A categorification of Yoshida's theorem for Mackey functors

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CENTRE DE RECERCA MATEMÀTICA

Recollection: ordinary Mackey functors

Fix a finite group G_0.
Fix a commutative ring of coefficients k ($= \mathbb{Z}$, a nice local ring, a field ...).

An ordinary **(k-linear) G_0-Mackey functor** M, is ...

- *Original definition* [Green 1971]:
  - a family of k-modules $M(G)$ for all subgroups $G \leq G_0$
  - with restriction, induction and conjugation k-linear maps between them
  - satisfying many relations (functoriality, commutativity, Mackey formula ...)
- *'Motivic' definition* [Dress 1973, Lindner 1976]:
  - simply a k-linear functor

    $$M\colon \mathrm{Span}_k(G_0\text{-set}) \to \mathrm{Mod}(k)$$

  - on the k-linear category of spans of G_0-sets:

$$\mathrm{Span}_k(G_0\text{-set}) := \begin{cases} \mathrm{Obj} = \text{finite } G_0\text{-sets} \\ \mathrm{Hom}(X,Y) = k \otimes_{\mathbb{N}} \left\{ X \swarrow Z \searrow Y \ \text{ in } G_0\text{-set} \right\} / \text{isos} \\ \text{composition via pullbacks.} \end{cases}$$

Recollection: cohomological Mackey functors

Definition

A Mackey functor is **cohomological** if it satisfies the relations

$$ind_H^G \circ res_H^G = [G : H]\, id$$

between maps $M(G) \to M(G)$ for all subgroup inclusions $H \hookrightarrow G \leq G_0$.

Examples:

- group cohomology $G \mapsto H^n(G; V|_G)$ for any $V \in \mathrm{Mod}(kG_0)$ and $n \geq 0$.
- homology, Tate cohomology, fixed point functors.

Non-examples:

- Burnside ring $G \mapsto B(G) = K_0(G\text{-set}, \sqcup)$.
- Representation ring $G \mapsto Rep(G) = K_0(\mathrm{mod}(\mathbb{C}G), \oplus)$.

Question: the cohomological relations are useful, but what do they mean?

Recollection: Yoshida's theorem

- $\mathrm{perm}_k(G_0) \subset \mathrm{mod}(kG_0)$: full subcategory of f.g. permutation kG_0-modules.
- **Observation:** There is a full, essentially surjective, k-linear (tensor) functor

$$\mathit{Yoshi}_{G_0} : \quad \mathrm{Span}_k(G_0\text{-set}) \longrightarrow \mathrm{perm}_k(G_0)$$

$$\left[X \xleftarrow{\alpha} Z \xrightarrow{\beta} Y \right] \mapsto \Big(k[X] \to k[Y],\, x \mapsto \textstyle\sum_{z\in\alpha^{-1}(x)} \beta(z) \Big).$$

Then:

Yoshida's theorem – [Yoshida 1983], [Panchadcharam-Street 2007]

A Mackey functor M is cohomological iff it factors (uniquely) through Yoshi_{G_0}.

Equivalently, the map-kernel of Yoshi_{G_0} is generated by the cohomological relations (for all $H \le G \le G_0$):

$$\underbrace{\left[G_0/H \xleftarrow{=} G_0/H \rightarrow G_0/G \right]}_{ind_H^G} \circ \underbrace{\left[G_0/G \leftarrow G_0/H \xrightarrow{=} G_0/H \right]}_{res_H^G} = [G:H]\, id.$$

Mackey 2-functors: the idea

Now let us categorify!

- **Naive idea:** want to axiomatize the numerous
  - families of k-linear categories $\mathcal{M}(G)$ for finite groups G
  - with restriction, induction, conjugation (and possibly other kinds of) k-linear functors between them
  - satisfying the most common relations, including:
    - adjunctions between functors
    - (coherent) isomorphism version of the various Mackey axioms.
- The **actual axioms** are inspired by:
  - Ordinary Mackey functors (applying K_0 should yield such!)
  - Additive derivators [Grothendieck 1980's]
  - Similar, but complementary, to the Mackey $(\infty, 1)$-functors of [Barwick 2017]:
    - he wants all higher coherences,
    - we want non-invertible 2-cells.

Mackey 2-functors: the definition

gpd : the 2-category of finite groupoids, functors, natural isos
ADD_k : the 2-category of (additive) k-linear categories, functors and nat. transf.

Definition [Balmer-D. 2020]

A **(k-linear) Mackey 2-functor** is a 2-functor

$$\mathcal{M}\colon gpd^{op} \longrightarrow ADD_k$$

satisfying the following axioms:

1. *Additivity:* $\mathcal{M}(G_1 \sqcup G_2) \overset{\sim}{\to} \mathcal{M}(G_1) \oplus \mathcal{M}(G_2)$ and $\mathcal{M}(\emptyset) \overset{\sim}{\to} 0$.
2. *Induction:* Every 'restriction' $i^* := \mathcal{M}(i)$ along a *faithful* functor between groupoids admits both a left adjoint $i_!$ and a right adjoint i_*.
3. *Base-Change:* left and right adjunctions satisfy base-change for pseudo-pullbacks.
4. *Ambidexterity:* There is a natural isomorphism $i_! \cong i_*$ for all faithful i.

Mackey 2-functors: comments

Explanations:

- **Additivity axiom 1:** groupoids decompose into groups $G \simeq \bigsqcup_i G_i$
  $\rightsquigarrow$ the data of the 2-functor $\mathcal{M}$ is determined by what it does on groups.
- **Induction 2:** as for derivators, co/induction $i_!$ and i_* are not part of the data.
- **Ambidexterity 4:** any isomorphisms $i_! \cong i_*$ will do, so the axiom is easy to check in examples!

  Fact (*rectification theorem*): if axiom 4 holds, there exist unique canonical isomorphisms $\theta_i \colon i_! \cong i_*$ fully compatible with given left and right adjunctions.

Variations are possible, e.g.:

- NB: The previous definition is actually more analogous to *inflation functors*, because it has 'restrictions' f^* along non-faithful maps.
- For the 'proper' *global Mackey functor* analogue: replace *gpd* by gpd_f (only allow faithful functors).
- For the 'local' version for fixed G_0: replace *gpd* with $gpd_f/G_0 \simeq G_0$-set (!).

Exemples of Mackey 2-functors

There is a Mackey 2-functor $\mathcal{M}$ for each of the following families of abelian or triangulated categories $\mathcal{M}(G)$:

- **In (linear) representation theory**:
  $\mathcal{M}(G) = \mathit{mod}\,kG$, $\mathit{Mod}\,kG$, $D(kG)$, $\mathit{stmod}(kG)$ ($\leftarrow$ only on gpd_f), ...
- **In topology**:
  $\mathcal{M}(G) = \mathit{Ho}(\mathcal{S}p^G)$, the homotopy category of genuine G-spectra.
- **In noncommutative geometry**:
  $\mathcal{M}(G) = KK^G$ or E^G, equivariant Kasparov theory or Higson-Connes E-theory of C*-algebras.
- **In geometry** (only defined 'locally' for a fixed group G_0):
  Fix X a locally ringed space (e.g. scheme) with a G_0-action.
  For $G \leq G_0$, $\mathcal{M}(G) = \mathit{Sh}(X /\!\!/ G)$ the category of G-equivariant $\mathcal{O}_X$-modules.
  Variants: take the derived category $D(\mathit{Sh}(X /\!\!/ G))$, or constructible sheaves, or coherent sheaves if X is a noetherian scheme, etc.
- ...

The motivic approach

Theorem (Mackey 2-motives)

There is a k-linear 2-category Mot_k of **Mackey 2-motives**, through which every k-linear Mackey 2-functor $\mathcal{M}$ factors uniquely as a k-linear 2-functor:

$$
\begin{tikzcd}
G \in \arrow[d, mapsto] & gpd^{op} \arrow[r, "\forall\, \mathcal{M}\ \ Mackey"] \arrow[d, "univ"'] & ADD_k \\
\text{(the motive of) } G \in & Mot_k \arrow[ur, dashed, "\exists!\ \widehat{\mathcal{M}}\ \ k\text{-linear}"'] &
\end{tikzcd}
$$

Corollary (motivic decompositions)

The 2-cell endomorphism ring $End_{Mot_k}(\mathrm{Id}_G)$ of a group G acts on the category $\mathcal{M}(G)$, for every k-linear Mackey 2-functor $\mathcal{M}$.
In particular, ring decompositions induce decompositions of the category:

$$1 = \underbrace{e_1 + \ldots + e_n}_{\text{orth. idemp. in } End(\mathrm{Id}_G)} \quad \stackrel{\widehat{\mathcal{M}}(-)}{\Longrightarrow} \quad \mathcal{M}(G) = e_1\mathcal{M}(G) \oplus \ldots \oplus e_n\mathcal{M}(G).$$

More concretely . . .

Mot_k has concrete models (see later), in which we can compute!

Theorem [Balmer-D. 2021 (+ Bouc)]

$End_{Mot_k}(\mathrm{Id}_G)$ is isomorphic to the **crossed Burnside k-algebra** [Yoshida 1997]

$$B_k^c(G) = k \otimes_{\mathbb{Z}} K_0\Big(G\text{-}sets/G^{conj}, \sqcup, \text{ a certain braided} \otimes\Big)$$

or concretely: the finite free k-module generated by G-conjugacy classes of pairs (H, a) with $H \le G$ and $a \in C_G(H)$, with multiplication given by:

$$(K, b) \cdot (H, a) = \sum_{[g] \in K \backslash G/H} (K \cap {}^gH, bgag^{-1})\,.$$

Example ($k = \mathbb{Z}$): consider $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$, on which the **Burnside ring** $B(G) = K_0(G\text{-set}) \cong End(S^0)$ acts because the sphere S^0 is the tensor unit.

But now the bigger ring $B_{\mathbb{Z}}^c(G) \supset B(G)$ with more idempotents also acts:

$$Ho(\mathcal{S}p^G) = \bigoplus_{e \in Prim\ Idem(B_{\mathbb{Z}}^c(G))} e \cdot Ho(\mathcal{S}p^G).$$

Cohomological Mackey 2-functors

Definition [Balmer-D. 2021]

A Mackey 2-functor $\mathcal{M}$ is **cohomological** if the composite

$$\mathrm{Id}_{\mathcal{M}(G)} \xrightarrow{\text{unit}} i_* i^* \underset{\sim}{\overset{\theta^{-1}}{\to}} i_! i^* \xrightarrow{\text{counit}} \mathrm{Id}_{\mathcal{M}(G)}$$

is multiplication by $[G : H]$ for every subgroup inclusion $i \colon H \hookrightarrow G$.

Examples:

- All those from representation theory: $Mod(kG)$, $D(kG)$, $stmod(kG)$,...
- But also equivariant sheaves: $Sh(X/\!\!/G)$, $D(X/\!\!/G)$, $coh(X/\!\!/G)$,...

Why?

Theorem (Hom-decategorification)

If $\mathcal{M}$ is a Mackey 2-functor and $U, V \in \mathcal{M}(G_0)$ two object at some G_0, then

$$G \mapsto M(G) := \mathrm{Hom}_{\mathcal{M}(G)}(Res^{G_0}_G U, Res^{G_0}_G V)$$

is an ordinary G_0-Mackey functor. If $\mathcal{M}$ is cohomological then so is M!

Cohomological Mackey 2-functors

Example:

- Cohomology: $H^n(G; V|_G) = \mathrm{Hom}_{D(kG)}(k, \Sigma^n V|_G)$ for any $V \in Mod(kG_0)$.

But why?

Categorified Cartan-Eilenberg formula [Maillard 2021]

Suppose $\mathcal{M}$ is cohomological, k-linear for a $\mathbb{Z}_{(p)}$-algebra k, and has idempotent-complete values. Then $\mathcal{M}$ is a 2-sheaf for the p-local topology, that is:

$$\forall G \qquad \mathcal{M}(G) \simeq \operatorname*{bilim}_{G/P \in \mathcal{O}_p(G)} \mathcal{M}(P)$$

with the bilimit taken in ADD_k over the orbit category of p-subgroups of G.

OK, but why? What is the sense of the cohomological relations?
Same as before: *they generate the kernel of 'linearization of spans'!*

Mackey 2-motives, concretely

Mackey 2-motives via spans [Balmer-D. 2020]

The universal 2-category Mot_k can be realized as follows:

- Objects: finite groupoids (or formal summands thereof ...)
- 1-Morphisms: spans of functors with faithful right-leg: $\begin{tikzcd} & K \arrow[dl] \arrow[dr, tail] & \\ H \arrow[rr, dotted] & & G \end{tikzcd}$
- 2-Morphisms: k-linearization of the monoid of iso-classes of diagrams

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr, tail] & \\ H & L \arrow[l] \arrow[r, tail] \arrow[u] \arrow[d] & G \\ & K' \arrow[ul] \arrow[ur, tail] & \end{tikzcd} \qquad \text{(in } gpd\text{).}$$

- Vertical and horizontal compositions: via iso-commas / pseudo-pullbacks.

Yoshida's theorem, categorified

$\mathit{biperm}_k^{rf} \subset \mathit{Bimod}$: bicategory of finite groupoids, permutation bimodules between them which are right-free, and equivariant maps (= natural transf.).

Yoshida's theorem [Balmer-D. 2021]

There is an equivalence of k-linear bicategories

$$\mathit{Mot}_k / \langle \text{cohomological relations} \rangle_{\text{2-cell ideal}} \xrightarrow{\sim} \mathit{biperm}_k^{rf}.$$

The pseudo-functor realizing it 'linearizes' spans at the level of 1- and 2-cells:

- It maps $\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, hook, "i"] & \\ H \arrow[rr, dotted] & & G \end{tikzcd}$ to $k[G(i-,-) \otimes_K H(-,f-)] \colon H^{op} \times G \to \mathit{Mod}(k)$.
- Vertically, it sends a span of equivariant maps to a sum-over-preimages homomorphism (exactly like *Yoshi*!).

Corollary

A *cohomological* mackey 2-functor $\mathcal{M}$ is the same as a k-linear pseudo-functor

$$\widehat{\mathcal{M}} \colon \mathit{biperm}_k^{rf} \to \mathit{ADD}_k.$$

Blocks of group algebras

For each G, the quotient pseudo-functor

$$Mot_k \longrightarrow biperm_k^{rf}$$

specializes to the 2-cell endomorphism rings of G, as the *surjective* ring map:

the crossed Burnside algebra $\rightsquigarrow$ $\begin{array}{c} B_k^c(G) \xrightarrow{\rho_G} Z(kG) \\ (H,a) \mapsto \sum_{x\in G/H} xax^{-1} \end{array}$ $\leftsquigarrow$ the center of the group algebra!

Some consequences:

- For every cohomological Mackey 2-functor, $\mathcal{M}(G)$ splits over the usual **blocks** $:=$ primitive idempotents of $Z(kG)$.
  Note: the full kG doesn't always act on $\mathcal{M}(G)$!
- If k is a complete local ring (e.g. a field), then primitive idempotents can be lifted along ρ_G (by a general lifting result).
- For instance, as soon as you 'massage' $Ho(Sp^G)$ so it becomes linear over a complete local ring, e.g. $\mathbb{Z}_p^\wedge$, it splits over the blocks of kG!
- But, which blocks have *non-zero* image on $\mathcal{M}(G)$? More work to be done ...

Thank you for your attention!

Reference:

1. Paul Balmer and Ivo Dell'Ambrogio. *Cohomological Mackey 2-functors.* Preprint 2021 (arXiv:2103.03974)

Further references on Mackey 2-functors:

1. Paul Balmer and Ivo Dell'Ambrogio. *Mackey 2-functors and Mackey 2-motives.* EMS Monographs in Mathematics. Zürich (2020), viii+227.
2. Paul Balmer and Ivo Dell'Ambrogio. *Green equivalences in equivariant mathematics.* Math. Ann. (2021)
3. Jun Maillard. *A categorification of the Cartan-Eilenberg formula.* Preprint 2021 (arXiv:2102.07554)
4. Ivo Dell'Ambrogio. *Green 2-functors.* Preprint June 2021 (on my homepage)