A survey of Mackey and Green 2-functors

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Idea: axiomatic representation theory of finite groups

- Classically, from the early 70's [Green, Dress, Lindner...]:

  Mackey functor := equivariant version of abelian group

  Green functor := equivariant version of a ring

  GOAL: to axiomatically capture the many restriction, conjugation and induction (trace) maps arising from the representation theory of finite groups.

- Derived versions, e.g. spectral Mackey and Green functors [Barwick]:

  replace: abelian group, ring $\rightsquigarrow$ spectrum, ring spectrum

- OUR GOAL: categorify, but remain algebraic using 2-categories:

  replace: abelian group, ring $\rightsquigarrow$ additive category, monoidal additive category

  Our theory should receive examples from all "derived" / "higher" theories, but should also remain purely algebraic and with lighter axioms.

Our axiomatization:

gpd_f : the 2-category of finite groupoids, faithful functors, natural isomorphisms
ADD : the 2-category of additive categories, additive functors, natural transf.

Definition [Balmer–D. 2020]

A **Mackey 2-functor** is a 2-functor

$$\mathcal{M}\colon gpd_f^{op} \longrightarrow ADD$$

satisfying the following four axioms.

1. **Additivity axiom**

$$\mathcal{M}(G_1 \sqcup G_2) \overset{\sim}{\to} \mathcal{M}(G_1) \oplus \mathcal{M}(G_2) \quad \text{and} \quad \mathcal{M}(\emptyset) \overset{\sim}{\to} 0.$$

$\rightsquigarrow$ by decomposing groupoids into groups $G \simeq \sqcup_n G_n$, we can reduce the *structure* of the Mackey 2-functor $\mathcal{M}$ to the data associated to: finite groups, injective group homomorphisms, and their conjugation relations.

A very nice induction

2. **Induction axiom:** For every faithful morphism $i\colon H \to G$, the 'restriction' functor $\mathcal{M}(i) = i^*$ has a left adjoint i_ℓ and a right adjoint i_r:

$$\begin{array}{ccc} G & \mapsto & \mathcal{M}(G) \\ {\scriptstyle i}\big\uparrow & & i_\ell \curvearrowright \big\downarrow {\scriptstyle i^*} \curvearrowleft i_r \\ H & \mapsto & \mathcal{M}(H) \end{array}$$

Note: the adjoints are not really part of the structure.

3. **Ambidexterity axiom:** For every faithful i, there is an isomorphism $i_\ell \cong i_r$.

The above are easy to check in examples, but we get more:

Rectification theorem

Under the four axioms, there exist for all i unique isomorphisms $\theta_i\colon i_\ell \cong i_r$ fully compatible with given left and right adjunctions. Thus $\rightsquigarrow i_* :=: i_\ell \cong i_r$.

Base-Change axiom = canonical Mackey formulas

4. **Base-Change axiom:** each iso-comma square γ in gpd_f, via $\mathcal{M}$ and the left/right adjunctions, defines two mates γ_ℓ and $(\gamma^{-1})_r$:

$$\gamma = \begin{array}{ccc} & \bullet & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ \bullet & \overset{\simeq}{\Rightarrow} & \bullet \\ {\scriptstyle i}\searrow & & \swarrow{\scriptstyle j} \\ & \bullet & \end{array} \quad \rightsquigarrow \quad \gamma_\ell := \begin{array}{ccc} & \bullet & \\ {\scriptstyle p^*}\nearrow & & \searrow{\scriptstyle q_\ell} \\ \bullet & \Downarrow & \bullet \\ {\scriptstyle i_\ell}\searrow & & \nearrow{\scriptstyle j^*} \\ & \bullet & \end{array} \quad \text{and} \quad (\gamma^{-1})_r := \begin{array}{ccc} & \bullet & \\ {\scriptstyle p^*}\nearrow & & \searrow{\scriptstyle q_r} \\ \bullet & \Uparrow & \bullet \\ {\scriptstyle i_r}\searrow & & \nearrow{\scriptstyle j^*} \\ & \bullet & \end{array}$$

We require both to be invertible: $j^* i_\ell \cong q_\ell p^*$ and $j^* i_r \cong q_r p^*$.

Convenient fact: via the rectification θ's, the two mates are mutual inverses!

Motivating example: for i, j two subgroup inclusions $K, L \leq G$

$$\text{iso-comma} \quad \begin{array}{ccc} & (i/j) & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ K & \overset{\simeq}{\Rightarrow} & L \\ {\scriptstyle i}\searrow & & \swarrow{\scriptstyle j} \\ & G & \end{array} \quad \rightsquigarrow \quad (i/f) \simeq \coprod_{[g]\in L\backslash G/K} L \cap {}^{g}K$$

get a Mackey decomposition

Reduction to finite groups

By Additivity, a Mackey 2-functor $\mathcal{M}$ can be reduced to what it does to groups:

- The **restriction**, **induction** and **conjugation** functors ($H \leq G$, $g \in G$):

$$\begin{array}{ccc} \mathcal{M}(G) & & \\ Ind \uparrow \;\; \downarrow Res & & \\ \mathcal{M}(H) & \xrightarrow[\sim]{Conj_g} & \mathcal{M}({}^g H) \end{array}$$

- The **adjunctions** $Ind \dashv Res \dashv Ind$
- **Conjugation natural isos** between composites, e.g.

$$conj_g \colon \; Conj_g \circ Res^G_H \cong Res^G_{{}^gH}$$

- Many relations, e.g. a canonical **Mackey formula** (for $K, L \leq G$):

$$Res^G_L \circ Ind^G_K \cong \bigoplus_{[g] \in L \backslash G / K} Ind^L_{L \cap {}^gK} \circ Conj_g \circ Res^K_{L^g \cap K} \, .$$

Further abstraction

There are useful (straightforward) variants, as 2-functors

$$\mathcal{M} \colon \mathbb{G}^{op} \longrightarrow \mathbb{A}$$

satisfying the same axioms, where

- the source $\mathbb{G}$ is some more general "extensive" (2,1)-category
- the target $\mathbb{A}$ is some more general "additive" 2-categoy
- we require inductions i_* for i's in some suitable class $\mathbb{J} \subseteq \mathbb{G}$.

Examples:

- For Mackey 2-functors for a fixed group G_0, use $\mathbb{G} = \mathbb{J} = gpd_f / G_0 \simeq G_0\text{-}set$.
- $\mathbb{A}$ could be any suitable 2-category of categories which are:
  abelian / exact / linear over a base ring k / triangulated ...
  or we could take: $\mathbb{A} =$ additive derivators / stable derivators ...

Some examples of Mackey 2-functors

Each of the following families of categories $\mathcal{M}(G)$ defines a Mackey 2-functor:

- **From (linear) representation theory**:
  $\mathcal{M}(G) = Mod(kG)$ linear representations over k
  $\mathcal{M}(G) = D(kG)$ the derived category
  $\mathcal{M}(G) = Stab(kG)$ stable module category
- **From (formal) representation theory**:
  $\mathcal{M}(G) = Mack_k(G)$ or $CoMack_k(G)$ ordinary (cohom.) Mackey functors!
- **From topology**:
  $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$ the homotopy category of genuine G-spectra
- **From noncommutative topology**:
  $\mathcal{M}(G) = KK^G$ the equivariant Kasparov category of G-C*-algebras
- **From geometry**: Fix X a locally ringed space with a G_0-action.
  For $G \leq G_0$, set $\mathcal{M}(G) = Sh(X /\!\!/ G)$ G-equivariant $\mathcal{O}_X$-modules.

These are all *tensor* categories, in fact they are "symmetric Green 2-functors" !

Green 2-functors

Definition [D. 2022]

A **Green 2-functor** is a Mackey 2-functor $\mathcal{M}$ equipped with a lifting

$$\begin{array}{ccc} & & PsMon(ADD) \\ & \nearrow & \downarrow \text{forget} \\ gpd_f^{op} & \xrightarrow{\mathcal{M}} & ADD \end{array}$$

to pseudo-monoids in ADD [or any "additive symmetric monoidal 2-category" $\mathbb{A}$], satisfying:

1. **Projection formulas**: the horizontal canonical mates are invertible for all i:

$$\begin{array}{ccc} X \otimes i_r(Y) & \xrightarrow{Rproj} & i_r(X \otimes i^* Y) \\ \simeq \uparrow \theta & & \theta \uparrow \simeq \\ X \otimes i_\ell(Y) & \xleftarrow{Lproj} & i_\ell(X \otimes i^* Y) \end{array} \qquad \begin{array}{ccc} i_r(Y) \otimes X & \xrightarrow{Rproj} & i_r(i^* Y \otimes X) \\ \theta \uparrow \simeq & & \theta \uparrow \simeq \\ i_\ell(Y) \otimes X & \xleftarrow{Lproj} & i_\ell(i^* Y \otimes X) \end{array}$$

For **braided** or **symmetric** Green 2-functors: replace

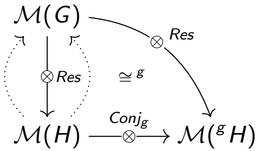
$$PsMon \quad \rightsquigarrow \quad BrPsMon \ \text{ or } \ SymPsMon$$

Remarks:

- The two squares are 'commutative', that is: the canonical Ambidexterity isomorphism θ identifies left and right projection maps as mutual inverses.
- The Projection formulas hold if and only if the **external products**

$$\mathcal{M}(G_1) \times \mathcal{M}(G_2) \xrightarrow{\overline{\otimes}} \mathcal{M}(G_1 \times G_2)$$

  associated with the given internal ones $\otimes$ "preserve inductions" in both variables separately. (Actually: this only works for $\mathbb{G}$ Cartesian!)
- When reduced to finite groups, a Green 2-functor amounts to:
  - each $\mathcal{M}(G)$ being **(braided, symmetric) monoidal** additive category,
  - restriction and conjugation being **strong (braided) monoidal functors**,
  - The conjugation natural isos are **monoidal natural transformations**,



  - satisfying coherent **projection formulas** (and their left-right reverse):

$$Ind_H^G(\ Res_H^G(X) \otimes_H Y\ ) \ \cong\ X \otimes_G Ind_H^G(Y)$$

Applications: produce ordinary Mackey and Green functors

A Mackey or Green 2-functor $\mathcal{M}$ can be 'decategorified' in at least two ways:

K-decategorification – [Dress 1973] [Balmer–D. 2020]

If $\mathcal{M}$ is essentially small, the composite $K_0^{add} \circ \mathcal{M}$ is an ordinary Mackey functor.
Variants: If $\mathcal{M}$ takes the appropriate values, we can use K_0^{triang}, K_0^{ex}, $K_*^{Quillen}$, ...

- If $\mathcal{M}$ is a Green 2-functor, its K-decategorifications are clearly *Green* functors.

Hom-decategorification – [Balmer–D. 2022] [D. 2022]

Given two objects $X, Y \in \mathcal{M}(G_0)$, there is an ordinary G_0-Mackey functor M with

$$G_0 \geq G \longmapsto M(G) := \mathrm{Hom}_{\mathcal{M}(G)}(\mathrm{Res}^{G_0}_G X, \mathrm{Res}^{G_0}_G Y).$$

If $\mathcal{M}$ is Green 2-functor, X a comonoid, Y a monoid, then M is a Green functor.

- Can obtain many variants, as well as modules over Green functors, etc.
- All classical Green functors arise as K- or Hom-decat. of Green 2-functors!

Applications: monadicity and p-local descent

Separable monadicity – [BD 2020, D 2022]

For an idempotent-complete Mackey 2-functor $\mathcal{M}$ and any faithful $i\colon H \rightarrowtail G$, the composite

$$\mathrm{Id}_{\mathcal{M}(H)} \to i^* i_\ell \cong i^* i_r \to \mathrm{Id}_{\mathcal{M}(H)}$$

is the identity. In particular, there are canonical equivalences:

$$\mathrm{Comod}_{\mathcal{M}(G)}(i^* i_\ell) \simeq \mathcal{M}(H) \simeq \mathrm{Mod}_{\mathcal{M}(G)}(i^* i_r).$$

If $\mathcal{M}$ is a Green 2-functor, $A(i) := i_*(\mathbf{1})$ is a symmetric Frobenius object and the latter are monoidal equivalences of $\mathcal{M}(H)$ with co/modules over $A(i)$.

What about the *other* unit-counit composite?

Definition

The Mackey 2-functor $\mathcal{M}$ is **cohomological** if the composite

$$\mathrm{Id}_{\mathcal{M}(G)} \to i_r i^* \cong i_\ell i^* \to \mathrm{Id}_{\mathcal{M}(G)}$$

is multiplication by $[G : H]$ for every subgroup inclusion $i\colon H \rightarrowtail G$.

Applications: monadicity and p-local descent

Examples: $D(kG)$, $Stab(kG)$, $D(Sh\, X /\!\!/ G)$ are cohomological, but not $SH(G)$!

p-Local descent – [BD 2022]

If $\mathcal{M}$ is cohomological, $\mathbb{Z}_{(p)}$-linear (p a prime number) and idempotent complete, and $i\colon S \rightarrowtail G$ is a p-Sylow subgroup, then:

$$\mathrm{Comod}_{\mathcal{M}(S)}(i^* i_r) \simeq \mathcal{M}(G) \simeq \mathrm{Mod}_{\mathcal{M}(S)}(i^* i_\ell).$$

p-Local descent – [Maillard 2022]

More precisely: a Mackey 2-functor as above is a 2-sheaf for the p-local (or 'sipp') topology on gpd. In particular, there exists a canonical equivalence

$$\mathcal{M}(G) \simeq \lim_{P \in \mathcal{O}_p(G)} \mathcal{M}(P)$$

with the (pseudo-)limit taken in ADD over the orbit category of p-subgroups of G. Also, any $\mathcal{M}$ admits a 2-sheafification $\mathcal{M} \to \mathcal{M}^{p\text{-loc}}$.

Applications: Green equivalences and correspondences

Notation:

- $\mathcal{M}(G;P) := \{M \mid M \text{ is a retract of } \mathrm{Ind}(N) \text{ for some } N \in \mathcal{M}(P)\} \overset{\text{full}}{\subset} \mathcal{M}(G)$, the full subcategory of ***P*-objects**, for $P \leq G$ a subgroup.
- $\mathcal{M}(G;\mathbb{X})$ defined similarly for a set $\mathbb{X}$ of subgroups of G.

The Green equivalence – [BD 2021]

$\mathcal{M}$ any Mackey 2-functor for G, and $P \leq H \leq G$ subgroups with $H \supseteq N_G(P)$. Then induction yields an equivalence of idempotent-completed additive quotients:

$$\left(\frac{\mathcal{M}(H;P)}{\mathcal{M}(H;\mathbb{X})}\right)^{\natural} \xrightarrow[\sim]{\ \mathrm{Ind}\ } \left(\frac{\mathcal{M}(G;P)}{\mathcal{M}(G;\mathbb{X})}\right)^{\natural}$$

where $\mathbb{X} = \{P \cap {}^{g}P \mid g \in G \smallsetminus H\}$.

- The idempotent completion is not needed in examples (for different reasons).
- If $\mathcal{M}$ is Krull-Schmidt, get the **Green correspondence**, a bijection of iso-classes: indecs of $\mathcal{M}(H)$ with vertex $P \leftrightarrow$ indecs of $\mathcal{M}(G)$ with vertex P.

There is much more, but enough for today . . .

Thank you for your attention!

References:

1. Paul Balmer and Ivo Dell'Ambrogio. Mackey 2-functors and Mackey 2-motives. *EMS Monographs in Mathematics.* Zürich (2020)
2. Paul Balmer and Ivo Dell'Ambrogio. Green equivalences in equivariant mathematics. *Math. Ann.* (2021)
3. Jun Maillard. A categorification of the Cartan-Eilenberg formula. *Adv. Math.* (2022).
4. Ivo Dell'Ambrogio. Green 2-functors. *Trans. Amer. Math. Soc.* (2022)
5. Paul Balmer and Ivo Dell'Ambrogio. Cohomological Mackey 2-functors. *J. Inst. Math. Jussieu* (2022)