

Category theory and C*-algebras

A commented (very partial) bibliography

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To my knowledge, the language of category theory almost never appears in textbook treatments of the theory of C*-algebras, even in situations where it could simplify statements or otherwise help the exposition. The following topics, for example, could gain some clarity from a categorical formulation: Gelfand duality¹, universal constructions such as direct sums (= products), pullbacks, free or amalgamated products and free algebras; tensor products and representations; Hilbert modules and bimodules over C*-categories, strong Morita equivalence, etc. Even standard monographs on more advanced topics, such as Lance [20] on Hilbert modules, Raeburn-Williams [33] on Morita equivalence and continuous trace C*-algebras, or Wegge-Olsen [46] and Blackadar [1] on K-theory, tend to avoid functorial thinking. There is certainly some justification for this, but nonetheless one could gather the impression that many mathematicians who enjoy working with C*-algebras don’t have much time for such abstract nonsense as category theory – even when it could help.

In the published literature, I can find only two main research sub-fields which provide a definite exception to this trend: one is the topic of **C*-categories** (which are, roughly speaking, C*-algebras with several objects) and the other one is **KK-theory**, also called Kasparov theory or bivariant K-theory – this is an important but rather technical generalization of the topological K-theory of C*-algebras. (Even the later topic though has been treated virtually without any mention of categories for some 20 years since its invention, despite the fact that KK-theory itself forms a category, and that it can be uniquely characterized as a certain universal functor on separable C*-algebras!)

C*-categories

The first systematic study of C*-categories (focusing on the aspects of the theory more related to von Neumann algebras) is Ghez-Lima-Roberts [11], and is still quite enjoyable. In Doplicher-Roberts [8,9] one can find the original important application of the theory; here C*-categories feature in what is now called Doplicher-Roberts duality, which is a variant of Tannaka-Krein duality that allows one to reconstruct any compact group G from its tensor C*-category of finite dimensional unitary representations. The two latter articles contain much categorical thinking, which however is presented in a rather non-categorical way (e.g. there are virtually no commutative diagrams!); a gentler and more recent exposition, which exploits category theory in a more standard way, is given by Müger [32].

One role of C*-categories, also acknowledged in the previous references, is that several kinds of representations of C*-algebras (various flavours of Hermitian modules and Hilbert modules) assemble naturally into C*-categories. Because of this C*-categories often appear in connection with KK-theory (see next

¹See [4] for a (too) detailed categorical treatment of Gelfand duality.

section), which is defined in terms of Hilbert modules. Authors who have extended or applied the theory of C^* -categories include Kandelaki [16–19], Mitchener [28–31], Vasselli [42–44], Dell’Ambrogio [7], Zito [47], ...

Although C^* -categories are not mentioned, some of the early works on the important theory of (strong) Morita equivalence of C^* -algebras make some (rather modest) use of categories and functors: see [2, 34, 35]. The recent monograph [10], as its title promises, takes a more categorical approach.

KK-theory

A certain amount of categorical language is unavoidable when dealing with topological K-theory of C^* -algebras. Textbooks tend to minimize this, but some classical papers such as Schochet’s series [37–40], introducing homotopy-theoretical methods into the theory of C^* -algebras, or the much-cited Schochet-Rosenberg [36], make a generous use of it.

The literature on variations and generalizations of K-theory and KK-theory of C^* -algebras is nowadays very extensive, and much of it is quite categorical in nature. Indeed, each of these theories is a functor on some specific category of C^* -algebras, often defined by some universal property or constructed by some categorical machinery, such as localization. It would be quite hard to even partially exhaust the existing literature, so we only single out the following papers for their importance coupled with their categorical point of view: Higson [12], which discovers the universal property of KK-theory; Higson [13], introducing E-theory²; Meyer-Nest [23], where G -equivariant KK-theory is shown to form a tensor triangulated category and where localization of categories is used to reformulates the Baum-Connes conjecture.

Meyer and Nest have produced several more works where they systematically exploit the fact that KK-theories are triangulated categories: [22, 24–27]. We recommend the introductory articles [21] and [25], as well as Chapters 8 and 13 of the textbook [3], as starting points for these ideas. Other authors have pursued this trend: Dell’Ambrogio [5, 6], Inassaridze-Kandelaki-Meyer [14, 15], Voigt [45], ...

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²See also [41] for an even more categorical analysis of E-theory, and for an early introduction of triangulated categories in the subject.

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