An equivariant Lefschetz formula for smooth actions of compact groups

Ivo Dell'Ambrogio
(joint work with Heath Emerson and Ralf Meyer)

Université de Lille 1

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References:

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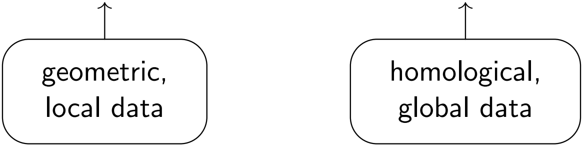
The classical Lefschetz formula (smooth version)

- X a smooth compact manifold
- $f\colon X \to X$ a smooth self-map
- Assume: f has isolated non-singular fixed points p
- $\operatorname{index}_f(p) := \deg_p(\mathrm{id}_X - f) = \operatorname{sign}\det(\mathrm{id}_X - \mathrm{d}_p f)$

Theorem (Lefschetz fixed-point formula [Lefschetz, Hopf 20's])

$$\sum_p \operatorname{index}_f(p) \;=\; \sum_i (-1)^i \operatorname{Tr} H^i(f;\mathbb{Q}) =: \Lambda(f)$$



geometric, local data

homological, global data

$\Lambda(f)$ = Lefschetz number of f.

The classical Lefschetz formula, K-theory version

Consider:

- $K^*(X) = K^0(X) \oplus K^1(X)$, the topological complex K-theory of X.
- Rationally, we have the Chern isomorphism:

$$K^0(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ even}} H^n(X;\mathbb{Q}) \quad \text{and} \quad K^1(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ odd}} H^n(X;\mathbb{Q})\,.$$

Hence we may reformulate the Lefschetz formula in K-theoretic terms:

Theorem (Lefschetz fixed-point formula)

$$\sum_p \mathrm{index}_f(p) \ = \ \mathrm{Tr}\, K^0(f) \otimes \mathbb{Q} - \mathrm{Tr}\, K^1(f) \otimes \mathbb{Q} =: \mathrm{sTr}\big(K^*(f) \otimes \mathbb{Q}\big)$$

sTr = super-trace ($\mathbb{Z}/2$-graded trace)
What is the Lefschetz number $\Lambda(f)$, conceptually?

Conceptually: traces in symmetric tensor categories

According to [Dold-Puppe 1978], we need:

- A **symmetric monoidal category** $\mathcal{C} = (\mathcal{C}, \otimes, \mathbf{1})$.
  If $\mathcal{C}$ is additive, then $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is a commutative ring.
- A **dualizable object** $X \in \mathcal{C}$, i.e., one admitting a *dual* object $X^\vee$ and maps $\eta\colon \mathbf{1} \to X^\vee \otimes X$ and $\epsilon\colon X \otimes X^\vee \to \mathbf{1}$ yielding and adjunction:

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes -, -) \cong \mathrm{Hom}_{\mathcal{C}}(-, X^\vee \otimes -)\,.$$

- Any **endomorphism** $f\colon X \to X$ in $\mathcal{C}$.

Then we can define the (monoidal or categorical) trace of f to be:

$$\mathrm{Tr}_{\otimes}(f) := \left(\begin{tikzcd}
\mathbf{1} \arrow[r, "\eta"] & X^\vee \otimes X \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "\cong"'] & X^\vee \otimes X \arrow[d, "\cong"] & \\
 & X \otimes X^\vee \arrow[r, "f\otimes \mathrm{id}"] & X \otimes X^\vee \arrow[r, "\epsilon"] & \mathbf{1}
\end{tikzcd}\right) \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})\,.$$

Conceptually: traces in symmetric tensor categories

Examples:

1. $\mathcal{C} = \mathrm{Mod}(R)$ for a commutative ring R, $\otimes = \otimes_R$, $\mathbf{1} = R$, $\mathrm{End}(\mathbf{1}) = R$ $\rightsquigarrow$ dualizable $\equiv$ f. g. projective, $\mathrm{Tr}_\otimes(f) =$ usual matrix trace $\mathrm{Tr}(f)$.
2. $\mathcal{C} = \mathrm{D}(R)$, the derived category of R, $\otimes = \otimes^{\mathrm{L}}_R$, $\mathbf{1} = R$, $\mathrm{End}(\mathbf{1}) = R$. $\rightsquigarrow$ dualizable $\equiv$ perfect complexes, $\mathrm{Tr}_\otimes(f) = \sum_i (-1)^i \,\mathrm{Tr}\, H^i(f)$.
3. $\mathbb{Z}/2$-graded modules $\rightsquigarrow$ $\mathrm{Tr}_\otimes(f) =$ super trace $\mathrm{sTr}(f)$.
4. stable homotopy category $\mathcal{C} = \mathrm{SH}$, $\otimes = \wedge$, $\mathbf{1} = \Sigma^\infty S^0$ …
5. $\mathcal{C} = KK$, the **Kasparov category**, a.k.a. KK-theory (see later).

The Lefschetz number $\Lambda(f)$ can be seen as a monoidal trace in any of the above tensor categories …

Conceptually: traces in symmetric tensor categories

General ideal: find a symmetric monoidal functor F:

$$\begin{tikzcd} (\{\text{“compact spaces”}\}, \times, \mathrm{pt}) \arrow[r, "F"] & (\mathcal{C}, \otimes, \mathbf{1}) \\ f\colon X \to X \arrow[r, phantom, "\mapsto"] & F(f) \in \mathrm{End}_{\mathcal{C}}(FX) \end{tikzcd}$$

Just make sure that:

- FX is dualizable in $\mathcal{C}$, for X suitably small.
- Traces can be computed effectively in $\mathcal{C}$.

Note: any symmetric monoidal functor $F\colon \mathcal{C} \to \mathcal{C}'$ preserve traces!

- E.g., if $\mathrm{End}(\mathbf{1})$ is torsion-free then $\mathrm{End}(\mathbf{1}) \hookrightarrow \mathrm{End}(\mathbf{1}) \otimes \mathbb{Q}$, so to simplify computations we can rationalize (localize): $\mathcal{C} \to \mathcal{C}_{\mathbb{Q}}$.

Question

What if X comes with a smooth action by a Lie group? Are there interesting fixed-point formulas that take the action into consideration?

GOALS

Our goal is to generalize the Lefschetz formula in the context of KK-theory.

The generalization is in two different directions:

1. by allowing a compact Lie group G to act on X.
   For G-*equivariant* f, we will have $\mathrm{Tr}_{\otimes}(f) \in R(G) =$ the complex representation ring (character ring) of G.
2. by allowing not only maps $f\colon X \to X$, but general *smooth* G-*equivariant correspondences* $\varphi = \big[X \xleftarrow{f} (M, \xi) \xrightarrow{g} X\big]$.

Prototype of the generalized formula:

For $\varphi\colon X \to X$ a correspondence (e.g. $\varphi = [f]$ from a map f) we compute $\mathrm{Tr}_{\otimes}(\varphi)$ in the G-equivariant Kasparov category KK^G in two ways:

$$\mathrm{Tr}_{\otimes}(f) \quad = \quad \boxed{\begin{array}{c}\text{geometric}\\ \text{\& local}\end{array}} \quad = \quad \boxed{\begin{array}{c}\text{homological}\\ \text{\& global}\end{array}} \quad \in \quad R(G)\,.$$

The equivariant Kasparov category

Let G be a compact Lie group.

The *G-equivariant Kasparov category*, KK^G, has the following properties:

1. Its objects are the separable complex G-C*-algebras.
   For instance: X (locally) compact G-space $\Rightarrow$ $C_0(X) \in KK^G$.
2. Hom-sets: $KK^G(A, B)$, Kasparov's G-equivariant KK-theory groups.
   E.g: $KK^G(\mathbb{C}, C_0(X)) \cong K^0_G(X)$ is equivariant K-theory.
3. KK^G is additive, indeed it is a triangulated category ($\Sigma^2 \cong \mathrm{id}$),
4. KK^G is a tensor category: $A \otimes B = A \hat{\otimes}_{\mathrm{min}} B$, $\mathbf{1} = \mathbb{C}$,
   $\mathrm{End}_{KK^G}(\mathbf{1}) \cong R(G)$, the complex representation ring of G.
5. We have a symmetric monoidal functor:

$$(\{\text{(locally) compact } G\text{-spaces}\}, \times, \mathrm{pt})^{\mathrm{op}} \xrightarrow{C_0(-)} KK^G$$

If X is a smooth compact G-manifold then $C_0(X) = C(X)$ is dualizable (can construct a dual geometrically).

The Geometric side of the equation (sketch)

[Emerson-Meyer 2006-10] Category of smooth correspondences, $\widehat{KK}^G$:

- Objects: smooth G-manifolds.
- Morphisms: smooth G-equivariant correspondences

$$\widehat{KK}^G(X,Y) \ni \varphi = \left[X \overset{f}{\leftarrow} M \overset{g}{\rightarrow} Y, \xi\right]$$

  Roughly speaking:
    - $f\colon M \to X$ a smooth G-map
    - $g\colon M \to Y$ a G-map + a nice factorization + K_G-orientation
    - $\xi \in RK^*_{G,X}(M)$ a K-theory class with X-compact supports
- Composition: intersection product (pull-back).
- Have monoidal functors:

$$\begin{tikzcd}
\{G\text{-manifolds}\}^{\mathrm{op}} \arrow[r, "f \mapsto {[f,\, g=\mathrm{id},\, \xi=1]}"'] \arrow[rr, bend left, "C_0(-)"] & \widehat{KK}^G \arrow[r] & KK^G
\end{tikzcd}$$

- $\widehat{KK}^G(X,Y) \overset{\cong}{\to} KK^G(C_0(X), C_0(Y))$, if X has finite orbit type, e.g. if X is compact.

The Geometric side of the equation (sketch)

X a compact smooth G-manifold.

$\varphi = [X \xleftarrow{f} M \xrightarrow{g} X, \xi]$ a correspondence $X \to X$.

- Coincidence space $Q := \{m \in M \mid f(m) = g(m)\}$
- Assume that f, g *intersect smoothly*:
  $Q \subseteq M$ smooth sub-mf, and $\forall\, v \in \mathrm{T}M$, $\mathrm{d}f(v) = \mathrm{d}g(v) \Rightarrow v \in \mathrm{T}Q$.
- Excess bundle $\eta := \mathrm{Coker}(\mathrm{d}f - \mathrm{d}g) \in K^0_G(Q)$, a G-bundle on Q.

Theorem 1 (Emerson-Meyer 2013)

With the above hypotheses and notations, we have:

$$\mathrm{Tr}_{\otimes}(\varphi) = \left(\begin{array}{c} \textit{the index of the Dirac operator on } Q \\ \textit{twisted by the bundle } \xi|_Q \otimes e(\eta) \in K^*_G(Q) \end{array} \right) \in R(G)$$

For $G = 1$ and $\varphi = [f, \mathrm{id}, 1]$ where f has isolated non-singular fixed-points, this reduce to $\sum_{p \in Q} \mathrm{index}_f(x)$.

The Homological side: preliminaries

$S^{-1}R(G)$: the total ring of fractions, $S := \{\text{nonzero divisors of } R(G)\}$.
Obtain an inclusion $R(G) \hookrightarrow S^{-1}R(G)$ into a product of fin. many fields:

$$S^{-1}R(G) \cong \prod_{\mathfrak{p}_i \text{ minimal prime}} \underbrace{\mathrm{Frac}\big(R(G)/\mathfrak{p}_i\big)}_{=: F_i, \text{ a field.}}$$

Have a bijection: $\{\mathfrak{p}_i\} \leftrightarrow \{\text{cyclic subgroups of } G/G^0\}$
In particular, $S^{-1}R(G)$ is a field iff G is connected.

Since $R(G) = \mathrm{End}(\mathbf{1})$ acts on KK^G, we can localize the whole category KK^G at S to obtain a (simpler!) tensor-triangulated category:

$$KK^G \xrightarrow{\text{symm. mon.}} S^{-1}KK^G$$

By injectivity, it will suffice to compute traces on the right-hand side.

The Homological side: abstract result

For any $A \in KK^G$, there is a decomposition of $S^{-1}R(G)$-modules:

$$S^{-1}K_*^G(A) \overset{\text{def.}}{=} S^{-1}KK_*^G(\mathbb{C}, A) \cong \bigoplus_i K_{*,i}^G(A)$$

where $K_{*,i}^G(A)$ is a $\mathbb{Z}/2$-graded vector space over F_i.

Theorem 2 ([D.-E.-M. 2013])

Assume that $A \in KK^G$ belongs to the thick subcategory generated by $\mathbb{C}$, $\mathrm{Thick}(\mathbb{C})$. Then A is dualizable and for any $\varphi \in KK^G(A, A)$ we have

$$\mathrm{Tr}_\otimes(\varphi) = \Big(\mathrm{sTr}\, K_{*,i}^G(\varphi) \Big)_i \quad \in \quad \prod_i F_i = S^{-1}R(G)$$

where $K_{*,i}^G(\varphi) \colon K_{*,i}^G(A) \to K_{*,i}^G(A)$ is the F_i-linear map induced by φ.

The Homological side: Hodgkin Lie groups

Assume G is a Hodgkin Lie group: compact, connected, $(\pi_1 G)_{\mathrm{tors}} = 0$.
Examples: $G = \mathbb{T}^n$ a torus, $U(n)$, $SU(n)$, $\mathrm{Sp}(n)$, $\mathrm{Spin}(n)$ $(n \geq 2)$, ...

Theorem 3 ([DEM 2013])

Let $A \in KK^G$ for G Hodgkin. Then $A \in \mathrm{Thick}(\mathbb{C})$ iff:
(1) A is dualizable, and (2) $\mathrm{Res}^G_1(A)$ belongs to the Bootstrap category.

(2) is satisfied e.g. if $A = C_0(X)$ is commutative.

(1) is satisfied if moreover X is a smooth G-manifold ($\exists$ explicit dual).

Corollary 4 ([DEM 2013])

Let G be a Hodgkin Lie group and X a smooth compact G-manifold.

For any correspondence $\varphi \in \widehat{KK}^G(X, X)$ we have

$$\mathrm{Tr}_{\otimes}(\varphi) = \mathrm{sTr}\left(S^{-1}K^G_*(\varphi)\right) \in S^{-1}R(G) = \mathrm{Frac}\, R(G)\,.$$

The Homological side: other groups?

This fails badly for non-connected groups!

Example

Let $G \neq 1$ be any finite group, e.g. $G = \mathbb{Z}/2$. Then $A = C(G) \in KK^G$ is *not* contained in $\mathrm{Thick}(\mathbb{C})$ (although $C(G)$ is certainly dualizable). Worse: the given trace formula is wrong for $\varphi \in KK^G(C(G), C(G)) \cong \mathbb{Z}[G]$!

For $\varphi \equiv g \in G$, we have $\mathrm{Tr}_\otimes(\varphi) = \begin{cases} [\mathbb{C}G] \in R(G) & \text{if } g = 1 \\ 0 & \text{if } g \neq 1. \end{cases}$

But $K_*^G(C(G)) = K_G^*(G) \cong \mathbb{Z}$ and $K_*^G(\varphi) = \mathrm{id}$ for all g.

Solution: consider $K_*^H(\varphi)$ for sufficiently many subgroups $H \leq G$: this solves both issues at the same time! In fact, it suffices to consider a certain subroup H_i for each minimal prime ideal $\mathfrak{p}_i$ of $R(G)$.

The Homological side: general groups

Now let G be any compact Lie group.

Recall from [Segal 1968]:

$$\big\{\text{minimal primes } \mathfrak{p} \subset R(G)\big\} \overset{\sim}{\longleftrightarrow} \big\{\text{cyclic subgps. } C \leq G/G^0\big\}/_{\text{conj.}}$$
$$\overset{\sim}{\longleftrightarrow} \big\{\text{Cartan subgps. } H \leq G\big\}/_{\text{conj.}}$$

H Cartan: topologically cyclic ($H \cong \mathbb{T}^r \times \mathbb{Z}/k$) and $|N_G(H)/H| < \infty$.

$$S^{-1}R(G) \cong \prod_{H \text{ Cartan } / \text{ conj.}} \underbrace{\mathrm{Frac}\big(R(G)/\mathfrak{p}_H\big)}_{=:F_H}$$

If $A \in KK^G$, then:

- $K_*^H(A) \overset{\text{def.}}{=} \mathrm{KK}_*^H(\mathbb{C}, \mathrm{Res}_H^G A)$ is an $R(G)$-module via $R(G) \overset{\text{res}}{\to} R(H)$.
- $K_*^H(A) \otimes_{R(G)} F_H$ is an F_H-vector space.

The Homological side: general groups

Definition (The G-equivariant Bootstrap class)

$\mathcal{B}^G := \mathrm{Loc}\{\mathrm{Ind}_H^G(M_n(\mathbb{C})) : H \leq G \text{ closed}, H \curvearrowright M_n(\mathbb{C}) \text{ any action}\} \subseteq KK^G$

Note that $\mathcal{B}^G$ contains $C(X)$ for all compact G-manifolds, but also many non-commutative G-C*-algebras.

Theorem 5 ([DEM 2013])

Let $A \in KK^G$ be a dualizable algebra in $\mathcal{B}^G$. Then for any $\varphi \in KK^G(A, A)$ we have

$$\mathrm{Tr}_\otimes(\varphi) = \Big(\mathrm{sTr}\left(K_*^H(\varphi) \otimes_{R(G)} F_H\right)\Big)_H \in \prod_{H \text{ Cartan}/_{conj.}} F_H = S^{-1}R(G)$$

where $K_*^H(\varphi) \otimes_{R(G)} F_H$ is the map induced by φ on $K_*^H(A) \otimes_{R(G)} F_H$.