Categories of C*-algebras

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Abstract

This handout is part of the course *An introduction to C*-algebras*. The goal is to learn the language of categories, functors and natural transformations – using numerous examples encountered in the lectures – well enough to express Gelfand duality as an equivalence of categories. I have also included an appendix on Alexandroff compactification and a detailed discussion of Gelfand duality for non-compact spaces/ non-unital algebras.

1 Categories and functors

Definition 1 (Category)**.** A *category* $\mathcal{C}$ consists of the following data:

- a collection of "objects" $X, Y, Z, \ldots$
- for every two objects X, Y, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of "morphisms" $f, g, h, \ldots$ To indicate that $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we often write $f : X \to Y$.
- a "composition law" for morphisms, that is: for every three objects X, Y, Z of $\mathcal{C}$ we are given a function $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ which is typically written $(g, f) \mapsto g \circ f$ or just gf.
- for every object X, a distinguished morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, called the "identity of X".

Moreover, the following two axioms must hold:

- *Associativity*: $(h \circ g) \circ f = h \circ (g \circ f)$ for all $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$.
- *Identity*: $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$ for every $f : X \to Y$.

A morphism $f : X \to Y$ in $\mathcal{C}$ is called an *isomorphism* (or we say that f *is invertible*) if there is a morphism $g : Y \to X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Exercise 2. Note that the identity of each object is uniquely determined: if $f : X \to X$ and $g : X \to X$ both satisfy the identity axiom, then $f = f \circ g = g$. Similarly, use the axioms to show that if f is invertible, then its inverse g is unique. It is usually written f^{-1}.

Examples 3. 1. Given any category $\mathcal{C}$, we define the *opposite category* $\mathcal{C}^{\mathrm{op}}$ as the category with the same objects as $\mathcal{C}$, and with a morphism $f^{\mathrm{op}} \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y)$ for every $f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$. The composition in $\mathcal{C}^{\mathrm{op}}$ is given via that of $\mathcal{C}$ according to the formula $f^{\mathrm{op}} \circ g^{\mathrm{op}} := g \circ f$; then $\mathrm{id}_X^{\mathrm{op}}$ is the identity of X in $\mathcal{C}^{\mathrm{op}}$. Thus $\mathcal{C}^{\mathrm{op}}$ is the category obtained from $\mathcal{C}$ by considering the arrows (morphisms) as going in the opposite direction.

2. The collection of all sets $X, Y, \ldots$ and all functions $f, g, \ldots$ between sets form a category, denoted by **Set**, where composition and identities are the usual composition of functions and identity maps. The isomorphisms in **Set** are precisely the bijective functions.

3. The collection of groups and group homomorphisms form a category, **Gps**, again with the usual composition and identities. The isomorphisms of groups are the bijective homomophisms.

4. More generally, given a concept of algebraic structure (abelian group / ring / unital ring / K-vector space / R-module / etc.) we can form a category where the objects are sets equipped with a structure of the given type, and where morphisms are the functions that preserve the structure, in the appropriate sense.

5. The category **Top** of all topological spaces and continuous maps. Note that a bijective and continuous map is not necessarily an isomorphism in **Top**, because the (unique) inverse map may not be continuous.

6. The category **Cpt** of compact[1] spaces and continuous maps between them. By a well-known topological lemma, every bijective continuous map is an isomorphism in **Cpt**.

7. Let **LCpt** be the category of locally compact spaces, where morphisms are the *proper* continuous maps between them ($f : X \to Y$ is proper if $f^{-1}K$ is compact for every compact subset K of Y).

8. We denote by $\mathbf{Cpt}_\bullet$ the category of pointed compact spaces. The objects are pairs (X, x_0) where X is a compact space and $x_0 \in X$ a chosen "base-point". A morphism $f : (X, x_0) \to (Y, y_0)$ is by definition a continuous map $f : X \to Y$ preserving the base-point: $f(x_0) = y_0$.

9. (Optional.) So far, we have only seen "concrete" categories, where objects are sets equipped with structure and morphisms are some kind of function between them. But there are very interesting "abstract categories". For example, let G be a group. Let $\mathcal{C}$ be the category with a unique object $*$ and with morphism set given by $\mathrm{Hom}_{\mathcal{C}}(*, *) := G$. Composition is defined to be the group operation of G, and $\mathrm{id}_* = 1_G$ is the identity element of the group. We see in this way that groups are the same thing as categories with a unique object and where every morphism is invertible.

We can define at least four interesting categories of C*-algebras:

C*alg: objects are all C*-algebras and morphisms are all *-homomorphisms between them.

$\mathbf{C^*alg_1}$: objects are *unital* C*-algebras and morphisms are *unital* *-homomorphisms.

C*com: objects are *commutative* C*-algebras and morphisms are all *-homomorphisms between them.

[1]We use the convention that *compact* and *locally compact* always include the Hausdorff separation property.

C*com$_1$: objects are *unital commutative* C*-algebras and morphisms are *unital* *-homomorphisms between them.

Note that, in each category, a morphism is invertible iff (= if and only if) it is bijective as a function iff it is an isometric *-isomorphism – as we have proved in the lecture – that is, iff it identifies the norms as well as the algebraic structure. By definition though, it is required to identify the two multiplicative units only if it belongs to **C*alg**$_1$ or **C*com**$_1$, of course.

Definition 4 (Functor)**.** Let $\mathcal{C}, \mathcal{D}$ be two categories. A *functor F from $\mathcal{C}$ to $\mathcal{D}$*, written $F : \mathcal{C} \to \mathcal{D}$, is a function assigning to every object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and to every morphism $f : X \to Y$ in $\mathcal{C}$ a morphism $F(f) : F(X) \to F(Y)$ of $\mathcal{D}$, so that the composition law and all identities are preserved:

$$F(g \circ f) = F(g) \circ F(f) \quad \text{and} \quad F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is by definition a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, or equivalently, $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ (i.e., one that "inverts the direction of the arrows"). A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* if, for every two objects X, Y in $\mathcal{C}$, the function $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective. If they are all injective instead, F is said to be *faithful*. A *fully faithful* functor is one that is full and faithful.

Remark 5. If $F : \mathcal{C} \to \mathcal{D}$ is a functor and f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism with inverse $F(f^{-1})$.

Examples 6.

1. Every category $\mathcal{C}$ has an *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that leaves everything unchanged. Check the following fact: the collection of all categories and functors between them, with the evident composition law and the above identities, satisfy the axioms of a category.

2. There is an obvious inclusion functor $F : \mathbf{Cpt} \hookrightarrow \mathbf{Top}$, that is, **Cpt** is a *subcategory* of **Top**. Note that this subcategory is *full*, i.e., every morphism $f : X \to Y$ in **Top** between compact spaces is also a morphism in **Cpt**. On the other hand, the subcategory $\mathbf{LCpt} \hookrightarrow \mathbf{Top}$ is *not* full, because not every continuous map between locally compact spaces is proper (find an example!).

3. The Alexandroff compactification (see section 4) defines a functor $(\cdot)_+ : \mathbf{LCpt} \to \mathbf{Cpt}$. It can also be considered as a functor $\mathbf{LCpt} \to \mathbf{LCpt}$, if we so wish; or as a functor $(\cdot)_+ : \mathbf{LCpt} \to \mathbf{Cpt}_\bullet$, if we choose the point at infinity as the base-point of X_+ (see ex. 3.8). Show that these functors are faithful but not full, i.e., that there are continuous (base-point preserving) functions $X_+ \to Y_+$ that do not have the form f_+ for any $f : X \to Y$.

4. The unitization of C*-algebras defines a functor $(\cdot)^+ : \mathbf{C^*alg} \to \mathbf{C^*alg_1}$.

5. There are four inclusions forming a commutative diagram of functors:

$$\begin{array}{ccc} \mathbf{C^*com_1} & \xrightarrow{\text{full}} & \mathbf{C^*alg_1} \\ \downarrow & & \downarrow \\ \mathbf{C^*com} & \xrightarrow{\text{full}} & \mathbf{C^*alg} \end{array} \tag{1}$$

Note that the horizontal inclusions are full (see def. 4) by definition, but not the vertical ones, because not all *-homomorphisms between unital algebras are unital (find an example!).

6. Let $\mathbf{vec}_K$ be the category of finite dimensional vector spaces over the field K, with K-linear maps as morphisms. The *dual space* functor $(\cdot)' : (\mathbf{vec}_K)^{\mathrm{op}} \to \mathbf{vec}_K$ sends a vector space V to its dual $V' := \mathrm{Hom}_K(V, K)$, and a linear map $f : V \to W$ to the linear map (in the opposite direction!) $f' : W' \to V'$, $\alpha \mapsto \alpha \circ f$.

7. The spectrum $\widehat{A}$ of a unital commutative C*-algebra A defines a functor $\widehat{(\cdot)} : (\mathbf{C^*com_1})^{\mathrm{op}} \to \mathbf{Cpt}$. Note that the morphism $\hat{f}$, just like the map $C(f)$ in example 6.9, is given by the same trick that produces f' in the previous example of the dual vector space (think: "pre-composition").

8. More generally, the spectrum is also a functor $\widehat{(\cdot)} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cpt}$, where $\mathcal{C}$ is an appropriate category of (unital commutative) Banach algebras. What should the morphisms in $\mathcal{C}$ be, in order for this functor to be well-defined?

9. The function algebra $C(X)$ defines a functor $C : (\mathbf{Cpt})^{\mathrm{op}} \to \mathbf{C^*com_1}$. Similarly, we have the functor $C_0 : \mathbf{LCpt}^{\mathrm{op}} \to \mathbf{C^*com}$ defined on objects by $C_0(X) = C(X_+, \infty) := \{f \in C(X_+) \mid f(\infty) = 0\}$.

10. (Optional.) Let G be a group, considered as a category with a unique object $*$ (see example 3.9). Verify the following: a left G-set is the same thing as a functor $G \to \mathbf{Set}$, and a right G-set a functor $G^{\mathrm{op}} \to \mathbf{Set}$. In the same way, a (complex, linear, and finite dimensional) representation of G is the same thing as a functor $G \to \mathbf{vec}_{\mathbb{C}}$.

2 Naturality and equivalences

Definition 7 (Natural transformation)**.** Consider any two "parallel" functors $F, G : \mathcal{C} \to \mathcal{D}$. A *natural transformation*, or *morphism of functors*, between F and G is a family of morphims $\alpha = \{\alpha_X : F(X) \to G(X)\}_{X \in \mathrm{Obj}(\mathcal{C})}$ in $\mathcal{D}$ indexed by the objects of $\mathcal{C}$, such that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

is commutative (that is: $G(f) \circ \alpha_X = \alpha_Y \circ F(f)$) for every $f : X \to Y$ in $\mathcal{C}$. (Roughly: "α commutes with changes in the variable X".) The notation $\alpha : F \to G$ (or sometimes $\alpha : F \Rightarrow G$ or $\alpha : F \dot{\to} G$) means that α is a morphism from F to G. We say that α is an *isomorphism*[2] if every component α_X is an isomorphism in $\mathcal{D}$.

[2]or also *natural isomorphism*, or *natural equivalence*; the latter should not be confused with an equivalence of categories, see the next definition.

Examples 8. 1. Let X_+ be the Alexandroff compactification of X. The dense open embeddings $j_X : X \to X_+$ define a natural transformation $j : \mathrm{id}_{\mathbf{LCpt}} \to (\cdot)_+$.

2. Let $\mathcal{C}$ be the category of commutative unital Banach algebras of example 6.8. The Gelfand transforms $\Gamma_A : A \to C(\hat{A})$ define a morphism $\Gamma : \mathrm{id}_{\mathcal{C}} \to C(\hat{\cdot})$. By Gelfand's theorem, the restriction of Γ to the subcategory $\mathbf{C^*com_1}$ yields an isomorphism $\Gamma : \mathrm{id}_{\mathbf{C^*com_1}} \cong C(\hat{\cdot})$.

3. Let A be a C*-algebra, A^+ its unitization. There is a short exact sequence

$$A \overset{\iota_A}{\rightarrowtail} A^+ \overset{\pi_A}{\twoheadrightarrow} \mathbb{C}$$

where the maps are $\iota_A(a) = (a, 0) \in A^+$ and $\pi_A(a, z) := z \in \mathbb{C}$. This is an *extension* of C*-algebras, meaning that ι_A is the inclusion of an ideal and $A^+/A \cong \mathbb{C}$; or, equivalently, that π_A is surjective and $A = \mathrm{Ker}(\pi_A)$. Moreover, the extension is *natural*: for every *-homomorphism $f : A \to B$ the following diagram in $\mathbf{C^*alg}$ is commutative:

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[d, "f"'] & A^+ \arrow[r, "\pi_A"] \arrow[d, "f^+"'] & \mathbb{C} \arrow[d, equal] \\ B \arrow[r, "\iota_B"] & B^+ \arrow[r, "\pi_B"] & \mathbb{C} \end{tikzcd}$$

In other words, $\iota : \mathrm{id}_{\mathbf{C^*alg}} \to (\cdot)^+$ and $\pi : (\cdot)^+ \to \mathbb{C}$ are natural transformation (here $\mathbb{C}$ is the constant functor $A \mapsto \mathbb{C}$, $f \mapsto \mathrm{id}_{\mathbb{C}}$). Note that π is unital, so it can also be seen as a morphism between functors from $\mathbf{C^*alg}$ to $\mathbf{C^*alg_1}$.

4. (Optional.) Show that, for any two categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ having the functors $F : \mathcal{C} \to \mathcal{D}$ as objects and natural transformations $\alpha : F \to G$ as morphisms. Show that a morphism α in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is invertible precisely when it is an isomorphism in the sense of definition 7.

5. (Optional.) Let G be a group. Show that the category G-**Set** of (left) G-sets and G-equivariant maps between them can be identified with the category $\mathbf{Fun}(G, \mathbf{Set})$ of exercise 8.4, where now we consider G as a category (see example 3.9). Similarly, the category $\mathbf{rep}_{\mathbb{C}}(G)$ of representations of G identifies with $\mathbf{Fun}(G, \mathbf{vec}_{\mathbb{C}})$. We see that two representations are "equivalent", in the usual sense, iff they are isomorphic in $\mathbf{Fun}(G, \mathbf{vec}_{\mathbb{C}})$ in the categorical sense.

Exercise 9 (The spectrum)**.** A *character* of a commutative C*-algebra A is, as in the unital case, a non-zero algebra homomorphism $A \to \mathbb{C}$. Let the *spectrum* $\hat{A} := \mathrm{Hom}(A, \mathbb{C}) \setminus \{0\}$ be the space of characters, with the weak* topology induced from A'. Show that $\hat{A}$ can be identified with the subspace $\widehat{A^+} \setminus \{\pi_A\}$ (*Hint:* show that every character $A \to \mathbb{C}$ extends uniquely to a character $A^+ \to \mathbb{C}$, and that the only $\chi \in \widehat{A^+}$ that cannot be obtained in this way is π_A). Show moreover that there is a homeomorphism $\widehat{A^+} \cong (\hat{A})_+$ identifying π_A with the point at infinity ∞ (*Hint:* note that $\hat{A} \cup \{0\} = \mathrm{Hom}(A, \mathbb{C})$ is closed in $B_{\leq 1}(A')$

for the weak* topology and thus the subsets $\{\gamma : |\gamma(a)| \geq \varepsilon\} \subset \mathrm{Hom}(A, \mathbb{C})$ are all compact). The spectrum defines a functor $\widehat{(\cdot)} : \mathbf{C^*com}^{\mathrm{op}} \to \mathbf{LCpt}$.

As it turns out, the concept of isomorphism (= invertible functor) is not very useful for (big) categories. Another phenomenon seems to be much more common in Nature:

Definition 10 (Equivalence of categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* if there exists some functor $G : \mathcal{D} \to \mathcal{C}$ (called a *quasi-inverse of* F), and two isomorphisms of functors $\alpha : GF \cong \mathrm{id}_{\mathcal{C}}$ and $\beta : FG \cong \mathrm{id}_{\mathcal{D}}$. A contravariant equivalence, i.e., one of the form $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, is sometimes called a *duality*.

Remark 11. The quasi-inverse of an equivalence F is unique only "up to isomorphism", meaning: if G, G' are both quasi-inverses of F, then we find an isomorphism of functors $G \cong G'$ (check this!).

Exercise 12. (Optional.) Prove the following important theorem by using the axiom of choice: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful (see def. 4) and also *essentially surjective* (i.e., for every object $D \in \mathcal{D}$ there is an isomorphism $f : D \cong F(C)$ for some object C of $\mathcal{C}$).

Examples 13.

1. The dual space functor $(\cdot)' : \mathbf{vec}_K \to (\mathbf{vec}_K)^{\mathrm{op}}$ is an equivalence with quasi-inverse $(\cdot)' : (\mathbf{vec}_K)^{\mathrm{op}} \to \mathbf{vec}_K$, since the double-dual map $\epsilon : V \to V''$, $v \mapsto \mathrm{ev}_v$, yields an isomorphism $\mathrm{id}_{\mathbf{vec}_K} \cong (\cdot)''$.
2. Let $\mathbf{M}_K$ be the category with objects $0, 1, 2, \ldots$ and with morphisms given by $\mathrm{Hom}_{\mathbf{M}_K}(n, m) := M_{m \times n}(K)$, the set of $m \times n$ matrices over K. Composition is defined to be the usual multiplication of matrices. Then, according to exercise 12, the inclusion $\mathbf{M}_K \hookrightarrow \mathbf{vec}_K$ is an equivalence.
3. (Optional; cf. example 8.5.) Consider the functor $(\cdot)^{-1} : G \to G^{\mathrm{op}}$ sending g to g^{-1} (and $*$ to $*$). It is invertible with itself as inverse. Therefore, it induces by pre-composition an equivalence (actually, an isomorphism) of categories G^{op}-**Set** $\simeq G$-**Set** between right and left G-sets.

The next section finally describes the example we are all waiting for.

3 Gelfand duality

Theorem 14 (Gelfand duality)**.** *The functor* $C : \mathbf{Cpt}^{\mathrm{op}} \to \mathbf{C^*com_1}$ *and the functor* $\widehat{(\cdot)} : (\mathbf{C^*com_1})^{\mathrm{op}} \to \mathbf{Cpt}$ *are contravariant equivalences of categories, quasi-inverse to each other, between compact spaces and commutative unital C*-algebras.*

Proof. By Gelfand's theorem, the Gelfand transform provides an isomorphism $\mathrm{id}_{\mathbf{C^*com_1}} \cong C(\widehat{\cdot})$. We have also proved in the lecture that the map $i_X : x \mapsto \mathrm{ev}_x$, $X \cong \widehat{C(X)}$, defines an isomorphism $i : \mathrm{id}_{\mathbf{Cpt}} \cong \widehat{C(\cdot)}$. □

Remark 15. Now we can answer the deep[3] philosophical question: Should $0 := \{0\}$ be considered a *unital* C*-algebra? Yes, of course, because $0 \cong C(\emptyset)$ and the empty set is a perfectly nice and useful compact space.

[3] I'm joking

Remark 16. It follows from Gelfand duality that every definition, construction, and theorem for compact spaces that can be expressed in the language of categories (that is, arrows and composition of arrows) has a translation for commutative C*-algebras, and vice-versa (but recall to invert the arrows!!). This includes typical constructions such as limits and colimits, but also many other more specific notions and results. Also, in view of the inclusions (1), it is natural to ask: A) What happens in the non-unital case? [see below], and B) What topological (= commutative) notions generalize to general noncommutative C*-algebras? This approach to $\mathbf{C^*alg}$ is often called *noncommutative topology.*

Definition 17. Let $\mathcal{C}$ be a category and $B \in \mathrm{Obj}(\mathcal{C})$ a fixed object of $\mathcal{C}$. Define the *comma category of objects of $\mathcal{C}$ over B*, denoted by $\mathcal{C}/B$, to be the category with objects the pairs (X, f), where $X \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, B)$, and where a morphism $g : (X, f) \to (X', f')$ is by definition a morphism $g : X \to X'$ in $\mathcal{C}$ such that $f' \circ g = f$.

Dually, the *comma category $B/\mathcal{C}$ under B* is $(\mathcal{C}^{\mathrm{op}}/B)^{\mathrm{op}}$; explicitly, its objects are pairs (X, f) with $X \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(B, X)$, and a morphism $g \in \mathrm{Hom}_{B/\mathcal{C}}((X, f), (X', f'))$ is a morphism $g : X \to X'$ such that $g \circ f = f'$.

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, "g"] & & X' \end{tikzcd}$$

Examples 18.

1. The category $\mathbf{Cpt}_\bullet$ of pointed compact spaces is nothing but the comma category $\mathrm{pt}/\mathbf{Cpt}$ under the one-point space.

2. The objects (A, p) of $\mathbf{C^*alg_1}/\mathbb{C}$ are sometimes called augmented C*-algebras. Using example 8.3, it is easy to see that unitisation, considered as the functor $(\cdot)^+ : \mathbf{C^*alg} \to \mathbf{C^*alg_1}/\mathbb{C}$ sending A to (A^+, π_A), is an equivalence with quasi-inverse the functor Ker sending the augmented algebra (A, p) to the kernel $\mathrm{Ker}(p) \subset A$ of the augmentation (check this!). The equivalence restricts to commutative algebras: $\mathbf{C^*com_1}/\mathbb{C} \simeq \mathbf{C^*com}$.

Corollary 19 (Gelfand duality, non-unital version)**.** *The category $\mathbf{C^*com}$ of all commutative C*-algebras is contravariantly equivalent to the category $\mathbf{Cpt}_\bullet$ of pointed compact spaces: $(\mathbf{Cpt}_\bullet)^{\mathrm{op}} \simeq \mathbf{C^*com}$.*

Proof. More precisely, there are two equivalences

$$\big(\mathbf{Cpt}_\bullet\big)^{\mathrm{op}} = \big(\mathrm{pt}/\mathbf{Cpt}\big)^{\mathrm{op}} = \mathbf{Cpt}^{\mathrm{op}}/\mathrm{pt} \xrightarrow[\simeq]{C/\mathrm{pt}} \mathbf{C^*com_1}/\mathbb{C} \xrightarrow[\simeq]{\mathrm{Ker}} \mathbf{C^*com} \tag{2}$$

and we conclude by noting that the composition of two equivalences is again an equivalence (check this!). The first one is induced by the equivalence C of theorem 14, because $C(\mathrm{pt}) = \mathbb{C}$. The second one is example 18.2. ☐

Remark 20. Note that the composition $\mathrm{Ker} \circ C/\mathrm{pt}$ in (2) is the functor sending the pointed space (X, x_0) to the commutative C*-algebra $C(X, x_0)$ of continuous functions $f : X \to \mathbb{C}$ with $f(x_0) = 0$. For example, if X is locally compact, we see that $C(X_+, \infty) = C_0(X)$.

Theorem 21 (Gelfand's theorem, non-unital version). *Every commutative C^*-algebra is isomorphic to one of the form $C_0(X)$, with X a locally compact space. We can choose X to be the spectrum of A, an in exercise 9.*

Proof. By Gelfand's theorem and exercise 9 we have an isomorphism Γ_{A^+} : $A^+ \cong C(\widehat{A^+}) = C(\hat{A}_+)$. By direct inspection, we see that $\mathrm{ev}_\infty \circ \Gamma_{A^+} = \pi_A$.

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, dotted] & A^+ \arrow[r, two heads, "\pi_A"] \arrow[d, "\Gamma_{A^+}"', "\cong"] & \mathbb{C} \arrow[d, equal] \\ C_0(\hat{A}) \arrow[r, tail] & C(\hat{A}_+) \arrow[r, two heads, "\mathrm{ev}_\infty"] & \mathbb{C} \end{tikzcd}$$

Hence the Gelfand isomorphism restricts to the kernels: $A \cong C_0(\hat{A})$. ☐

Warning 22. Note that Theorem 21 does *not* extend to a contravariant equivalence $C_0 : \mathbf{LCpt}^{\mathrm{op}} \simeq \mathbf{C^*com}$. The theorem says exactly that C_0 is essentially surjective (see exercise 12). But C_0 is not an equivalence: consider the commutative diagram of functors (cf. remark 20)

$$\begin{tikzcd} \mathbf{LCpt} \arrow[rr, "(\cdot)_+"] \arrow[dr, "C_0"'] & & \mathbf{Cpt}_\bullet \arrow[dl, "C(\ ,\ )", "\simeq"'] \\ & (\mathbf{C^*com})^{\mathrm{op}} & \end{tikzcd} \tag{3}$$

If C_0 were an equivalence, then also $(\cdot)_+$ must be one, but we know from example 6.3 that $(\cdot)_+$ is not full, so this cannot be the case. The correct functorial statement is that of corollary 19, in terms of pointed compact spaces.

4 Appendix: The Alexandroff compactification

Let X be a locally compact space (that is: X is Hausdorff, and every $x \in X$ has a compact neighborhood). The *Alexandroff compactification of* X (also called *one-point* compactification of X) is the topological space X_+ defined as follows. As a set, it is simply the disjoint union $X \coprod \{\infty\}$ of X and a new point denoted by ∞. A subset $U \subset X_+$ is defined to be open if it is an open subset of X, or, in case $\infty \in U$, if its complement $X_+ \smallsetminus U$ is compact and closed[4] in X.

The verification of the following facts is left as a series of easy familiarizing exercises.

1. If X is not compact, then X_+ is compact and the inclusion $j_X : X \hookrightarrow X_+$ identifies the space X with an open dense subset of X_+ (this is what *compactification* means in general).
2. If X is already compact, then $X_+ = X \coprod \{\infty\}$ is the topological sum (coproduct) of the space X and a disjoint point. (So, in this case, X_+ is not really a compactification of X).

[4]actually, closed is automatic; recall also that a subset of a compact (Hausdorff) space is closed iff it is compact)

3. The assignment $X \mapsto X_+$ extends to a functor $(\cdot)_+ : \mathbf{LCpt} \to \mathbf{Cpt}$ (see ex. 3). In fact, the evident function $f_+ : X_+ \to Y_+$ is always defined but it is continuous only if $f : X \to Y$ is continuous *and* proper.

4. The maps $j_X : X \to X_+$ form a natural transformation $j : \mathrm{id}_{\mathbf{LCpt}} \to (\cdot)_+$.

Remark 23. Every open subset of a compact space is locally compact. Conversely, by the Alexandroff compactification every locally compact space is an open subset of a compact space.

Example 24. There are homeomorphims $(\,]0,1[\,)_+ \cong \mathbb{R}_+ \cong S^1$. More generally, $(\,]0,1[^n\,)_+ \cong (\mathbb{R}^n)_+ \cong S^n$ for all $n \geq 1$ via the stereographic projection.

Proposition 25. *Let X be any locally compact space. Then $C_0(X)^+ \cong C(X_+)$. More precisely, there is an isomorphism $(\cdot)^+ \circ C_0 \cong C \circ (\cdot)_+$ of functors $\mathbf{LCpt}^{\mathrm{op}} \to \mathbf{C^*com_1}$.*

Proof. The two maps

$$\begin{array}{rcl} C_0(X)^+ = C_0(X) \oplus \mathbb{C} & \to & C(X_+) \\ (f, z) & \mapsto & f + z \cdot 1 \end{array}$$

and

$$\begin{array}{rcl} C(X_+) & \to & C_0(X)^+ \\ g & \mapsto & \big(g - g(\infty) \cdot 1, g(\infty)\big) \end{array}$$

are well-defined natural unital *-homomorphisms, and they are clearly inverse to each other. ☐

Remark 26. Another way to express the proposition is to say that the diagram of functors

$$\begin{tikzcd} \mathbf{LCpt} \arrow[r, "(\cdot)_+"] \arrow[d, "C_0"'] & \mathbf{Cpt} \arrow[d, "C"] \\ (\mathbf{C^*com})^{\mathrm{op}} \arrow[r, "(\cdot)^+"] & (\mathbf{C^*com_1})^{\mathrm{op}} \end{tikzcd}$$

commutes "up to natural isomorphism". Similarly, the homeomorphism $\hat{A}_+ \cong \widehat{A^+}$ of exercise 9 is natural and thus testifies that the square

$$\begin{tikzcd} \big(\mathbf{C^*com}\big)^{\mathrm{op}} \arrow[r, "(\cdot)^+"] \arrow[d, "\widehat{(\cdot)}"'] & \big(\mathbf{C^*com_1}\big)^{\mathrm{op}} \arrow[d, "\widehat{(\cdot)}"] \\ \mathbf{LCpt} \arrow[r, "(\cdot)_+"] & \mathbf{Cpt} \end{tikzcd}$$

commutes up to isomorphism.