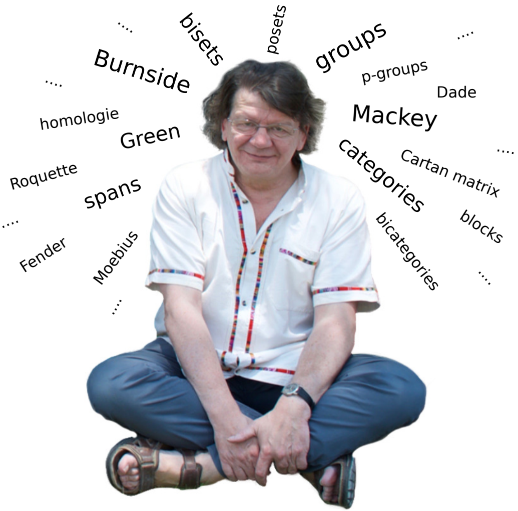
Spans, bisets and blocks

Ivo Dell'Ambrogio

Université de Lille
FACULTÉ DES SCIENCES ET TECHNOLOGIES
Département de Mathématiques
Laboratoire Paul Painlevé

The Work of Serge Bouc
4-6 August 2021

Universidad Nacional Autónoma de México
UNAM
CENTRO DE CIENCIAS MATEMÁTICAS
UNIVERSIDAD DE GUANAJUATO



Thank you Serge for all the inspiration!

Recollection: ordinary Mackey functors

Fix a finite group G_0
Fix a commutative ring of coefficients k ($= \mathbb{Z}$, a nice local ring, a field ...)

Recall the “motivic” definition of classical Mackey functors:

Definition – [Dress 1973, Lindner 1976]

A **(k-linear) G_0-Mackey functor** M is a k-linear representation

$$M\colon \mathrm{Span}_k(G_0\text{-set}) \longrightarrow \mathrm{Mod}(k)$$

of the k-linear category of spans of G_0-sets (the “Burnside category”):

$$\mathrm{Span}_k(G_0\text{-set}) := \begin{cases} \mathrm{Obj} = \text{finite } G_0\text{-sets} \\ \mathrm{Hom}(X,Y) = k \otimes_{\mathbb{N}} \left\{ \begin{smallmatrix} & Z & \\ X \swarrow & & \searrow Y \end{smallmatrix} \ \text{ in } G_0\text{-set} \right\}_{/\mathrm{iso}} \\ \text{composition via pullbacks.} \end{cases}$$

Recollection: cohomological Mackey functors

Definition

A Mackey functor is **cohomological** if it satisfies the relation

$$ind_H^G \circ res_H^G = [G : H]\, id$$

between maps $M(G_0/G) \to M(G_0/G)$ for all subgroup inclusions $H \hookrightarrow G \leq G_0$.

Examples:

- group cohomology $G \mapsto H^n(G; V|_G)$ for any $V \in \mathrm{Mod}(kG_0)$ and $n \geq 0$.
- Variations: homology, Tate cohomology, fixed point functors.

Non-examples:

- Burnside ring $G \mapsto B(G) = K_0(G\text{-set}, \sqcup, \times)$.
- Representation ring $G \mapsto Rep(G) = K_0(\mathrm{mod}(\mathbb{C}G), \oplus, \otimes_k)$.

Question: the cohomological relations are useful, but what do they mean?

Recollection: Yoshida's theorem

- $\mathrm{perm}_k(G_0) \subset \mathrm{mod}(kG_0)$: full subcategory of f.g. permutation kG_0-modules.
- **Observation:** There is a full, essentially surjective, k-linear functor

$$\mathit{Yoshi}_{G_0}: \quad \mathrm{Span}_k(G_0\text{-set}) \longrightarrow \mathrm{perm}_k(G_0)$$

$$\left[\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dotted] & & Y \end{tikzcd}\right] \longmapsto \Big(\, k[X] \to k[Y],\; x \mapsto \textstyle\sum_{z\in\alpha^{-1}(x)} \beta(z) \,\Big).$$

Then:

Yoshida's theorem – [Yoshida 1983], [Panchadcharam-Street 2007]

A Mackey functor M is cohomological iff it factors (uniquely) through Yoshi_{G_0}.

Equivalently, the kernel of Yoshi_{G_0} is generated as a k-linear ideal of morphisms by the cohomological relations (for all $H \leq G \leq G_0$):

$$\underbrace{\left[\begin{tikzcd} & G_0/H \arrow[dl, "="'] \arrow[dr] & \\ G_0/H \arrow[rr, dotted] & & G_0/G \end{tikzcd}\right]}_{\mathit{ind}_H^G} \circ \underbrace{\left[\begin{tikzcd} & G_0/H \arrow[dl] \arrow[dr, "="] & \\ G_0/G \arrow[rr, dotted] & & G_0/H \end{tikzcd}\right]}_{\mathit{res}_H^G} = [G:H]\,\mathit{id}.$$

Mackey 2-functors: the idea

Now let us categorify:

k-modules $\rightsquigarrow$ k-linear categories

homomorphisms $\rightsquigarrow$ functors

equalities $\rightsquigarrow$ (coherent) isomorphisms

functors $\rightsquigarrow$ 2-functors

Naive idea: similarly to the *original* definition of Mackey functors [Green 1971], we want to axiomatize:

- families of k-linear categories $\mathcal{M}(G)$ for finite groups G
- with k-linear restriction, induction, conjugation functors between them (and possibly others: inflation functors ...)
- satisfying the most basic relations:
  - adjunctions between functors
  - (coherent) isomorphism versions of the various Mackey axioms.

Mackey 2-functors: the definition

gpd : the 2-category of finite groupoids, functors, natural isos
ADD_k : the 2-category of (additive) k-linear categories, functors, natural transf.

Definition – [Balmer-D. 2020]

A **(k-linear) Mackey 2-functor** is a 2-functor

$$\mathcal{M} \colon gpd^{op} \longrightarrow ADD_k$$

satisfying the following axioms:

1. *Additivity:* $\mathcal{M}(G_1 \sqcup G_2) \xrightarrow{\sim} \mathcal{M}(G_1) \oplus \mathcal{M}(G_2)$ and $\mathcal{M}(\emptyset) \xrightarrow{\sim} 0$.
2. *Induction:* Every 'restriction' $i^* := \mathcal{M}(i)$ along a *faithful* functor i between groupoids admits both a left adjoint $i_!$ and a right adjoint i_*.
3. *Base-Change / Mackey formula:* the left and right adjunctions satisfy base-change with respect to pseudo-pullbacks (iso-comma squares).
4. *Ambidexterity:* There is a natural isomorphism $i_! \cong i_*$ for all faithful i.

Mackey 2-functors: comments

Explanations:

- **Additivity axiom:** groupoids decompose into groups $G \simeq \coprod_i G_i$
  $\rightsquigarrow$ the data of the 2-functor $\mathcal{M}$ is determined by what it does on groups.
- **Induction:** as for derivators, (co)induction $i_!$ and i_* are not part of the data.
- **Ambidexterity:** any isomorphisms $i_! \cong i_*$ will do, so the axiom is easy to check in examples!

  Fact (rectification theorem): the axioms imply there exist unique canonical isomorphisms $\theta_i \colon i_! \cong i_*$ fully compatible with given left and right adjunctions.

Variations are possible:

- NB: The previous definition is actually more analogous to inflation functors, because it has 'restrictions' f^* along non-faithful morphisms f.
- For the 'correct' analogue of global Mackey functor: replace *gpd* by gpd_f (only allow faithful functors).
- For the 'local' version for fixed G_0: replace *gpd* with $gpd_f / G_0 \simeq G_0$-set (!).

Base-Change = canonical Mackey formulas

Base-Change axiom: an iso-comma square γ in *gpd* with two faithful sides defines, via $\mathcal{M}$ and the left/right adjunctions, two mates $\gamma_!$ and $(\gamma^{-1})_*$:

$$\gamma = \begin{tikzcd}[column sep=small] & {} \arrow[dl, tail, "p"'] \arrow[dr, tail, "q"] & \\ {} \arrow[dr, tail, "i"'] \arrow[rr, phantom, "\overset{\sim}{\Rightarrow}"] & & {} \arrow[dl, tail, "f"] \\ & {} & \end{tikzcd} \quad \rightsquigarrow \quad \gamma_! = \begin{tikzcd}[column sep=small] & {} \arrow[dr, "q_!"] & \\ {} \arrow[ur, "p^*"] \arrow[dr, "i_!"'] \arrow[rr, phantom, "\Downarrow"] & & {} \\ & {} \arrow[ur, "f^*"'] & \end{tikzcd} \quad \text{and} \quad (\gamma^{-1})_* = \begin{tikzcd}[column sep=small] & {} \arrow[dr, "q_*"] & \\ {} \arrow[ur, "p^*"] \arrow[dr, "i_*"'] \arrow[rr, phantom, "\Uparrow"] & & {} \\ & {} \arrow[ur, "f^*"'] & \end{tikzcd}$$

The axiom requires both to be invertible: $f^*i_! \cong q_!p^*$ and $f^*i_* \cong q_*p^*$.
Convenient fact: via the rectification isos θ, they are mutual inverses!

Motivating example: for i, f two subgroup inclusions $K, L \leq G$

$$\text{iso-comma} \quad \begin{tikzcd}[column sep=small] & (i/f) \arrow[dl, tail, "p"'] \arrow[dr, tail, "q"] & \\ K \arrow[dr, tail, "i"'] \arrow[rr, phantom, "\overset{\sim}{\Rightarrow}"] & & L \arrow[dl, tail, "f"] \\ & G & \end{tikzcd} \quad \rightsquigarrow \quad (i/f) \simeq \coprod_{[g] \in L \backslash G / K} L \cap {}^{g}K$$

get a Mackey formula!

Note: the iso-comma groupoid (i/f) and the Base-Change isos are canonical, but the decomposition into groups depends on choices!

Exemples of Mackey 2-functors

There is a Mackey 2-functor $\mathcal{M}$ for each of the following families of abelian or triangulated categories $\mathcal{M}(G)$:

- **In (linear) representation theory**:
  $\mathcal{M}(G) = mod(kG)$, $Mod(kG)$, $D(kG)$, $stmod(kG)$ ($\leftsquigarrow$ only on gpd_f), ...
- **In formal representation theory**:
  $\mathcal{M}(G) = Mack_k(G)$ or $CoMack_k(G)$, categories of ordinary Mackey functors!
- **In topology**:
  $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$, the homotopy category of genuine G-spectra.
- **In geometry** (only defined 'locally' for a fixed group G_0):
  Fix X a locally ringed space (e.g. scheme) with a G_0-action.
  For $G \leq G_0$, $\mathcal{M}(G) = Sh(X /\!\!/ G)$ the category of G-equivariant $\mathcal{O}_X$-modules.
  Variants: take the derived category $D(Sh(X /\!\!/ G))$, or constructible sheaves, or coherent sheaves if X is a noetherian scheme, etc.
- ...

The motivic approach

Theorem [Balmer-D. 2020]

There is a k-linear 2-category Mot_k of **Mackey 2-motives**, through which every k-linear Mackey 2-functor $\mathcal{M}$ factors uniquely as a k-linear 2-functor:

$$\begin{tikzcd} G \in \arrow[d, mapsto] & gpd^{op} \arrow[r, "\forall\ \mathcal{M}\ \ Mackey"] \arrow[d, "univ"'] & ADD_k \\ \text{(the motive of) } G \in & Mot_k \arrow[ur, dashed, "\exists!\ \widehat{\mathcal{M}}\ \ k\text{-linear}"'] & \end{tikzcd}$$

Corollary (motivic decompositions)

The 2-cell endomorphism k-algebra $End_{Mot_k}(\mathrm{Id}_G)$ of a group G acts on the category $\mathcal{M}(G)$, for every k-linear Mackey 2-functor $\mathcal{M}$.
In particular, ring decompositions induce decompositions of the category:

$$1 = \underbrace{e_1 + \ldots + e_n}_{\text{orth. idemp. in } End(\mathrm{Id}_G)} \quad \overset{\widehat{\mathcal{M}}(-)}{\Longrightarrow} \quad \mathcal{M}(G) = e_1\mathcal{M}(G) \oplus \ldots \oplus e_n\mathcal{M}(G).$$

More concretely ...

Mot_k has concrete models (see later), in which we can compute!

Theorem [Balmer-D. 2020 (+ Bouc)]

$End_{Mot_k}(\mathrm{Id}_G)$ is isomorphic to the **crossed Burnside k-algebra** [Yoshida 1997]

$$B_k^c(G) = k \otimes_{\mathbb{Z}} K_0\big(\ G\text{-sets}/G^{conj}\ ,\ \sqcup\ ,\ \ \text{a certain braided} \otimes\ \big)$$

or concretely: the finite free k-module generated by G-conjugacy classes of pairs (H, a) with $H \leq G$ and $a \in C_G(H)$, with multiplication given by:

$$(K, b) \cdot (H, a) = \sum_{[g] \in K \backslash G / H} (K \cap {}^gH, bgag^{-1}) .$$

Example: consider $\mathcal{M}(G) = Mack_k(G)$, the category of Mackey functors for G. Well-known that the **Burnside algebra** $B_k(G) = k \otimes_{\mathbb{Z}} K_0(G\text{-set})$ acts on it.

But now the bigger ring $B_k^c(G) \supset B_k(G)$ with more idempotents also acts:

$$Mack_k(G) = \bigoplus_{e \in Prim\ Idem\ B_k^c(G)} e \cdot Mack_k(G).$$

Cohomological Mackey 2-functors

Definition [Balmer-D. 2021]

A Mackey 2-functor $\mathcal{M}$ is **cohomological** if the composite

$$\mathrm{Id}_{\mathcal{M}(G)} \xrightarrow{\text{unit}} i_* i^* \underset{\sim}{\xrightarrow{\theta^{-1}}} i_! i^* \xrightarrow{\text{counit}} \mathrm{Id}_{\mathcal{M}(G)}$$

is multiplication by $[G : H]$ for every subgroup inclusion $i \colon H \hookrightarrow G$.

Examples:

- All those from linear representation theory: $Mod(kG)$, $D(kG)$, $stmod(kG)$,...
- Usual cohomological Mackey functors: $CoMack_k(G)$ (but not $Mack_k(G)$!)
- Equivariant sheaves: $Sh(X /\!\!/ G)$, $D(X /\!\!/ G)$, $coh(X /\!\!/ G)$,...

Why?

Theorem (Hom-decategorification)

If $\mathcal{M}$ is a Mackey 2-functor and $U, V \in \mathcal{M}(G_0)$ two object at some G_0, then

$$G \quad \mapsto \quad M(G) := \mathrm{Hom}_{\mathcal{M}(G)}(Res^{G_0}_G U, Res^{G_0}_G V)$$

is an ordinary G_0-Mackey functor M. And if $\mathcal{M}$ is cohomological then so is M!

Cohomological Mackey 2-functors

Example:

- Cohomology: $H^n(G; V|_G) = \mathrm{Hom}_{D(kG)}(k, \Sigma^n V|_G)$ for any $V \in Mod(kG_0)$.

There are other uses, e.g.:

- **Categorified Cartan-Eilenberg formula** [Maillard 2021]:
  If $\mathcal{M}$ is cohomological with idempotent-complete values, and k is a $\mathbb{Z}_{(p)}$-algebra for a prime p, there is an equivalence

$$\forall G \qquad \mathcal{M}(G) \simeq \operatorname*{bilim}_{G/P \in \mathcal{O}_p(G)} \mathcal{M}(P)$$

  with the bilimit taken in ADD_k over the orbit category of p-subgroups of G.
- **Green correspondence** [Balmer-D. 2021]
  There is a Green-equivalence / Green-correspondence for every Mackey 2-functor, and it works especially well for cohomological ones.

OK, but why? What is the sense of these cohomological relations?
Same as before: *they generate the kernel of 'linearization of spans'!*

A concrete model of Mackey 2-motives

Mackey 2-motives via spans [Balmer-D. 2020]

The universal 2-category Mot_k can be realized as follows:

- Objects: finite groupoids (or formal direct summands thereof ...)
- 1-Morphisms: spans of functors with faithful right-leg: $\begin{tikzcd}[column sep=small] & K \arrow[dl] \arrow[dr, tail] & \\ H \arrow[rr, dotted] & & G \end{tikzcd}$
- 2-Morphisms: k-linearization of the monoids of iso-classes of diagrams

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr, tail] & \\ H \arrow[ur, phantom, "\simeq", near end] & L \arrow[l] \arrow[r, tail] \arrow[u] \arrow[d] & G \arrow[ul, phantom, "\simeq", near end] \\ & K' \arrow[ul] \arrow[ur, tail] \arrow[ul, phantom, "\simeq", near end] \arrow[ur, phantom, "\simeq", near end] & \end{tikzcd} \qquad \text{(in } gpd\text{).}$$

- Vertical and horizontal compositions: via iso-commas / pseudo-pullbacks.

Yoshida's theorem, categorified

$\color{red}{biperm_k^{rf}} \subset Bimod$: bicategory of finite groupoids, permutation bimodules between them which are right-free, and equivariant maps (= natural transf.).

Yoshida's theorem [Balmer-D. 2021]

There is an equivalence of k-linear bicategories

$$Mot_k / \langle \text{cohomological relations} \rangle_{\text{2-cell ideal}} \xrightarrow{\sim} biperm_k^{rf}\,.$$

The 2-functor realizing it 'linearizes' spans at the level of 1- and 2-cells:

- It maps $\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, hook, "i"] & \\ H \arrow[rr, dotted] & & G \end{tikzcd}$ to $k\big[G(i-,-) \otimes_K H(-,f-)\big] \colon H^{op} \times G \to Mod(k)$.
- Vertically, it sends a span of equivariant maps to a sum-over-preimages homomorphism (exactly like *Yoshi*!).

Corollary

A *cohomological* Mackey 2-functor $\mathcal{M}$ is the same as a k-linear pseudo-functor

$$\widehat{\mathcal{M}} \colon biperm_k^{rf} \to ADD_k\,.$$

Blocks of group algebras

For each G, the quotient 2-functor of the theorem

$$Mot_k \longrightarrow biperm_k^{rf}$$

specializes to the 2-cell endomorphism rings of Id_G, as the *surjective* ring map:

the crossed Burnside algebra	$\rightsquigarrow$	$B_k^c(G) \xrightarrow{\rho_G} Z(kG)$ $(H,a) \mapsto \sum_{x \in G/H} xax^{-1}$	$\leftsquigarrow$	the center of the group algebra!

Some consequences:

- For every cohomological Mackey 2-functor $\mathcal{M}(G)$, the category $\mathcal{M}(G)$ splits according to the usual **blocks** := primitive idempotents of $Z(kG)$.
  Note: the full kG doesn't necessarily act on $\mathcal{M}(G)$!
- If k is a complete local ring (e.g. a field), then primitive idempotents can be lifted along ρ_G (by a general lifting result).
- For instance take $Mack_k(G)$, not cohomological as a Mackey 2-functor. But for k a field, or $k = \mathbb{Z}_p^\wedge$ etc., it still splits over the blocks of kG!
- But, which blocks have *non-zero* image on $\mathcal{M}(G)$? More work to be done...

Thank you for your attention!

Reference:

1. Paul Balmer and Ivo Dell'Ambrogio. *Cohomological Mackey 2-functors.* Preprint 2021 (arXiv:2103.03974)

Further references on Mackey 2-functors:

1. Paul Balmer and Ivo Dell'Ambrogio. *Mackey 2-functors and Mackey 2-motives.* EMS Monographs in Mathematics. Zürich (2020), viii+227
2. Paul Balmer and Ivo Dell'Ambrogio. *Green equivalences in equivariant mathematics.* Math. Ann. (2021)
3. Jun Maillard. *A categorification of the Cartan-Eilenberg formula.* Preprint 2021 (arXiv:2102.07554)
4. Ivo Dell'Ambrogio. *Green 2-functors.* Preprint 2021 (arXiv:2107.09478)