd}$$

commutes for all i.
Unicity: Consider $\tau = [m,t]$ and $\tau' = [m',t']$ such that for all i

$$[m,t][0,\eta_i] = [m',t'][0,\eta_i] = [n_i,t_i] \ .$$

As before, for an N big enough we get a collection of diagrams

$$\begin{tikzcd}
\Sigma^N F_i \arrow[rr, "\Sigma^{N-n_i} t_i"] \arrow[dr, "\Sigma^N \eta_i"'] & & \Sigma^{N+n} T \\
& \Sigma^N \mathrm{colim} F \arrow[ur, shift left, "\Sigma^{N-m'} t'"] \arrow[ur, shift right, "\Sigma^{N-m} t"'] &
\end{tikzcd}$$

which commute (more precisely, both triangles in them commute). Then, since by hypothesis $\Sigma^N \mathrm{colim} F$ is the colimit of $\Sigma^N F$, the maps $\Sigma^{N-m} t$ and $\Sigma^{N-m'} t'$ must coincide, and so must τ and τ' by definition. $\square$

Corollary 4.4. *Let $\mathcal{I}$ be a finite category. If $\mathcal{C}$ has all $\mathcal{I}$-shaped limits (colimits) and $\Sigma : \mathcal{C} \to \mathcal{C}$ preserves them, then also $SW(\mathcal{C},\Sigma)$ has them and $\Sigma^{\pm 1} : SW \to SW$ preserve them.*

Proof: That $\Sigma^{\pm 1} : SW \to SW$ preserve (and reflect) limits and colimits is a very general fact, true for all invertible functors. Let's now prove the existential claim; for variation, we now consider limits. Let $F : \mathcal{I} \to SW$ be a finite diagram in SW, and write

$$F(i) =: (F_i, m_i) \qquad (i \in \mathrm{Ob}\mathcal{I})$$

$$F\big(\ i \xrightarrow{v} j\ \big) =: [n_v, f_v] \qquad (v \in \mathrm{Mor}\mathcal{I})$$

for the finitely many objects and morphisms of its image. Then, for an N big enough the diagram in $\mathcal{C}$ consisting of the objects

$$\Sigma^{N+m_i} F_i \qquad (i \in \mathrm{Ob}\mathcal{I})$$

and the morphisms

$$\Sigma^{N-n_v} f_v \qquad (v \in \mathrm{Mor}\mathcal{I})$$

has a limit in $\mathcal{C}$, say

$$\big(G, \big(\ G \xrightarrow{p_i} \Sigma^{N+m_i} F_i\ \big)_i\big) \ .$$

By the above lemma, the functor $\Sigma^{-N}\iota$ preserves limiting cones, so by applying it to the above limit we obtain a limiting cone

$$\big((G,-N), \big(\ (G,-N) \xrightarrow{[N,p_i]} (F_i, m_i)\ \big)_i\big)$$

in SW for the original diagram F. $\square$

5 The Spanier-Whitehead category is additive

Recall our convention. Instead of working with the "real" homotopy category of a pointed model category and its suspension, we assumed that our pair $(\mathcal{H}, \Sigma)$ satisfies the conditions **(H0)-(H6)**. So let us now consider again such a pair. First though we need to recall one more useful fact. (I should also warn the reader that this section might seem rather pedantic, but I thought it best to be precise and give all the constructions explicitly.)

Lemma 5.1. *Fix a small category $\mathcal{I}$, and consider functors $F : \mathcal{I} \to \mathcal{A}\mathrm{b}$. Let U be the forgetful functor $\mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{et}$. Then there exists a well defined map of sets*

$$\theta : \mathrm{colim}\,(UF) \to U(\mathrm{colim}\, F)$$

which is a natural transformation of functors $\mathrm{colim}\, U, U\mathrm{colim} : \mathcal{A}\mathrm{b}^{\mathcal{I}} \to \mathcal{S}\mathrm{et}$. Morover, if $\mathcal{I} \simeq \mathbb{N}$, i.e. in the case of sequential colimits, θ is a natural isomorphism.

$$\begin{tikzcd} & \mathcal{A}\mathrm{b} \arrow[d, "U"] \\ \mathcal{I} \arrow[ur, "F"] \arrow[r, "UF"'] & \mathcal{S}\mathrm{et} \end{tikzcd}$$

Proof: The existence of the map θ is given simply by the universal property of colimits. Its naturality comes from the naturality of all the maps involved in its construction. One might like to see explicitly how this is done.
Recall then that

$$\mathrm{colim}\, UF = \big(\coprod_i UF_i\big)/\sim$$

where the equivalence relation $\sim$ is that induced by $a_i \sim f(a_i)$ for every map $f : F_i \to F_j$ in the diagram $UF(\mathcal{I})$; and recall that the colimit of F in $\mathcal{A}\mathrm{b}$ is

$$\mathrm{colim}\, F = \big(\bigoplus_i F_i\big)/K$$

where K is the abelian subgroup generated by elements of the form $e_i(a_i) - e_j(f(a_i))$ for all maps $f : F_i \to F_j$ in the diagram $F(\mathcal{I})$ (here e_i denotes the canonical inclusion $F_i \hookrightarrow \bigoplus_j F_j$). Consider the map of sets

$$\amalg_i e_i : \ \coprod_i F_i \longrightarrow \bigoplus_i F_i$$

defined by the canonical inclusions e_i. Composing with the canonical projection onto the set of cosets provides a map

$$\theta' : \ \coprod_i F_i \longrightarrow \bigoplus_i F_i \twoheadrightarrow (\bigoplus_i F_i)/K$$

which sends (i, a) to the coset $e_i(a) + K$. Now, say that $a_j = f(a_i)$ for some f in the diagram. Then θ' sends (i, a_i) to $e_i(a_i) + K$ and (j, a_j) to $e_j(a_j) + K = e_j(f(a_i)) + K$, and the difference of their images is in K. Thus there is a well defined map of sets

$$\theta : \ (\coprod_i F_i)/\sim \longrightarrow (\bigoplus_i F_i)/K$$

as wished which sends the class of (i,a) to the coset $e_i(a)+K$.
In the case when $\mathcal{I} \simeq \mathbb{N}$, this map of sets

$$\theta : \operatorname{colim}(UF) \to U(\operatorname{colim} F)$$

$$[i,a] \longmapsto e_i(a)+K$$

is easily seen to have as a two-sided inverse the map which sends

$$\sum_{i_1<\ldots<i_k} e_{i_j}(a_j)+K = e_{i_k}\big(\Sigma^{i_k-i_1}a_1+\ldots+\Sigma^{i_k-i_{k-1}}a_{k-1}+a_k\big)+K$$

to the equivalence class

$$\big[i_k, \Sigma^{i_k-i_1}a_1+\ldots+\Sigma^{i_k-i_{k-1}}a_{k-1}+a_k\big]\,.$$

The statement in this lemma can be generalized to filtered colimits, but we have no use here for this extra generality. $\square$

Lemma 5.2. *The Spanier-Whitehead category $SW(\mathcal{H},\Sigma)$ is additive.*

Proof: What we need is an abelian group structure on the hom sets $SW((X,i),(Y,j))$ which is distributive with respect to the composition (which would make SW into a 'pre-additive' category, or 'Ab-category'), a zero object, and direct sums for any two objects. For the abelian group structure, it would suffice to find a lift $SW'(-,-)$ of the hom bifunctor $SW(-,-)$ to the category $\mathcal{A}$b of abelian groups along the forgetful functor $U : \mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{et}$.

$$\begin{tikzcd} & \mathcal{A}\mathrm{b} \arrow[d, "U"] \\ SW^{op}\times SW \arrow[ur, dotted, "SW'(-,-)"] \arrow[r, "SW(-,-)"'] & \mathcal{S}\mathrm{et} \end{tikzcd} \tag{5}$$

Indeed, the functoriality of $SW'(A,B)$ in the first and second variable just means that precomposition and composition (that is $f^* = SW'(f,B)$ and $g_* = SW'(A,g)$ for an $f : A' \to A$ resp. a $g : B \to B'$) are group homomorphisms, i.e. that $(h+h')f = hf+h'f$ and $g(h+h') = gh+gh'$.

Condition **(H0)(b)** says that the sets in the sequence whose colimit defines $SW((X,i),(Y,j))$, except possibly for the first two, are abelian groups (see Def. (4.1)). Moreover, because of **(H0)(c)** and Remark (3.1), the maps in that sequence are group homomorphisms, except possibly for the leftmost one. Therefore, it is just natural to equip the hom sets $SW((X,i),(Y,j))$ with a functorial abelian group structure simply by taking the colimit of the sequence in $\mathcal{A}$b instead of $\mathcal{S}$et. More precisely, one can define the bifunctor $SW'((X,i),(Y,j))$ to be the following colimit in $\mathcal{A}$b:

$$\operatorname*{colim}_{n\geq -i+2,-j}\left(\ldots \xrightarrow{\Sigma} \mathcal{H}(\Sigma^{n+i}X,\Sigma^{n+j}Y) \xrightarrow{\Sigma} \mathcal{H}(\Sigma^{n+1+i}X,\Sigma^{n+1+j}Y) \xrightarrow{\Sigma} \ldots\right)$$

(Notice that the necessary "+2" after the index i doesn't matter much, since every class $[n,f]$ in the colimit as taken in Definition (4.1) has representatives also in the hom groups $\mathcal{H}(\Sigma^{n+i}X, \Sigma^{n+j}Y)$ for arbitrarily big n's.) There is a (little) problem here though: the forgetful functor $U : \mathcal{A}\mathrm{b} \to \mathcal{S}\mathrm{et}$ doesn't commute with colimits in general (only with limits, having the free abelian group functor for left adjoint), so diagram (5) doesn't have to commute. In other words: by taking the colimits of Def. (4.1) in the category $\mathcal{A}\mathrm{b}$ of abelian groups instead of in $\mathcal{S}\mathrm{et}$ we would define a category "$SW'(\mathcal{H}, \Sigma)$" which is a priori different from $SW(\mathcal{H}, \Sigma)$. Fortunately, Lemma (5.1) applies here, so that diagram (5) commutes up to natural isomorphism, i.e. there is an isomorphism of bifunctors $\theta : SW(-,-) \simeq U \circ SW'(-,-)$. (Notice that here, instead of looking at all sequences $F : \mathbb{N} \to \mathcal{A}\mathrm{b}$, we confine our attention to a family with parameter a pair of objects $(X, i), (Y, j)$ of SW.) In other words, Lemma (5.1) implies that the categories $SW(\mathcal{H}, \Sigma)$ and "$SW'(\mathcal{H}, \Sigma)$" are equivalent. Thus the Spanier-Whitehead category SW is pre-additive. One can use the map θ, given explicitly in the proof of Lemma (5.1), to see how the group structure on the hom sets works. This turns out to be very simply

$$[n, f] + [m, g] := [n + m + 2, \Sigma^{m+2} f + \Sigma^{n+2} g]$$

for any two parallel arrows $f, g : (X, i) \to (Y, j)$. (The "+2" on the right hand side can be omitted whenever $n, m \geq -i + 2$.)

What has been shown so far is that SW is a preadditive category (or "Ab-category"). In such a category, an object is a zero object iff it is initial and iff it is final (see [3], Prop. 1 p. 190). Therefore SW has a zero object, which is the image under the canonical functor ι (which preserves finite colimits by Lemma (4.3)) of the zero object (thus in particular the initial object, i.e. the empty coproduct) $*$ in $\mathcal{H}$. (Here we have used that $\mathcal{H}$, by **(H0)(a)**, has a zero object and that Σ preserves finite coproducts.)

Also in a preadditive category, two objects have a product or a coproduct if and only if they have a direct sum (see [3] Thm 2, p. 190). Because SW has all finite coproducts (by **(Ho)(a)** and Corollary (4.4)), we have shown that SW is additive.
More precisely, direct sums are as follows. Given a coproduct diagram

$$(X, i) \xrightarrow{\eta_1} (X, i) \vee (Y, j) \xleftarrow{\eta_2} (Y, j)$$

in SW, its definition provides a unique map

$$(X, i) \vee (Y, j) \xrightarrow{\pi_1 = (1,0)} (X, i)$$

whose components are $\pi_1 \eta_1 = 1_{(X,i)}$ and $\pi_1 \eta_2 = 0$, and a unique

$$(X, i) \vee (Y, j) \xrightarrow{\pi_2 = (0,1)} (Y, j)$$

with components $\pi_2 \eta_1 = 0$ and $\pi_2 \eta_2 = 1_{(Y,j)}$. It is immediate to check that the diagram

$$\begin{tikzcd} (X, i) \arrow[r, shift left, "\eta_1"] & (X, i) \vee (Y, j) \arrow[l, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & (Y, j) \arrow[l, shift right, "\eta_2"'] \end{tikzcd}$$

satisfies the direct sum equations

$$\pi_1\eta_1 = 1_{(X,i)}, \quad \pi_2\eta_2 = 1_{(Y,j)}, \quad \eta_1\pi_1 + \pi_2\eta_2 = 1_{(X,i)\vee(Y,j)} \; .$$

Hence in SW from now on we shall write $(X,i)\oplus(Y,j)$ instead of $(X,i)\vee(Y,j)$, and we shall denote the zero object $\iota(*)$ by 0, as usual in an additive category. □

Remark 5.3. Note that the suspension $\Sigma : SW(\mathcal{H},\Sigma) \to SW(\mathcal{H},\Sigma)$ is an additive functor. This is because, by **(H0)(a)** and Corollary (4.4), it preserves coproduct diagrams $(X,i)\xrightarrow{\eta_1}(X,i)\vee(Y,j)\xleftarrow{\eta_2}(Y,j)$; hence it preserves also direct sum diagrams (see the construction at the end of the above proof) and is therefore additive. In particular Σ preserves the zero object, $\Sigma(0) = 0$.

6 The Spanier-Whitehead category is triangulated

6.1 Triangulated categories

Let's now recall the exact definition of a triangulated category (Verdier [6]).

Definition 6.1 (Triangulated category)**.** Call a diagram of the form

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$$

a *triangle*. We will occasionally abbreviate such a triangle by (u,v,w), or we will picture it as

$$\begin{tikzcd} & C \arrow[dl, "w"', "\circ" marking] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd}$$

(A circled arrow $f : X \longrightarrow\!\!\circ\!\!\to Y$ denotes a "morphism of degree one" $f : X \to \Sigma Y$.) A *morphism of triangles* is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, "h"'] & \Sigma A \arrow[d, "\Sigma f"'] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \Sigma A' \end{tikzcd}$$

Then one defines the notion of *isomorphism of triangles* in the obvious way. Now, a *(Verdier or classical) triangulated category* is an additive category $\mathcal{K}$ together with an additive self-equivalence Σ called *suspension (or translation or shift)* and together with a *(classical) triangulation* on $\mathcal{K}$, that is a collection $\mathcal{T}$ of triangles in $\mathcal{K}$, called *distinguished triangles*, which satisfy the following four axioms.

(TR1) (i) Every morphism u in $\mathcal{K}$ fits into some distinguished triangle (u,v,w).
(ii) The collection $\mathcal{T}$ is "replete", i.e., any triangle in $\mathcal{K}$ isomorphic (in the above sense) to a distinguished triangle, is also distinguished.

(iii) For every object A in $\mathcal{K}$, the triangle $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \Sigma A$ is distinguished. (Or equivalently in the presence of (TR2), the triangle $0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0$ is distinguished.)

(TR2) (*Rotation Axiom*) If $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ is distinguished, then also

$$B \xrightarrow{v} C \xrightarrow{w} \Sigma A \xrightarrow{-\Sigma u} \Sigma B$$

$$\Sigma^{-1} C \xrightarrow{-\Sigma^{-1} w} A \xrightarrow{u} B \xrightarrow{v} C$$

are distinguished. One says that $(v, w, -\Sigma u)$ is "(u, v, w) rotated to the left" and that $(-\Sigma^{-1} w, u, v)$ is "(u, v, w) rotated to the right"[1]. (Here Σ^{-1} is the quasi-inverse of the suspension Σ, i.e. $\Sigma\Sigma^{-1} \simeq \mathrm{Id}_{\mathcal{K}} \simeq \Sigma^{-1}\Sigma$ naturally. In the practice, one tends to treat these natural isos as equalities.)

(TR3) (*Morphism Axiom*) Given a (solid) diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dotted] & \Sigma A \arrow[d, "\Sigma f"] \\
A' \arrow[r, "u'"'] & B' \arrow[r] & C \arrow[r] & \Sigma A
\end{tikzcd}$$

where the rows are distinguished triangles and the left square commutes, there exists a (nonunique) dotted arrow which makes the rest of the diagram commute. Somewhat criptically, one could say that every morphism of morphisms u and u' (i.e., the commuting square on the left) completes to a morphism of distinguished triangles for any choice of distinguished triangles containing u resp. u'.

(TR4) (*Composition or Octahedron Axiom*)

Given a composition $h = (\ X \xrightarrow{f} Y \xrightarrow{g} Z\ )$ of two morphisms and given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{f'} U \xrightarrow{f''} \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{g'} W \xrightarrow{g''} \Sigma Y$$

$$X \xrightarrow{h} Z \xrightarrow{h'} V \xrightarrow{h''} \Sigma X$$

on f, g and h respectively, there exist two morphisms $s : U \to V$, $s' : V \to W$, such that the triangle

$$U \xrightarrow{s} V \xrightarrow{s'} W \xrightarrow{\Sigma f' \circ g''} \Sigma U$$

[1] Some authors' terminology has 'right' and 'left' swapped. *Exercise*: Find an epistemological explanation of this curious phenomenon.

is distinguished, and such that

$$\begin{aligned} & h''s = f'' \qquad s'h' = g' \\ \text{and} \quad & sf' = h'g \qquad g''s' = (\Sigma f)h'' \end{aligned} \tag{6}$$

There are many ways to visualize this information. Here is a “flat octahedron” (with $s'' := \Sigma f' \circ g''$):

$$\begin{tikzcd}
& & V \arrow[dr, dotted, "s'"] \arrow[ddll, bend right, "h''"', "\circ" marking] & & \\
& U \arrow[ur, dotted, "s"] \arrow[dl, "f''"', "\circ" marking] & & W \arrow[ll, "s''"', "\circ" marking] \arrow[dl, "g''"', "\circ" marking] & \\
X \arrow[rr, "f"'] \arrow[rrrr, bend right, "h"'] & & Y \arrow[ul, "f'"'] \arrow[rr, "g"'] & & Z \arrow[ul, "g'"'] \arrow[uull, bend right, "h'"']
\end{tikzcd}$$

The equations (6) make the diagram commutative (wherever it can), while the four ‘triangles’ in it of the form

$$\begin{tikzcd}
& \cdot \arrow[dl, "\circ" marking] & \\
\cdot \arrow[rr] & & \cdot \arrow[ul]
\end{tikzcd}$$

are all distinguished. We will call such a diagram with the above good properties *an octahedron*. Thus this axiom can be rephrased as: Any choice of distinguished triangles on the three morphisms of a composition can be completed to an octahedron.
In the presence of (TR1-3), this is known to be equivalent to the following (otherwise weaker) version.

(TR4’) Given a composition $h = g \circ f$, there is an octahedron (i.e. four distinguished triangles with the good properties as above) containing it.

6.2 The triangulation on $SW(\mathcal{H}, \Sigma)$

We shall now see how one can use the collection of cofiber sequences in $\mathcal{H}$ to produce a triangulation on SW in a very natural way. Let’s begin with a technical definition.

Definition 6.2. Given a cofiber sequence (3) in $\mathcal{H}$, one can always ‘shorten’ it to a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\partial} \Sigma A$$

using the boundary map $\partial = (0, 1) \circ \nu$ of (3). Call such triangles in $\mathcal{H}$ *shortened cofiber sequences.*

Definition 6.3 (The triangulation $\mathcal{T}$ on $SW(\mathcal{H}, \Sigma)$)**.** A triangle

$$(X, i) \xrightarrow{\alpha} (Y, j) \xrightarrow{\beta} (Z, k) \xrightarrow{\gamma} \Sigma(X, i) = (X, i + 1) \tag{7}$$

in $SW(\mathcal{H},\Sigma)$ belongs to the collection $\mathcal{T}$ of *distinguished* triangles if and only if, up to an even number of suspensions, it is isomorphic to the canonical image of some shortened cofiber sequence (f,g,∂). That is, (α,β,γ) is distinguished if there exists a cofiber sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad C \xrightarrow{\nu} C \vee \Sigma A$$

in $\mathcal{H}$, an $n \in \mathbb{Z}$ and an isomorphism of triangles

$$\begin{tikzcd}
\Sigma^{2n}(X,i) \arrow[r, "\Sigma^{2n}\alpha"] \arrow[d, "\phi_1"', "\simeq"] & \Sigma^{2n}(Y,j) \arrow[r, "\Sigma^{2n}\beta"] \arrow[d, "\phi_2"', "\simeq"] & \Sigma^{2n}(Z,k) \arrow[r, "\Sigma^{2n}\gamma"] \arrow[d, "\phi_3"', "\simeq"] & \Sigma^{2n}(X,i+1) \arrow[d, "\Sigma\phi_1"', "\simeq"] \\
\iota A \arrow[r, "\iota f"] & \iota B \arrow[r, "\iota g"] & \iota C \arrow[r, "\iota\partial"] & \iota\Sigma A = \Sigma\iota A
\end{tikzcd} \tag{8}$$

where ∂ is the boundary map of the cofiber sequence.

Remark 6.4. By definition, the boundary map of a cofiber sequence (3) is

$$\partial : \ C \xrightarrow{\nu} C \vee \Sigma A \xrightarrow{(0,1)} \Sigma A$$

and is therefore determined by the action ν. By sending the cofiber sequence in the additive category SW, one sees that the canonical image $\iota\nu$ of the action is in turn determined by the canonical image $\iota\partial$ of the boundary map. This is because in an additive category a map into a direct sum is uniquely determined by its components (this is not true in general for a map into a coproduct!). In SW one can write

$$\iota\partial : \ (C,0) \xrightarrow{\binom{1}{\iota\partial}} (C,0) \oplus (A,1) \xrightarrow{(0\ 1)} (A,1)$$

(remember that ι preserves finite limits and colimits by Lemma (4.3)). Indeed, the second component of $\iota\nu$ is $\iota\partial$ by definition, and the first must by the identity of C by the first axiom of a coaction (see Def. (2.5)): The image in SW of the unit axiom of a coaction, that is of the triangle on the right of diagram (2), is

$$\begin{tikzcd}
\iota X \oplus \iota A \arrow[r, "(1\ 0)"] & \iota X \oplus 0 \\
\iota X \arrow[u, "\nu=\binom{?}{\partial}"] \arrow[ur, "\binom{1}{0}"'] &
\end{tikzcd}$$

hence $? = 1_{\iota X}$. Because of this, when sending cofiber sequences in some additive category (i.e., up to suspension, when studying distinguished triangles), one can forget about their actions and all the complications thereof, since it is enough to keep track of the boundary maps (i.e., the 'third maps' γ of distinguished triangles), which contain just the same amount of information. This is one of the many advantages of working in an additive category.

Now time has come to prove our main theorem, stated below. Since we have proven already that $SW(\mathcal{H},\Sigma)$ is additive and that the suspension $\Sigma : SW(\mathcal{H},\Sigma) \to SW(\mathcal{H},\Sigma)$ is an additive functor (Lemma (5.2), Remark (5.3)), we now have to check that the axioms (TR1)-(TR4) are satisfied.

Theorem 6.5 (Main theorem)**.** *The Spanier-Whitehead category $SW(\mathcal{H},\Sigma)$ is triangulated, where the triangulation $\mathcal{T}$ is that of Definition (6.3).*

6.3 Proving the axioms

Lemma 6.6. *If (f, g, ∂) is a shortened cofiber sequence in $\mathcal{H}$ (see Def. (6.2)), then so is $(\Sigma^2 f, \Sigma^2 g, \Sigma^2 \partial)$.*

Proof: By definition, there is a cofiber sequence (3) whose shortened version is (f, g, ∂). By **(H4)**, one obtains out of (3) the following commuting diagram, where the top row is another shortened cofiber sequence:

$$\begin{tikzcd}
B \arrow[r, "g"] & C \arrow[r, "\partial"] & \Sigma A \arrow[rr, "\partial_f"] \arrow[d, "m_{\Sigma A}"'] \arrow[drr, "\nu_f"] & & \Sigma B \\
& & \Sigma A \vee \Sigma A \arrow[rr, "1\vee(i_{\Sigma B}\circ\Sigma f)"'] & & \Sigma A \vee \Sigma B \arrow[u, "(0,1)"']
\end{tikzcd}$$

As should be clear, $m_{\Sigma A}$ is the comultiplication of the cogroup ΣA, $i_{\Sigma B}$ is the coinverse of the cogroup ΣB, and ν_f is the coaction belonging to the sequence. Let's now take a closer look at ∂_f; consider the following diagram, where the unnamed isomorphism is the canonical one (i.e. the inverse of $(0, 1)$).

$$\begin{tikzcd}
\Sigma A \arrow[r, "m_{\Sigma A}"] \arrow[dr, "\simeq"'] & \Sigma A \vee \Sigma A \arrow[rr, "1\vee(i_{\Sigma B}\circ\Sigma f)"] \arrow[d, "0\vee 1"] & & \Sigma A \vee \Sigma B \arrow[d, "(0,1)"] \\
& 0 \vee \Sigma A \arrow[rr, "(0,i_{\Sigma B}\circ\Sigma f)"'] & & \Sigma B
\end{tikzcd}$$

The square on the right obviously commutes, and the triangle on the left commutes also because it is part of the counit axiom for the cogroup ΣA (see (2.3); notice that e is forced to be the zero map, since the initial object 0 here is also final). Since one path from ΣA to ΣB is ∂_f by definition and another one is $(0, i_{\Sigma B} \circ \Sigma f)\circ \simeq = i_{\Sigma B} \circ \Sigma f$, we have $\partial_f = i_{\Sigma B} \circ \Sigma f$. With this knowledge, we can now apply **(H4)** twice more (to the 'unshortened' cofiber sequence (g, ∂, ν_f), of course) in order to get the shortened cofiber sequence $(i_{\Sigma B} \circ \Sigma f, i_{\Sigma C} \circ \Sigma g, i_{\Sigma^2 A} \circ \Sigma\partial)$. Applying it three more times we get the shortened cofiber sequence

$$(i_{\Sigma^2 B} \circ \Sigma(i_{\Sigma B} \circ \Sigma f), i_{\Sigma^2 C} \circ \Sigma(i_{\Sigma C} \circ \Sigma g), i_{\Sigma^3 A} \circ \Sigma(i_{\Sigma^2 A} \circ \Sigma\partial)) \, . \tag{9}$$

Now we know by **(H0)(c)** that $\Sigma : \mathcal{C} \to \mathcal{C}$ preserves the structure morphisms of the cogroup ΣX; in particular $\Sigma(i_{\Sigma X}) = i_{\Sigma^2 X}$. We need also to know that $i_X \circ i_X = 1_X$ for any cogroup X. This follows from the axioms of a usual group and the Yoneda Lemma by the same argument used in Remark (3.1). Everything said, we obtain that the shortened cofiber sequence (9) is equal to $(\Sigma^2 f, \Sigma^2 g, \Sigma^2 \partial)$. □

Lemma 6.7. *$(SW(\mathcal{H}, \Sigma), \Sigma, \mathcal{T})$ satisfies (TR1).*

Proof: (i) Let $\alpha = [m, f] : (X, i) \to (Y, j)$ be an arbitrary morphism in SW. Choose n with $2n \geq m$. Then

$$\Sigma^{2n}\alpha : \Sigma^{2n}(X, i) \longrightarrow \Sigma^{2n}(Y, j)$$

is the canonical image of

$$\Sigma^{2n-m} f : \Sigma^{2n+i} X \longrightarrow \Sigma^{2n+j} Y \ .$$

By **(H3)** the above morphism fits into some shortened cofiber sequence

$$\Sigma^{2n+i} X \xrightarrow{\Sigma^{2n-m} f} \Sigma^{2n+j} Y \xrightarrow{g} C \xrightarrow{\partial} \Sigma^{2n+i+1} X$$

in $\mathcal{C}$, whose canonical image is

$$(X, i+2n) \xrightarrow{[m-2n,f]} (Y, j+2n) \xrightarrow{[0,g]} (C, 0) \xrightarrow{[0,\partial]} (X, i+1+2n) \ .$$

Thus, after applying Σ^{-2n}, our map α fits into the distinguished triangle

$$(X, i) \xrightarrow{\alpha} (Y, j) \xrightarrow{[2n,g]} (C, -2n) \xrightarrow{[2n,\partial]} (X, i+1) \ .$$

(ii) $\mathcal{T}$ is obviously replete because of how distinguished triangles are defined in SW. Notice that, by Remark (3.2), a morphism of cofiber sequences translates neatly into a morphism of triangles.
(iii) Consider $0 \longrightarrow (X, i) \xrightarrow{1} (X, i) \longrightarrow 0$. Choose an n such that $n \geq -i$. Then the diagram

$$\begin{tikzcd}
0 = \Sigma^{2n} 0 \arrow[r] \arrow[d, equal] & (X, i+2n) \arrow[r, "1"] \arrow[d, equal] & (X, i+2n) \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 = \iota(*) \arrow[r] & \iota\Sigma^{i+2n} X \arrow[r, "1"] & \iota\Sigma^{i+2n} \arrow[r, "\iota\partial = 0"] & 0
\end{tikzcd}$$

commutes, where the bottom line is the canonical image of a cofiber sequence in $\mathcal{C}$, by courtesy of **(H2)**. $\square$

Lemma 6.8. *$(SW(\mathcal{H}, \Sigma), \Sigma, \mathcal{T})$ satisfies (TR2), i.e. Rotation.*

Proof: Consider the diagram (8) (copied below) as a certification that the triangle (α, β, γ) is distinguished, where the bottom row is the canonical image of a shortened cofiber sequence (f, g, ∂) in $\mathcal{H}$.

$$\begin{tikzcd}
\Sigma^{2n}(X, i) \arrow[r, "\Sigma^{2n}\alpha"] \arrow[d, "\phi_1"', "\simeq"] & \Sigma^{2n}(Y, j) \arrow[r, "\Sigma^{2n}\beta"] \arrow[d, "\phi_2"', "\simeq"] & \Sigma^{2n}(Z, k) \arrow[r, "\Sigma^{2n}\gamma"] \arrow[d, "\phi_3"', "\simeq"] & \Sigma^{2n}(X, i+1) \arrow[d, "\Sigma\phi_1"', "\simeq"] \\
\iota A \arrow[r, "\iota f"] & \iota B \arrow[r, "\iota g"] & \iota C \arrow[r, "\iota\partial"] & \iota\Sigma A = \Sigma\iota A
\end{tikzcd}$$

Applying **(H4)** to (the unshortened version of) (f, g, ∂) yields the shortened cofiber sequence $(g, \partial, \partial_f = i_{\Sigma B} \circ \Sigma f)$, as shown in the proof of Lemma (6.6). If we can show that in SW

$$\iota(i_{\Sigma B} \circ \Sigma f) = -\iota\Sigma f \ , \tag{10}$$

then the triangle $(\beta, \gamma, -\Sigma\alpha)$ is also distinguished, as shown by the following commuting diagram.

$$\begin{tikzcd}
(Y, j+2n) \arrow[r, "\Sigma^{2n}\beta"] \arrow[d, "\phi_2"'] & (Z, k+2n) \arrow[r, "\Sigma^{2n}\gamma"] \arrow[d, "\phi_3"'] & (X, i+1+2n) \arrow[r, "-\Sigma^{2n+1}\alpha"] \arrow[d, "\Sigma\phi_1"'] & (Y, j+1+2n) \arrow[d, "\Sigma\phi_2"] \\
\iota B \arrow[r, "\iota g"] & \iota C \arrow[r, "\iota\partial"] & \iota\Sigma A \arrow[r, "\iota\partial_f = -\iota\Sigma f"] & \iota\Sigma B
\end{tikzcd}$$

(In particular then the rightmost square would commute: $(\Sigma\phi_2)(-\Sigma^{2n+1}\alpha) = -\Sigma(\phi_2 \circ \Sigma^{2n}\alpha) = -\Sigma(\iota f \circ \phi_1) = (-\iota\Sigma f)(\Sigma\phi_1)$, where the second equality is the commutativity of the first square in (8).) Thus the first half of (TR2) would be proven.

We proceed to prove (10). Let's begin by considering an arbitrary cogroup (G, m, e, i) in $\mathcal{H}$. Since the category SW is additive, the canonical images of the commuting diagrams which define the cogroup have the following form:

$$\begin{tikzcd}
G \oplus G \oplus G & G \oplus G \arrow[l, "\left(\begin{smallmatrix} 1 & \\ & m \end{smallmatrix}\right)"'] \\
G \oplus G \arrow[u, "\left(\begin{smallmatrix} m & \\ & 1 \end{smallmatrix}\right)"] & G \arrow[l, "m"] \arrow[u, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G & G \oplus G \arrow[l, "(i\ 1)"'] \\
0 \arrow[u, "0"] & G \arrow[l, "e=0"] \arrow[u, "m"']
\end{tikzcd}$$

$$\begin{tikzcd}
0 \oplus G & G \oplus G \arrow[l, "\left(\begin{smallmatrix} e & \\ & 1 \end{smallmatrix}\right)"'] \arrow[r, "\left(\begin{smallmatrix} 1 & \\ & e \end{smallmatrix}\right)"] & G \oplus 0 \\
& G \arrow[ul, "\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)"] \arrow[u, "m"] \arrow[ur, "\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)"'] &
\end{tikzcd}$$

It is now easy to determine the components m_1, m_2 of $m = \binom{m_1}{m_2}$. Since $e = 0$ (because here the initial object 0 is also final), the third diagram says that $m_1 = m_2 = 1_G$. Thus $m = \binom{1}{1}$. The second diagram then says that

$$0 = \begin{pmatrix} i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = i + 1$$

i.e. that $i = -1_G$.[2] Hence choosing $G = \Sigma B$ we get $\iota(i_{\Sigma B} \circ \Sigma f) = (-1_{\Sigma B}) \circ (\iota\Sigma f) = -\iota\Sigma f$, as wished.

Let's now prove the second half of (TR2). We must show that

$$(Z, k-1) \xrightarrow{-\Sigma^{-1}\gamma} (X, i) \xrightarrow{\alpha} (Y, j) \xrightarrow{\beta} (Z, k) \tag{11}$$

is distinguished. Let's apply five times 'rotation to the left' (i.e. the first half of the axiom) on the distinguished triangle (α, β, γ). Thus we obtain the triangle

$$(Z, k+1) \xrightarrow{-\Sigma\gamma} (X, i+2) \xrightarrow{\Sigma^2\alpha} (Y, j+2) \xrightarrow{\Sigma^2\beta} (Z, k+2)$$

[2]Thus, in particular, all the structure morphisms of a cogroup object in an additive category are uniquely determined. This follows also by Prop. (2.6) and the fact that the additive structure of an additive category is intrinsic to the category, i.e. if it exists, it is unique.

which is distinguished, that is $\exists n$ etc. such that

$$\begin{tikzcd}
(Z,(2n+2)+k-1) \arrow[r, "-\Sigma^{(2n+2)-1}\gamma"] \arrow[d, "\simeq"'] & (X,(2n+2)+i) \arrow[r, "\Sigma^{2n+2}\alpha"] \arrow[d, "\simeq"'] & (Y,(2n+2)+j) \arrow[r, "\Sigma^{2n+2}\beta"] \arrow[d, "\simeq"'] & (Z,(2n+2)+k) \arrow[d, "\Sigma\simeq"] \\
(A,0) \arrow[r] & (B,0) \arrow[r] & (C,0) \arrow[r] & (A,1)
\end{tikzcd}$$

By definition, this means also that (11) is distinguished. $\square$

Lemma 6.9. $(SW(\mathcal{H},\Sigma),\Sigma,\mathcal{T})$ *satisfies (TR3), i.e. Morphism.*

Proof: Consider the following (solid) diagram in SW, where the rows are distinguished and the square on the left commutes.

$$\begin{tikzcd}
(X,i) \arrow[r, "\alpha"] \arrow[d, "\psi_1"'] & (Y,j) \arrow[r, "\beta"] \arrow[d, "\psi_2"'] & (Z,k) \arrow[r, "\gamma"] \arrow[d, dotted] & (X,i+1) \arrow[d, "\Sigma\psi_1"] \\
(X',i') \arrow[r, "\alpha'"] & (Y',j') \arrow[r, "\beta'"] & (Z',k') \arrow[r, "\gamma'"] & (X',i'+1)
\end{tikzcd} \tag{12}$$

Consider also diagram (8) as a certification that the triangle (α,β,γ) is distinguished, and a similar diagram (with everything primed) for (α',β',γ'). Now suspend these three diagrams so as to make them coherent, as follows.

$$\begin{tikzcd}
(A,2n') \arrow[r, "\iota\Sigma^{2n'}f"] & (B,2n') \arrow[r, "\iota\Sigma^{2n'}g"] & (C,2n') \arrow[r, "\iota\Sigma^{2n'}\partial"] & (A,1+2n') \\
(X,i+2(n+n')) \arrow[u, "\Sigma^{2n'}\phi_1"', "\simeq"] \arrow[r, "\Sigma^{2(n+n')}\alpha"] \arrow[d, "\Sigma^{2(n+n')}\psi_1"'] & (Y,j+2(n+n')) \arrow[u, "\Sigma^{2n'}\phi_2"', "\simeq"] \arrow[r, "\Sigma^{2(n+n')}\beta"] \arrow[d, "\Sigma^{2(n+n')}\psi_2"'] & (Z,k+2(n+n')) \arrow[u, "\Sigma^{2n'}\phi_3"', "\simeq"] \arrow[r, "\Sigma^{2(n+n')}\gamma"] \arrow[d, dotted] & (X,i+1+2(n+n')) \arrow[u, "\Sigma^{2n'+1}\phi_1"', "\simeq"] \arrow[d, "\Sigma^{2(n+n')+1}\psi_1"'] \\
(X',i'+2(n'+n)) \arrow[r, "\Sigma^{2(n+n')}\alpha'"'] \arrow[d, "\Sigma^{2n}\phi_1'"', "\simeq"] & (Y',j'+2(n'+n)) \arrow[r, "\Sigma^{2(n+n')}\beta'"'] \arrow[d, "\Sigma^{2n}\phi_2'"', "\simeq"] & (Z',k'+2(n'+n)) \arrow[r, "\Sigma^{2(n+n')}\gamma'"'] \arrow[d, "\Sigma^{2n}\phi_3'"', "\simeq"] & (X',i'+1+2(n'+n)) \arrow[d, "\Sigma^{2n+1}\phi_1'"', "\simeq"] \\
(A',2n) \arrow[r, "\iota\Sigma^{2n}f'"'] & (B',2n) \arrow[r, "\iota\Sigma^{2n}g'"'] & (C',2n) \arrow[r, "\iota\Sigma^{2n}\partial'"'] & (A',1+2n)
\end{tikzcd} \tag{13}$$

Use the following notation for the vertical compositions in (13).

$$\epsilon_1 := (\Sigma^{2n}\phi_1')(\Sigma^{2(n+n')}\psi_1)(\Sigma^{2n'}\phi_1)^{-1}$$

$$\epsilon_2 := (\Sigma^{2n}\phi_2')(\Sigma^{2(n+n')}\psi_2)(\Sigma^{2n'}\phi_2)^{-1}$$

$$\text{and} \quad \Sigma\epsilon_1 \ .$$

Hence we can abbreviate diagram (13) as follows.

$$\begin{tikzcd}
(A,2n') \arrow[r, "\iota\Sigma^{2n'}f"] \arrow[d, "\epsilon_1"'] & (B,2n') \arrow[r, "\iota\Sigma^{2n'}g"] \arrow[d, "\epsilon_2"'] & (C,2n') \arrow[r, "\iota\Sigma^{2n'}\partial"] \arrow[d, dotted] & (A,2n'+1) \arrow[d, "\Sigma\epsilon_1"'] \\
(A',2n) \arrow[r, "\iota\Sigma^{2n}f'"'] & (B',2n) \arrow[r, "\iota\Sigma^{2n}g'"'] & (C',2n) \arrow[r, "\iota\Sigma^{2n}\partial'"'] & (A',2n+1)
\end{tikzcd} \tag{14}$$

By possibly suspending some more (even) number of times we can assume that this diagram is the canonical image of the following (solid) diagram in $\mathcal{H}$.

$$\begin{tikzcd}
\Sigma^{2n'}A \arrow[r, "\Sigma^{2n'}f"] \arrow[d, "e_1"'] & \Sigma^{2n'}B \arrow[r, "\Sigma^{2n'}g"] \arrow[d, "e_2"'] & \Sigma^{2n'}C \arrow[r, "\Sigma^{2n'}\partial"] \arrow[d, dotted] & \Sigma^{2n'+1}A \arrow[d, "\Sigma e_1"'] \\
\Sigma^{2n}A' \arrow[r, "\Sigma^{2n}f'"'] & \Sigma^{2n}B' \arrow[r, "\Sigma^{2n}g'"'] & \Sigma^{2n}C' \arrow[r, "\Sigma^{2n}\partial'"'] & \Sigma^{2n+1}A'
\end{tikzcd} \tag{15}$$

By asking again that $2n, 2n'$ be big enough, we can assume that the square on the left commutes. Now, the iterated application of Lemma (6.6) says that the two rows in (15) are shortened cofiber sequences. Hence **(H5)** applies to their 'unshortened' versions (which are cofiber sequences), yielding some dotted map $e_3 : \Sigma^{2n'}C \to \Sigma^{2n}C'$ which makes the square on its left commute in (15) and which is Σe_1-equivariant. Because of Remark (3.2) on boundary maps, the Σe_1-equivariance of e_3 implies that also the square on its right commutes. Hence the whole diagram (15) commutes, and so does (14) in SW with ιe_3 the dotted map. Then the map

$$x := (\Sigma^{2n}\phi_3')^{-1}(\iota e_3)(\Sigma^{2n'}\phi_3)$$

completes (13) and makes it commutative, because a diagram isomorphic to a commutative diagram is commutative. Finally we obtain a dotted arrow $\psi_3 := \Sigma^{-2(n+n')}x$ which completes (12) to a morphism of distinguished triangles in SW. $\square$

Lemma 6.10. *$(SW(\mathcal{H},\Sigma),\Sigma,\mathcal{T})$ satisfies (TR4), i.e. Composition.*

Proof: We prove the weak version (TR4'). Consider the composition of two morphisms of SW, that is a commutative diagram of the following form.

$$\begin{tikzcd}
(X,i) \arrow[r, "{[n,f]}"] \arrow[rr, bend right, "{[l,h]}"'] & (Y,j) \arrow[r, "{[m,g]}"] & (Z,k)
\end{tikzcd} \tag{16}$$

Then there is an $N \in \mathbb{Z}$ even and big enough, so as to make the diagram below commute in $\mathcal{H}$.

$$\begin{tikzcd}
\Sigma^{N+i}X \arrow[r, "\Sigma^{N-n}f"] \arrow[rr, bend right, "\Sigma^{N-l}h"'] & \Sigma^{N+j}Y \arrow[r, "\Sigma^{N-m}g"] & \Sigma^{N+k}Z
\end{tikzcd}$$

Now apply **(H6)** to this composition of morphisms and fit the yielded information into a "flat octahedron" shape as in (TR4):

$$\begin{tikzcd}
 & & V \arrow[dr, "s'"] \arrow[ddll, dotted, bend right, "h''"'] & & \\
 & U \arrow[ur, "s"] \arrow[dl, dotted, "f''"'] & & W \arrow[ll, dotted, "s''"'] \arrow[dl, dotted, "g''"'] & \\
\Sigma^{N+i}X \arrow[rr, "\Sigma^{N-n}f"'] \arrow[rrrr, bend right, "\Sigma^{N-l}h"'] & & \Sigma^{N+j}Y \arrow[ul, "f'"'] \arrow[rr, "\Sigma^{N-m}g"'] & & \Sigma^{N+k}Z \arrow[ul, "g'"'] \arrow[uull, bend right, "h'"']
\end{tikzcd} \tag{17}$$

More precisely, by **(H6)** one gets shortened cofiber sequences

$$\Sigma^{N+i}X \xrightarrow{\Sigma^{N-n}f} \Sigma^{N+j}Y \xrightarrow{f'} U \xrightarrow{f'':=\partial} \Sigma^{N+i+1}X$$

$$\Sigma^{N+j}Y \xrightarrow{\Sigma^{N-m}g} \Sigma^{N+k}Z \xrightarrow{g'} W \xrightarrow{g'':=\partial} \Sigma^{N+j+1}Y$$

$$\Sigma^{N+i}X \xrightarrow{\Sigma^{N-l}h} \Sigma^{N+k}Z \xrightarrow{h'} V \xrightarrow{h'':=\partial} \Sigma^{N+i+1}X$$

and

$$U \xrightarrow{s} V \xrightarrow{s'} W \xrightarrow{s'':=\partial} \Sigma U$$

(where the ΣU-coaction on W is the composite

$$\nu_s : \; W \xrightarrow{\nu_g} W \vee \Sigma(\Sigma^{N+j}Y) \xrightarrow{1\vee\Sigma f'} W \vee \Sigma U \tag{18}$$

where ν_g is the $\Sigma(\Sigma^{N+j}Y)$-coaction on W), such that

$$s'h' = g' \qquad \text{and} \qquad h' \circ \Sigma^{N-m}G = sf' \tag{19}$$

and such that:

$$\begin{aligned} &s \text{ is } \Sigma^{N+i+1}1_X\text{-equivariant,} \\ &s' \text{ is } \Sigma^{N-n+1}f\text{-equivariant.} \end{aligned} \tag{20}$$

Now we check that the canonical image of (17) is an octahedron in SW. The above shortened cofiber sequences become the required distinguished triangles. The equations (19) make the upper right 'triangle'[3] and the 'right square' commute. Because of Remark (3.2), (20) means that we have the following commuting diagram, which in SW becomes two morphisms of distinguished triangles.

$$\begin{tikzcd}
\Sigma^{N+i}X \arrow[r, "\Sigma^{N-n}f"] \arrow[d, equal] & \Sigma^{N+j}Y \arrow[r, "f'"] \arrow[d, "\Sigma^{N-m}g"'] & U \arrow[r, "f''"] \arrow[d, "s"] & \Sigma^{N+i+1}X \arrow[d, equal] \\
\Sigma^{N+i}X \arrow[r, "\Sigma^{N-l}h"'] \arrow[d, "\Sigma^{N-n}f"'] & \Sigma^{N+k}Z \arrow[r, "h'"'] \arrow[d, equal] & V \arrow[r, "h''"'] \arrow[d, "s'"] & \Sigma^{N+i+1}X \arrow[d, "\Sigma(\Sigma^{N-n}f)"] \\
\Sigma^{N+j}Y \arrow[r, "\Sigma^{N-m}g"'] & \Sigma^{N+k}Z \arrow[r, "g'"'] & W \arrow[r, "g''"'] & \Sigma^{N+j+1}Y
\end{tikzcd}$$

Because in particular the two rightmost squares commute, we have

$$h''s = f'' \qquad \text{and} \qquad g''s' = (\Sigma^{N-n+1}f) \circ h'' \, ,$$

i.e. in (17) also the left 'triangle' and the left 'square' are commutative.
We are left with checking that the inner 'triangle' of the canonical image of (17) commutes in SW, i.e. that

$$\iota s'' = \iota(\Sigma f' \circ g'') \, .$$

[3] Meaning here a diagram shaped as a triangle. I apologize for the awkwardness of the terminology.

To see this, it suffices to calculate explicitly $\iota s''$ in SW, using (18) and the fact that $\iota\nu_g = \binom{1}{g''}$:

$$\iota s'' = \iota\Big(W \xrightarrow{\nu_s} W \vee \Sigma U \xrightarrow{(0,1)} \Sigma U \Big)$$

$$= \Big(\iota W \xrightarrow{\binom{1}{\iota g''}} \iota W \oplus \iota\Sigma(\Sigma^{N+j}Y) \xrightarrow{\begin{pmatrix}1 & \\ & \iota\Sigma f'\end{pmatrix}} \iota W \oplus \iota\Sigma U \xrightarrow{(0\ 1)} \iota\Sigma U \Big)$$

$$= (0\ 1)\begin{pmatrix}1 & \\ & \iota\Sigma f'\end{pmatrix}\binom{1}{\iota g''} = \iota(\Sigma f' \circ g'')\ .$$

So now we have proven that the image of (17) in SW is an octahedron for the N^{th} suspension of the composition (16). To get an octahedron for (16), it suffices to desuspend N times the one we have. In fact, it is true in general that an octahedron which is suspended or desuspended an even number of times remains an octahedron. Indeed, the required commutativity of the 'squares' and 'triangles' is clear, and the fact that the distinguished triangles remain distinguished when suspended or desuspended an even number of times is true in every triangulated category. For instance, see what happens when one suspends twice a distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} \Sigma A$ (compare with Lemma (6.6)!). What one gets is $\Sigma^2 A \xrightarrow{\Sigma^2 u} \Sigma^2 B \xrightarrow{\Sigma^2 v} \Sigma^2 C \xrightarrow{\Sigma^2 w} \Sigma^3 A$, which is the same as the original triangle rotated six times to the right, and so it is distinguished by (TR2) (here Lemma (6.8)). □

With this last lemma, the proof of Theorem (6.5) is completed.

Part II
Application to homotopical algebra

The second part of this work is devoted to prove that the homotopy category of a pointed model category, together with its suspension endofunctor and its collection of cofiber sequences (defined in Section 8), satisfies the conditions **(H0)-(H6)**, although the “proofs” will mainly consist of pointers to the literature. Once we succeed in this, Theorem (6.5) applies, showing that the Spanier-Whitehead category obtained by inverting the suspension of the homotopy category of a model category is always triangulated.

The material of the next section originates from Quillen's paper [4], and was inspired by the tools of homotopy in algebraic topology, whence the terminology. Modern presentations of this notions can be found in e.g. Dwyer and Spalinsky [1] and Hovey [2].

7 Model categories

This section is just a collection of the very basic definitions and the very first results in the theory of model categories, and it serves to fix notation (in the broadest sense of the word). The more experienced reader may want to skip it. The more unexperienced reader is strongly advised to consult also the very readable introductory paper by Dwyer and Spalinski [1]. The middle reader may go on reading.

Definition 7.1. Given two maps $i : A \to B$ and $p : X \to Y$ in a category $\mathcal{C}$, one says that i *has the left lifting property (LLP) with respect to* p, or that p *has the right lifting property (RLP) with respect to* i if for all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

there exists a map $h : B \to X$ which commutes with the rest of the diagram. Given a collection $\mathcal{K}$ of maps in $\mathcal{C}$, denote by $\mathrm{LLP}(\mathcal{K})$ the collection of maps having the left lifting property with respect to all k in $\mathcal{K}$. Dually, denote by $\mathrm{RLP}(\mathcal{K})$ the collection of maps having the right lifting property with respect to all k in $\mathcal{K}$.

Definition 7.2 (Model category)**.** A *model category* is a quadruple $\mathcal{M} = (\mathcal{M}, \mathcal{W}eq, \mathcal{C}of, \mathcal{F}ib)$, where $\mathcal{M}$ is a category and $\mathcal{W}eq$, $\mathcal{C}of$ and $\mathcal{F}ib$ are classes of morphisms of $\mathcal{M}$ called *weak equivalences*, *cofibrations* and *fibrations* respectively, and satisfying the four axioms (MC1)-(MC5) stated below. In diagrams, it is usual to denote weak equivalences by $\xrightarrow{\sim}$, cofibrations by $\rightarrowtail$ and fibrations by $\twoheadrightarrow$.

(MC1) The category $\mathcal{M}$ has all (small) limits and colimits, in particular products $X \times Y$, coproducts $X \vee Y$, pullbacks $X \times_Z Y$, pushouts $X \vee_Z Y$, an initial object 0 and a final object 1.

(MC2) The class $\mathcal{W}eq$ satisfies the 2-out-of-3 property: Given a composition $g \circ f$, if two out of f, g and $g \circ f$ are weak equivalences, so is the third.

(MC3) The classes $\mathcal{W}eq$, $\mathcal{C}of$ and $\mathcal{F}ib$ are closed under retract. That is, if f is a *retract* of g, i.e., if there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

such that the two rows are the identity, and if g is in one of the classes, so is f.

(MC4) *(a)* $\mathcal{C}of \subseteq \mathrm{LLP}(\mathcal{W}eq \cap \mathcal{F}ib)$
(b) $\mathcal{F}ib \subseteq \mathrm{RLP}(\mathcal{W}eq \cap \mathcal{C}of)$

In words, if one calls a map in $\mathcal{W}eq \cap \mathcal{F}ib$ a *trivial fibration* and a map in $\mathcal{W}eq \cap \mathcal{C}of$ a *trivial cofibration*, (a) says that cofibrations have the left lifting property with respect to trivial fibrations, and (b) says that fibrations have the right lifting property with respect to trivial cofibrations.

(MC5) Every morphism f of $\mathcal{M}$ can be factorized as
(a) $f = p \circ i$, a cofibration i followed by a trivial fibration p, and also as
(b) $f = p \circ i$, a trivial cofibration followed by a fibration.

We assume that these factorisations are functorial in f (see [2] Def. 1.1.1). In particular, in both (a) and (b) a morphism of maps f and g, that is a commutative square as the one below on the left, induces the dotted map which 'links' the factorisations of f and g by making the diagram on the right commute.

$$\begin{tikzcd}
\cdot \arrow[r, "f"] \arrow[d] & \cdot \arrow[d] \\
\cdot \arrow[r, "g"] & \cdot
\end{tikzcd}
\qquad \rightsquigarrow \qquad
\begin{tikzcd}
\cdot \arrow[rr, "f"] \arrow[dr] \arrow[dd] & & \cdot \arrow[dd] \\
& \cdot \arrow[ur] \arrow[dd, dotted] & \\
\cdot \arrow[rr, "g"] \arrow[dr] & & \cdot \\
& \cdot \arrow[ur] &
\end{tikzcd}$$

The definition of a model category may vary in different authors. The one given above is that of Hovey [2] and is slightly stronger than those of [4] and [1].

The following two lemmata are the basic tools for proving things in a model category.

Lemma 7.3. (i) $\mathcal{C}of = \mathrm{LLP}(\mathcal{W}eq \cap \mathcal{F}ib)$
(ii) $\mathcal{W}eq \cap \mathcal{C}of = \mathrm{LLP}(\mathcal{F}ib)$
(iii) $\mathcal{F}ib = \mathrm{RLP}(\mathcal{W}eq \cap \mathcal{C}of)$
(iv) $\mathcal{W}eq \cap \mathcal{F}ib = \mathrm{RLP}(\mathcal{C}of)$

Lemma 7.4. (i) *The classes* $\mathcal{C}$of *and* $\mathcal{W}eq \cap \mathcal{C}of$ *are closed under taking pushouts.*
(ii) *The classes* $\mathcal{F}ib$ *and* $\mathcal{W}eq \cap \mathcal{F}ib$ *are closed under taking pullbacks.*

The following definition identifies objects which turn out to be the "right ones" for doing homotopy, as we shall see below.

Definition 7.5. An object X of $\mathcal{M}$ is said to be *cofibrant* if the unique map $0 \to X$ is a cofibration. Dually, X is *fibrant* if the unique map $X \to 1$ is a fibration. The cofibrant objects, the fibrant objects, and objects which are both fibrant and cofibrant in $\mathcal{M}$ determine three full subcategories, denoted respectively by $\mathcal{M}_c$, $\mathcal{M}_f$ and $\mathcal{M}_{cf}$.

Applying the functorial factorisation (MC5)(a) to the the unique map $0 \to X$ yields

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, tail] & & X \\ & QX \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

The object QX is cofibrant and is naturally weakly equivalent to X (i.e., there is a natural map $QX \to X$ which is a weak equivalence). One calls the functor $X \mapsto QX$ from $\mathcal{M}$ to $\mathcal{M}_c$ *cofibrant replacement functor.*

Dually, applying the factorization (MC5)(b) to the unique map $X \to 1$ provides a functor $X \mapsto RX$, called *fibrant replacement functor*, from $\mathcal{M}$ to $\mathcal{M}_f$, with the property that RX is naturally weakly equivalent to X. The composition RQ obviously lands in $\mathcal{M}_{cf}$ and has the property that RQX is weakly equivalent to X.

Definition 7.6. Let A be an object of $\mathcal{H}$. A *cylinder object* for A is an object A' together with a diagram

$$\begin{tikzcd} A \vee A \arrow[r, tail, "(i_0,i_1)"] & A' \arrow[r, "s", "\sim"'] & A \end{tikzcd}$$

such that $s \circ (i_0, i_1) = (1, 1)$. Using (MC5)(a) to factorize $(1, 1) : A \vee A \to A$ yields a functorial cylinder object which is denoted by IA. In particular, there is always a cylinder object for any A. Two parallel maps $f, g : A \to X$ are said to be *left homotopic* (written $f \sim_l g$) if for some cylinder object A' for A there is a *left homotopy from f to g*, that is a map $H : A' \to X$ such that $H \circ (i_0, i_1) = (f, g)$.

Dually, a path object for X is an object X'' together with a diagram

$$\begin{tikzcd} X \arrow[r, "t", "\sim"'] & X'' \arrow[r, two heads, "(p_0,p_1)"] & X \times X \end{tikzcd}$$

such that $(p_0, p_1) \circ t = (1, 1)$. Applying (MC5)(b) to the diagonal map $(1, 1) : X \to X \times X$ yields a functorial path object PX. Then $f, g : A \to X$ are *right homotopic* ($f \sim_r g$) if for some path object X'' there is a *right homotopy* from f to g, i.e. a map $K : A \to X''$ such that $(p_0, p_1) \circ K = (f, g)$.

The following proposition explains why fibrant and cofibrant objects are nice, and it shows that the above defined relations are meaningful.

Proposition 7.7. *If A is cofibrant and X fibrant, the left homotopy relation and the right homotopy relation on $\mathcal{H}(A, X)$ coincide and are an equivalence relation, called* homotopy relation *and denoted by $\sim$. Moreover, $\sim$ is compatible with composition.*

Definition 7.8. Let $\mathcal{C}$ be an arbitrary category, and let $\mathcal{W}$ be a class of morphisms therein. A *localisation of $\mathcal{C}$ with respect to $\mathcal{W}$* is a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$

out of $\mathcal{C}$ such that
(a) γ carries morphisms of $\mathcal{W}$ to isomorphisms
(b) γ is universal for property (a): If $\alpha : \mathcal{C} \to \mathcal{D}$ is a functor such that $\gamma(f)$ is an isomorphism for all $f \in \mathcal{W}$, then there exists a unique functor $\overline{\alpha} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $\overline{\alpha}\gamma = \alpha$.

In the general case, the localisation can always be constructed using generators and relations, but nothing guarantees that the class of arrows from an object to another be a *set*. In a model category, the localisation with respect to the class of weak equivalences can be realized by means of the above defined homotopy relation, as the following theorem states.

Theorem 7.9. *If $\mathcal{M}$ is a model category, then the localisation of $\mathcal{M}$ with respect to the class of weak equivalences, called* the homotopy category of $\mathcal{M}$ *and denoted by* $\mathrm{Ho}\mathcal{M} := \mathcal{M}[\mathcal{W}eq^{-1}]$*, exists and is equivalent to the category $\mathcal{M}_{cf}/\sim$. The localizing functor γ is induced by the composition of the cofibrant and fibrant replacement functors.*

Remark 7.10. One should think of $\mathrm{Ho}\mathcal{M}$ as the category with the same objects as $\mathcal{M}$ and with maps $\mathrm{Ho}\mathcal{M}(X,Y) = \mathcal{M}(RQX, RQY)/\sim$. If A is cofibrant and B is fibrant, one has a natural isomorphism $\mathrm{Ho}\mathcal{M}(A,B) \simeq \mathcal{M}(A,B)/\sim$.

The homotopy category $\mathrm{Ho}(\mathcal{M})$ of a model category $\mathcal{M}$ is not just *any* category, but comes equipped with much extra structure. This structure, together with $\mathrm{Ho}(\mathcal{M})$, is often called the "homotopy theory" of $\mathcal{M}$. In the case the model category $\mathcal{M}$ is pointed, this begins with the suspension and the loop functors, with which one defines cofiber and fiber sequences; they are described in the next section. More in general one can define homotopy pushouts and homotopy pullbacks (see [1]), but there is more. Hovey [2] has proven that the homotopy category of a model category is naturally equipped with a closed action of the homotopy category of simplicial sets. Further on, we'll need to see a glimpse of this.

However, perhaps the first thing to remark about the homotopy category of a model category is that it inherits the products and coproducts of $\mathcal{M}$ ([2] Example 1.3.3, Prop. 1.3.5), in particular the empty product and coproduct, i.e. the initial object 0 and the final object 1. If $\mathcal{M}$ is pointed, i.e. $0 \simeq 1$, then $0 \simeq 1$ also in $\mathrm{Ho}\mathcal{M}$, thus also $\mathrm{Ho}\mathcal{M}$ is pointed. This takes care of the first half of **(H0)(a)**.

Let $F : \mathcal{M} \to \mathcal{C}$ be a functor from a model category. One of the prominent features of model categories is that, under mild assumptions, there exists the *left (resp. right) derived functor* LF (resp. RF): $\mathrm{Ho}\mathcal{M} \to \mathcal{C}$, which, up to natural transformation from one side or the other, are the best possible approximation to a factorisation of F through $\gamma : \mathcal{M} \to \mathrm{Ho}\mathcal{M}$. If $\mathcal{C}$ is also a model category, one defines the *total left (right) derived functor* of F as being the left (right) derived functor of $\gamma_{\mathcal{C}} \circ F : \mathcal{M} \to \mathrm{Ho}\mathcal{C}$. We will not go into detail here. It suffices to know the following

Example 1. The adjunction $|-| : \mathrm{s}\mathcal{S}\mathrm{et}_* \Longleftrightarrow \mathcal{T}\mathrm{op}_* : \mathrm{Sing}$ of the geometric realisation functor and the singular functor between the category of pointed simplicial sets and the category of pointed topological spaces (with the usual model structures, see [2] §2.4,§3) induces an adjoint equivalence of the total derived functors $L|-| : \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_* \Longleftrightarrow \mathrm{Ho}\,\mathcal{T}\mathrm{op}_* : R\mathrm{Sing}$ (see [2] Cor. 2.4.24,Thm. 3.6.7). Moreover, since $|-|$ preserves weak equivalences (by the very definition of the model structure on simplicial sets), the functor $\gamma \circ |-| : \mathrm{s}\mathcal{S}\mathrm{et}_* \to \mathrm{Ho}\,\mathcal{T}\mathrm{op}_*$ sends weak equivalences to isomorphisms, hence it does factorize through $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*$ and this factorisation is the same as $L|-|$. One writes $|-| = L|-| : \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_* \to \mathrm{Ho}\,\mathcal{T}\mathrm{op}_*$.

Such adjunctions of model categories, which induce adjoint equivalences on the homotopy categories, preserve all the known extra structure ("homotopy theory") of the respective categories. They are called *Quillen equivalences*, and are to be considered as a kind of 'weak equivalences' of model categories, in some hypothetical higher-order homotopy theory.

8 The homotopy category of a pointed model category

As we have seen, a model category $\mathcal{M}$ always has an initial object 0 and a final object 1. If the unique map $0 \to 1$ is an isomorphism, $\mathcal{M}$ is pointed. This is not always the case, but there is a simple procedure to construct a pointed category out of any category. Define $\mathcal{M}_*$ to be the category $\mathcal{M}$ under the terminal object 1, where an object of $\mathcal{M}_*$ is a map $x : 1 \to X$ of $\mathcal{M}$ (usually written (X, x)) and where a morphism $f : (X, x) \to (Y, y)$ of $\mathcal{M}_*$ is a morphism $f : X \to Y$ of $\mathcal{M}$ such that $f(x) = y$. Clearly the category $\mathcal{M}_*$ is pointed, and it inherits the limits, the colimits and the model structure of $\mathcal{M}$ in a straightforward way (one must just pay some attention to the recovery of the colimits; see [2] Prop. 1.1.8 and preceding remarks).

Thus we can assume that $\mathcal{M}$ is a *pointed* model category, without losing generality. We shall use the notation $\mathcal{H} := \mathrm{Ho}\mathcal{M}$ for the homotopy category of $\mathcal{M}$, which is also pointed.

Definition 8.1. Let $f : X \to Y$ be a map in $\mathcal{M}$. The *cofiber* of f is the map $g : Y \to Z$ in the pushout square below.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] & Z \end{tikzcd}$$

The *fiber* of f is the map $h : W \to X$ in the pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "h"] & * \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Equivalently, one can define the cofiber (resp. fiber) of f as the coequalizer (resp. equalizer) of f and the zero map $0 : X \to * \to Y$.

Theorem 8.2. [Quillen [4], Thm. 2, §I 2.9] *There is a functor*

$$(\mathcal{H})^{op} \times \mathcal{H} \longrightarrow \mathcal{G}\mathrm{rp}$$

$$(X, Y) \longmapsto [X, Y]_1$$

taking pairs of objects of the homotopy category to groups, which is uniquely determined up to canonical isomorphism by:

$$[X, Y]_1 = \pi_1(A, B) \quad \textit{for } A \textit{ fibrant and } B \textit{ cofibrant.}$$

Moreover, there are two functors $\Sigma, \Omega : \mathcal{H} \to \mathcal{H}$*, called the* suspension *and the* loop functor*, and natural equivalences, such that*

$$\mathcal{H}(\Sigma X, Y) \simeq \pi_1(X, Y) \simeq \mathcal{H}(X, \Omega Y) \ .$$

In particular, Σ *and* Ω *are adjoint and the functor* $[-,-]_1$ *is representable in* $\mathcal{H}$ *in both variables.*

The above group $\pi_1(A, B)$ for A fibrant and B cofibrant is constructed by Quillen as the set of homotopy classes of (left) homotopies from $0 : A \to B$ to itself (for an adequate notion of "homotopy of homotopies"). This turns out to be equipped with a group structure which is natural as A runs over $\mathcal{M}_c$ and B over $\mathcal{M}_f$, and the group homomorphisms induced in the first and the second variable commute, so that $\pi_1(-,-)$ is a functor from $(\mathcal{M}_c)^{op} \times \mathcal{M}_f$ to $\mathcal{G}$rp. The suspension Σ is first defined on $\mathcal{M}_c$ as the cofiber $IA \to \Sigma A$ of the map $(i_0, i_1) : A \vee A \to IA$ (where $A \vee A \xrightarrow{(i_0,i_1)} IA \xrightarrow{\sim} A$ is the functorial cylinder object of A). Thus, by definition, a map from ΣA to some B corresponds uniquely to a homotopy $IA \to B$ from $0 : A \to B$ to itself. Dually, the loop functor Ω is first defined on $\mathcal{M}_f$ as the fiber $\Omega B \to PB$ of the map $(p_0, p_1) : PB \to B$ of the path object. The two functors of the theorem are essentially the total left derived functor of this Σ and the total right derived functor of this Ω.

By the above theorem, the suspension Σ is left adjoint to Ω, so in particular it preserves the coproducts and the zero object of $\mathcal{H}$. This is the second part of **(H0)(a)**.

From now on we proceed by following Hovey [2]. As hinted in the previous section, he proves ([2] Thm. 5.6.2, Thm. 5.7.3) that the homotopy category $\mathcal{H}$ is naturally equipped with a closed (right) action of the homotopy category $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*$ of the category of pointed simplicial sets (with the standard model structure, see [2] §3). This action is basically just a functor

$$(\mathcal{H})^{op} \times \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{H}$$

$$(X, K) \longmapsto X \wedge K$$

together with an 'associativity' and a 'unit' natural isomorphisms

$$(X \wedge K) \wedge L \simeq X \wedge (K \wedge L)$$

$$X \wedge * \simeq X$$

such that some compatibility diagrams commute. The $\wedge$ in $K \wedge L$ is the smash product of pointed simplicial sets, while $*$ is the one point simplicial set. These serve respectively as multiplication and unit of a monoidal structure on $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*$ ([2] Prop. 4.2.8-9). The action being "closed" means that there are two functors Hom and Map and natural isomorphisms

$$\mathcal{H}(X, \mathrm{Hom}(K, L)) \simeq \mathcal{H}(X \wedge K, Y) \simeq \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*(K, \mathrm{Map}(X, Y)) \ .$$

(Hom and Map should be thought of as functorial 'hom objects', or 'function spaces', the first one living inside of $\mathcal{H}$ and the second one in $\operatorname{Ho} s\mathcal{S}\mathrm{et}_*$.) In particular, for all $X \in \mathrm{Ob}\mathcal{H}$ and $K \in \mathrm{Ob}\operatorname{Ho} s\mathcal{S}\mathrm{et}_*$ there are adjunctions

$$- \wedge K : \mathcal{H} \Longleftrightarrow \mathcal{H} : \mathrm{Hom}(K, -)$$

$$X \wedge - : \operatorname{Ho} s\mathcal{S}\mathrm{et}_* \Longleftrightarrow \mathcal{H} : \mathrm{Map}(X, -) \ .$$

Thus, since $- \wedge K$ and $X \wedge -$ have a right adjoint they preserve colimits, and in particular coproducts. This yields two more 'distributivity' natural isomorphisms

$$(X \vee Y) \wedge K \simeq (X \wedge K) \vee (Y \wedge K)$$

$$X \wedge (K \vee L) \simeq (X \wedge K) \vee (X \wedge L)$$

(see [2] §4 for details).

Hovey defines the suspension on $\mathcal{H}$ by the action of the pointed simplicial circle S^1, $\Sigma(-) := - \wedge S^1$, and the loop functor by $\Omega(-) := \mathrm{Hom}(S^1, -)$. A close inspection shows that these definitions are equivalent to Quillen's (see [2] Def. 6.1.1 and the following remarks).

Proposition 8.3. *$\Sigma^n X$ is a natural cogroup object in $\mathcal{H}$ for $n \geq 1$, abelian for $n \geq 2$. Dually, $\Omega^n X$ is a natural group object for $n \geq 1$, abelian for $n \geq 2$. In particular, $(\mathcal{H}, \Sigma)$ satisfies condition* **(H0)(b)**.

Proof: [cf. [2] Cor. 6.1.6, with notation of §6.5] By Proposition (2.6) and Remark (2.7) we must show that for all $X, Y \in \mathrm{Ob}\mathcal{H}$ and $n \geq 1$ the set $\mathcal{H}(\Sigma^n X, Y)$ is a group (abelian for $n \geq 2$) and that the maps induced in both variables are group homomorphisms. This would imply the part about Ω, because by composing the adjunction natural isomorphism (bijection) we get a natural isomorphism of sets $\mathcal{H}(\Sigma^n X, Y) \simeq \mathcal{H}(X, \Omega^n Y)$, by which we can transport the natural group structure from the left to the right. One has the following natural bijections:

$$\begin{aligned}
\mathcal{H}(\Sigma^n X, Y) &\simeq \mathcal{H}(X \wedge S^n, Y) \\
&\simeq \operatorname{Ho} s\mathcal{S}\mathrm{et}_*(S^n, \mathrm{Map}(X, Y)) \\
&\simeq \operatorname{Ho} \mathcal{T}\mathrm{op}_*(|S^n|, |\mathrm{Map}(X, Y)|) \\
&\simeq \operatorname{Ho} \mathcal{T}\mathrm{op}_*(S^n, Z) \\
&\simeq \pi_n(Z) \ .
\end{aligned}$$

The first natural bijection is the repeated application of the associativity of the $\operatorname{Ho} s\mathcal{S}\mathrm{et}_*$-action ($S^n$ here is the pointed simplicial n-sphere); the second one is the closedness of the action (see above); the third one is the adjoint equivalence between $\operatorname{Ho} s\mathcal{S}\mathrm{et}_*$ and $\operatorname{Ho} \mathcal{T}\mathrm{op}_*$ of Example (1) of the previous section; the next one (now S^n denotes the n-sphere as a pointed topological space, Z is just an abbreviation for the pointed topological space $|\mathrm{Map}(X, Y)|$) is induced by the homeomorphism $|S^n| \simeq S^n$; now, S^n is cofibrant in $\mathcal{T}\mathrm{op}_*$ and every object is fibrant, in particular Z. Hence by Remark (7.10) $\operatorname{Ho} \mathcal{T}\mathrm{op}_*(S^n, Z) \simeq \mathcal{T}\mathrm{op}_*(S^n, Z)/\sim$, which is by definition the usual homotopy group $\pi_n(Z)$ of algebraic topology. (Of course, in the model structure of $\mathcal{T}\mathrm{op}_*$ the homotpy equivalence relation $\sim$ as defined in the previous section corresponds to the usual homotopy relation rel basepoint.)

It is a well known fact of algebraic topology that the homotopy groups $\pi_n(Z)$ are natural groups, abelian for $n \geq 2$ (see e.g. Spanier [5] §1.6,§7.2). This shows that $\mathcal{H}(\Sigma^n X, Y)$ is a group. Maps f into or out of X and Y induce maps f_* or f^* into or out of $\mathcal{H}(\Sigma^n X, Y)$, which under the above composition of natural equivalences correspond to some g_* or g^* into or out of $\pi_n(Z)$, which are group homomorphisms by the naturality of the homotopy groups. $\square$

The greatest difficulty we met while proving Theorem (1.1) was that of finding a proof to the following proposition. The one we found ultimately boils down to translating back the abstract suspension functor Σ of Quillen to the classical suspension homomorphism of algebraic topology. In order to do this, we heavily used the closedness of the $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*$-action on $\mathcal{H}$; any attempt to avoid this failed miserably.

Proposition 8.4. *The homotopy category $\mathcal{H}$ of a pointed model category and its suspension Σ satisfy condition* **(H0)(c)**.

Proof: We prove **(H0)(c')** (see Remark (3.1)). In order to do this, it suffices to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{H}(X \wedge S^n, Y) \arrow[r, "\phi", "\simeq"'] \arrow[dd, "-\wedge S^1"'] & \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*(S^n, \mathrm{Map(X,Y)}) \arrow[d, "-\wedge S^1"] \\
 & \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*(S^{n+1}, \mathrm{Map}(X,Y) \wedge S^1) \arrow[d, "k_*"] \\
\mathcal{H}(X \wedge S^{n+1}, Y \wedge S^1) \arrow[r, "\phi", "\simeq"'] & \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*(S^{n+1}, \mathrm{Map}(X, Y \wedge S^1))
\end{tikzcd}$$

Let's see why this is true. (In this diagram, as well as in the rest of the proof, we shall consider the associativity isomorphism of the $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*$-action as equality.) The horizontal arrows are given by the natural bijection $\phi_{K,Y}$: $\mathcal{H}(X \wedge K, Y) \simeq \mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*(K, \mathrm{Map(X,Y)})$ given by the adjunction $X \wedge - :$ $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_* \Longleftrightarrow \mathcal{H} : \mathrm{Map(X,-)}$, and are group isomorphisms by definition (see the proof of Prop. (8.3)). The map k_* is induced by a map $k : \mathrm{Map}(X,Y) \wedge S^1 \to \mathrm{Map}(X, Y \wedge S^1)$ which we shall define below, and is a group homomorphism by naturality of the homotopy groups in $\mathrm{Ho}\,\mathrm{s}\mathcal{S}\mathrm{et}_*$ (see again the proof of Prop. (8.3)). The $-\wedge S^1$ on the right is easily seen to be a group homomorphism by translating it, via the Quillen equivalence of Example (1), into the classical "suspension homomorphism" $\Sigma : \pi_n(Z) \to \pi_{n+1}(\Sigma Z)$ of algebraic topology ([5] §8.5). If the diagram commutes, the $- \wedge S^1$ on the left (i.e., our suspension Σ) would be also a group homomorphism, and **(H0)(c')** would be proved.

Instead of proving that the above diagram commutes, we shall prove that the more general one below does, where S^n resp. S^1 are replaced by two arbitrary pointed simplicial sets K resp. L.

$$\begin{tikzcd}
\mathcal{H}(X \wedge K, Y) \arrow[rr, "\phi", "\simeq"'] \arrow[dd, "-\wedge L"'] & & \mathrm{Ho}\, \mathrm{s}\mathcal{S}\mathrm{et}_*(K, \mathrm{Map(X, Y)}) \arrow[d, "-\wedge L"] \\
& & \mathrm{Ho}\, \mathrm{s}\mathcal{S}\mathrm{et}_*(K \wedge L, \mathrm{Map}(X, Y) \wedge L) \arrow[d, "k_*"] \\
\mathcal{H}(X \wedge K \wedge L, Y \wedge K \wedge L) \arrow[rr, "\phi", "\simeq"'] & & \mathrm{Ho}\, \mathrm{s}\mathcal{S}\mathrm{et}_*(K \wedge L, \mathrm{Map}(X, Y \wedge L))
\end{tikzcd} \tag{21}$$

Recall now that the natural bijection ϕ can be completely recovered by the unit $\eta : \mathrm{Id}_{\mathrm{Ho}\, \mathrm{s}\mathcal{S}\mathrm{et}_*} \to \mathrm{Map}(X, X \wedge -)$ and counit $\epsilon : X \wedge \mathrm{Map}(X, -) \to \mathrm{Id}_{\mathcal{H}}$ natural maps of the adjunction, as follows:

$$\phi\Big(X \wedge K \xrightarrow{f} Y \Big) = \Big(K \xrightarrow{\eta_K} \mathrm{Map(X, X \wedge K)} \xrightarrow{\mathrm{Map}(X,f)} \mathrm{Map}(X, Y) \Big)$$

$$\phi^{-1}\Big(K \xrightarrow{g} \mathrm{Map}(X, Y) \Big) = \Big(X \wedge K \xrightarrow{X \wedge g} X \wedge \mathrm{Map}(X, Y) \xrightarrow{\epsilon_Y} Y \Big).$$

With this notation, we define the map $k = k_{X,Y,L}$ to be

$$\mathrm{Map}(X, \epsilon_Y \wedge L) \circ \eta_{\mathrm{Map}(X,Y) \wedge L} : \mathrm{Map}(X, Y) \wedge L \longrightarrow \mathrm{Map}(X, Y \wedge L).$$

This looks rather complicated, but in fact it is just the thing one gets when trying to obtain, out of the identity map $\mathrm{Map(X, Y)} \to \mathrm{Map(X, Y)}$, some map from $\mathrm{Map(X, Y)} \wedge L$ to $\mathrm{Map(X, Y \wedge L)}$. Now that we know explicitly all the maps of diagram (21), we can calculate the images of a map $h \in \mathcal{H}(X \wedge K, Y)$. Going first down and then to the right yields

$$\alpha \ := \ \phi(h \wedge L) \ = \ \mathrm{Map}(X, h \wedge L) \circ \eta_{K \wedge L}.$$

Going first right and then down yields the map

$$\begin{aligned}
\beta \ &:= \ k_*(\phi(h) \wedge L) \\
&= \ k_*((\mathrm{Map}(X, h) \circ \eta_K) \wedge L) \\
&= \ k_*((\mathrm{Map}(X, h) \wedge L) \circ (\eta_K \wedge L)) \\
&= \ \mathrm{Map}(X, \epsilon_Y \wedge L) \circ \eta_{\mathrm{Map}(X,Y) \wedge L} \circ (\mathrm{Map}(X, h) \wedge L) \circ (\eta_K \wedge L).
\end{aligned}$$

Now we must show that α and β are equal, which follows immediately from the commutativity of the diagram below (α and β are placed on the outer frame of the diagram).

$$\begin{tikzcd}
K \wedge L \arrow[r, "\eta_K \wedge L"] \arrow[dr, phantom, "③"] \arrow[ddr, bend right, "\eta_{K \wedge L}"'] & \mathrm{Map}(X, X \wedge K) \wedge L \arrow[r, "\mathrm{Map}(X,h) \wedge L"] \arrow[d, "\eta_{\mathrm{Map}(X, X \wedge K) \wedge L}"] \arrow[dr, phantom, "①"] & \mathrm{Map}(X, Y) \wedge L \arrow[d, "\eta_{\mathrm{Map}(X,Y) \wedge L}"] \\
& \mathrm{Map}(X, X \wedge \mathrm{Map}(X, X \wedge K) \wedge L) \arrow[r, "\mathrm{Map}(X, X \wedge \mathrm{Map}(X,h) \wedge L)"] \arrow[d, "\mathrm{Map}(X, \epsilon_{X \wedge K} \wedge L)"] \arrow[dr, phantom, "②"] & \mathrm{Map}(X, X \wedge \mathrm{Map}(X, Y) \wedge L) \arrow[d, "\mathrm{Map}(X, \epsilon_Y \wedge L)"] \\
& \mathrm{Map}(X, X \wedge K \wedge L) \arrow[r, "\mathrm{Map}(X, h \wedge L)"'] & \mathrm{Map}(X, Y \wedge L)
\end{tikzcd}$$

Square ① commutes because of the naturality of the unit η of the adjunction. The naturality of the counit ϵ makes the square

$$\begin{tikzcd} X \wedge \mathrm{Map}(X, X \wedge K) \arrow[r, "X\wedge \mathrm{Map}(X,h)"] \arrow[d, "\epsilon_{X\wedge K}"'] & X \wedge \mathrm{Map}(X, Y) \arrow[d, "\epsilon_Y"] \\ X \wedge K \arrow[r, "h"] & Y \end{tikzcd}$$

commutative. After applying to it the functor $\mathrm{Map}(X, - \wedge L)$, this becomes the commutative square ②. Let us copy here below (in solid arrows) the remaining square ③.

$$\begin{tikzcd} K \wedge L \arrow[r, "\eta_K \wedge L"] \arrow[d, "\eta_{K\wedge L}"'] \arrow[dr, phantom, "③"] & \mathrm{Map}(X, X \wedge K) \wedge L \arrow[d, "\eta_{\mathrm{Map}(X,X\wedge K)\wedge L}"] \\ \mathrm{Map}(X, X \wedge K \wedge L) \arrow[r, dotted, shift right, "\mathrm{Map}(X,X\wedge\eta_K\wedge L)"'] & \mathrm{Map}(X, X \wedge \mathrm{Map}(X, X \wedge K) \wedge L) \arrow[l, shift right, "\mathrm{Map}(X,\epsilon_{X\wedge K}\wedge L)"'] \end{tikzcd}$$

Recall now that, by any adjunction $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, one always has the so-called 'triangular identities' of natural maps $\epsilon_F \circ F\eta = \mathrm{Id}_F$ and $G\epsilon \circ \eta_G = \mathrm{Id}_G$ ([3] p. 80). In the case of our adjunction $X \wedge - : \mathrm{Ho}\, s\mathcal{S}\mathrm{et}_* \Longleftrightarrow \mathcal{H} : \mathrm{Map}(X, -)$, specializing the first identity for the object K yields $\epsilon_{X\wedge K} \circ (X \wedge \eta_K) = 1_{X\wedge K}$, which via application of the functor $\mathrm{Map}(X, - \wedge L)$ becomes the equality

$$\mathrm{Map}(X, \epsilon_{X\wedge K} \wedge L) \circ \mathrm{Map}(X, X \wedge \eta_K \wedge L) = 1_{\mathrm{Map}(X,X\wedge K\wedge L)} .$$

Thus, instead of considering square ③ we can as well consider the square obtained from ③ by forgetting the lower horizontal arrow, $\mathrm{Map}(X, \epsilon_{X\wedge K} \wedge L)$, and adding the dotted arrow $\mathrm{Map}(X, X \wedge \eta_K \wedge L)$. But this new square commmutes, once again, by the naturality of η. □

Hovey [2] dedicates chapter 6.2 to prove that in $\mathcal{H}$ there is a coaction $C \to C \vee \Sigma A$ of the natural group ΣA on the cofiber[4] C of a cofibration between cofibrant objects $f : A \to B$. Giving a name $g : B \to C$ to the cofiber, this can be written

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad C \longrightarrow C \vee \Sigma A \ . \tag{22}$$

Then *cofiber sequences* are defined as being all diagrams of the form

$$X \longrightarrow Y \longrightarrow Z \qquad Z \xrightarrow{\nu} Z \vee \Sigma X$$

where ν is also a coaction, which are isomorphic to such a (22), for some cofibration of cofibrant objects $f : A \to B$ (hence **(H1)** tautologically holds; here 'isomorphic' has the same meaning as in **(H1)**). A dual statement is true for fibrations between fibrant objects $f : A \to B$: there is an action $C \times \Omega B \to C$ of the natural group ΩB on the fiber C of f. The dual definition of *fiber sequence* is given. In the next chapter 6.3 the basic properties of cofiber and fiber

[4] Here we abuse notation, since the cofiber is properly a map $B \to C$.

sequences are proved. Among these are propositions which prove that the collection of cofiber sequences satisfies our conditions **(H2)-(H6)** (and that fiber sequences satisfy the dual conditions). Here are the exact references:

(H2) Lemma 6.3.2
(H3) Lemma 6.3.3
(H4) Prop. 6.3.4
(H5) Prop. 6.3.5
(H6) Prop. 6.3.6

Now we can finally apply Theorem (6.5) of Part I to the pair $(\mathcal{H}, \Sigma)$ consisting of the homotopy category of a pointed model category and its suspension Σ, thus proving Theorem (1.1) as stated in the introduction.

References

[1] W.G.Dwyer and J.Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology, pp. 73-126, North-Holland, Amsterdam (1995) (Also available on W.G.Dwyer's homepage, <www.nd.edu/∼wgd>, publication 75)

[2] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, Vol. 63, Am. Math. Soc. (1999)

[3] Saunders Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag (1971)

[4] Daniel Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, Vol. 43, Springer-Verlag (1967)

[5] E.H.Spanier, *Algebraic topology*, McGrow-Hill Book Co. (1966)

[6] J.-L.Verdier, *Catégories dérivées*, Cohomologie Etale (SGA $4\frac{1}{2}$) (P.Deligne, ed.), Springer Lecture Notes in Mathematics, Vol. 569, pp. 262-311 (1977)

[7] C.A.Weibel, *History of homological algebra*, author's homepage <www.math.rutgers.edu/∼weibel>