

The Green correspondence in algebraic geometry

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References:

- Paul Balmer and Ivo Dell'Ambrogio. *Green equivalences in equivariant mathematics*. Preprint 2020, 25 pages ([arXiv:2001.10646](https://arxiv.org/abs/2001.10646)).
- Paul Balmer and Ivo Dell'Ambrogio. *Mackey 2-functors and Mackey 2-motives*. Preprint 2019, 209 pages ([arXiv:1808.04902](https://arxiv.org/abs/1808.04902)). To appear in EMS Monographs in Mathematics.

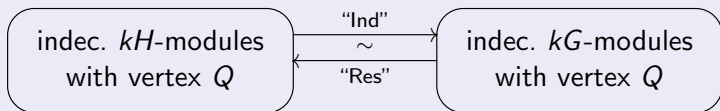
1. The Green correspondence

A classical result in modular representation theory of finite groups:

- k a field of characteristic $p > 0$
- G a finite group
- For M an indecomposable kG -module \rightsquigarrow the **vertex of M** : the minimal subgroup $Q \leq G$ such that $M \leq \text{Ind}_Q^G(N)$ for some kQ -module N (this Q is a p -group, unique up to conjugation)

The Green correspondence (J. A. Green 1958, 1964)

For all $Q \leq H \leq G$ with $N_G(Q) \subseteq H$, there is a bijection



where N and M correspond iff $N \leq \text{Res}_H^G(M)$ iff $M \leq \text{Ind}_H^G(N)$.

2. The classical Green equivalence

Notations:

- $\mathcal{M}(G) := kG\text{-mod}$, the category of finitely generated kG -modules
- $\mathcal{M}(G; S) := \{M \mid M \leq \text{Ind}(N) \text{ for some } N \in \mathcal{M}(S)\} \overset{\text{full}}{\subset} \mathcal{M}(G)$
the full subcategory of **S-objects**, for $S \leq G$ a subgroup
- $\mathcal{M}(G; \mathbb{S})$ defined similarly for a set \mathbb{S} of subgroups of G

The Green equivalence (J. A. Green 1974)

For $Q \leq H \leq G$ as before, there is an equivalence of “additive quotients”

$$\frac{\mathcal{M}(H; Q)}{\mathcal{M}(H; \mathbb{X})} \begin{array}{c} \xrightarrow{\text{Ind}} \\ \sim \\ \xleftarrow{\text{“Res”}} \end{array} \frac{\mathcal{M}(G; Q)}{\mathcal{M}(G; \mathbb{X})}$$

where $\mathbb{X} = \mathbb{X}(G, H, Q) := \{Q \cap gQg^{-1} \mid g \in G \setminus H\}$.

3. The classical Green equivalence: explanations

Recall **additive quotient categories**:

- Given \mathcal{A} an additive category, $\mathcal{B} \subset \mathcal{A}$ full additive subcategory
- \mathcal{A}/\mathcal{B} or $\frac{\mathcal{A}}{\mathcal{B}}$: category with same objects as \mathcal{A} and Hom groups:

$$\mathcal{A}/\mathcal{B}(X, Y) = \frac{\mathcal{A}(X, Y)}{\left\{ \varphi: X \rightarrow Y \mid \exists \begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & \searrow \tau & \nearrow \tau \\ & Z & \end{array} \text{ with } Z \in \mathcal{B} \right\}}$$

- Example: $kG\text{-mod} := kG\text{-mod}/kG\text{-proj}$, the stable module category
Not abelian, but a triangulated category!
- For similar reasons, the 'relative stable categories' in the Green equivalence are also triangulated

4. Green equiv + Krull-Schmidt \Rightarrow Green corr

- $kG\text{-mod}$ is a **Krull-Schmidt category**, in particular:

\forall object $M \quad \exists$ decomposition $M \simeq M_1 \oplus \cdots \oplus M_n$ with M_1, \dots, M_n indecomposable and unique (up to iso and a perm.)

- Property preserved by taking subcategories and quotient categories
- The Green equivalence preserves vertices
 \Rightarrow can derive the Green correspondence from the Green equivalence:

$$\begin{array}{ccc}
 \mathcal{M}(H) & \xrightarrow{\text{Ind}} & \mathcal{M}(G) \\
 \uparrow & & \uparrow \\
 N & \mathcal{M}(H; Q) \cdots \cdots \cdots \rightarrow & \mathcal{M}(G; Q) \\
 \downarrow \text{vert}=Q & \downarrow & \downarrow \\
 N & \mathcal{M}(H; Q)/\mathcal{M}(H; \mathbb{X}) \xrightarrow{\sim} & \mathcal{M}(G; Q)/\mathcal{M}(G; \mathbb{X})
 \end{array}
 \qquad
 \begin{array}{c}
 \text{Ind}(N) \geq M \\
 \downarrow \exists! M \text{ vert}=Q \\
 M
 \end{array}$$

5. Generalizations

Note:

- The *statement* of the Green eq. makes sense as soon as we have: add cats $\mathcal{M}(S)$ ($S \leq G$) & add functors $\text{Ind}, \text{Res} \rightsquigarrow \mathcal{M}(G; Q)$ etc.
- Green eq. \Rightarrow Green corr. always, as soon as the $\mathcal{M}(S)$ are all KS
- Can prove the Green eq. for other categories $\mathcal{M}(S) = ??$

Previous results:

- 1 Benson-Wheeler 2001: $\mathcal{M}(G) = kG\text{-Mod}$, ∞ -dim representations
 - 2 Carlson-Wang-Zhang 2020: $\mathcal{M}(G) =$ derived and homotopy categories of complexes (variously bounded) of kG -modules
- Note: $\mathcal{M}(G) = D^b(kG\text{-mod})$ is Krull-Schmidt
 \Rightarrow a Green correspondence for indecomposable complexes!

6. The 'right' context

Our contribution: use the setting of *Mackey 2-functors*!

- Idea: the *proofs* really only need additive categories $\mathcal{M}(S)$ ($S \leq G$), adjoint induction and restriction functors, the Mackey formula!
- More precisely:

Definition (Balmer-Dell'Ambrogio 2019)

A **Mackey 2-functor for G** is a 2-functor $\mathcal{M}: (\text{gpd}^f/G)^{op} \rightarrow \text{ADD}$ satisfying the axioms:

- 1 **Additivity:** $\mathcal{M}(G_1 \sqcup G_2) \simeq \mathcal{M}(G_1) \times \mathcal{M}(G_2)$
- 2 **Induction:** for $K \xrightarrow{i} L \leq G$ faithful (e.g. a subgroup inclusion), the functor $\text{Res}_K^L := \mathcal{M}(i): \mathcal{M}(L) \rightarrow \mathcal{M}(K)$ has a right-and-left adjoint Ind_K^L
- 3 **The Mackey formula:** Both the left and the right adjunctions satisfy Base-Change for iso-comma squares in gpd^f/G (\simeq pullbacks in G -sets).

7. The 'right' result

The general Green equivalence (Balmer-D. 2020)

\mathcal{M} any Mackey 2-functor for G , and $Q \leq H \leq G$ any subgroups.
Then the induction functor induces an equivalence of categories

$$\left(\frac{\mathcal{M}(H; Q)}{\mathcal{M}(H; \mathbb{X})} \right)^{\natural} \xrightarrow[\sim]{\text{Ind}} \left(\frac{\mathcal{M}(G; Q)}{\mathcal{M}(G; \mathbb{X})} \right)^{\natural}$$

where $\mathbb{X} = \{Q \cap {}^g Q \mid g \in G \setminus H\}$ and $(-)^{\natural}$ is idempotent-completion.

Remarks:

- No conditions on coefficients, or on $N_G(Q)$!
- No idempotent-completion needed in any of the examples, for different reasons: (e.g. if the $\mathcal{M}(S)$ are KS or nicely triangulated)
- In the Krull-Schmidt case, we always get the Green correspondence

8. Mackey 2-functors are everywhere!

There is a Mackey 2-functor \mathcal{M} for G for each of the following families of abelian or triangulated categories $\mathcal{M}(S)$ (for all $S \leq G$):

- 1 In (linear) representation theory:
 $\mathcal{M}(S) = kS\text{-mod}, kS\text{-Mod}, D^b(kS\text{-mod}), D(kS\text{-Mod}), kS\text{-mod} \dots$
- 2 In stable homotopy theory:
 $\mathcal{M}(S) = Ho(\mathcal{S}p^S)$, the homotopy category of genuine S -spectra
- 3 In noncommutative topology / operator algebras:
 $\mathcal{M}(S) = KK^S$ or E^S , equivariant Kasparov theory or E-theory
- 4 In geometry:
Fix X a locally ringed space (e.g. scheme) with a G -action
 $\mathcal{M}(S) = Sh(X//S)$ or $D(Sh(X//S))$, S -equivariant sheaves

9. Application to equivariant sheaves

In algebraic geometry:

- X a scheme
- Assume G acts on X (hence each $S \leq G$ too)
- There is a Mackey 2-functor \mathcal{M} such that $\mathcal{M}(S) = \text{Coh}(X//S)$, the category of **S-equivariant sheaves of coherent \mathcal{O}_X -modules**:
 $M = (M, \{\gamma_g: M \xrightarrow{\sim} g^*(M)\}_{g \in S})$ with $\begin{cases} M \in \text{Coh}(X) \\ \text{cocycle condition on } \gamma_g \text{'s} \end{cases}$
- For $X = \text{Spec}(k)$ we have $\text{Coh}(X//S) = kS\text{-mod}$
- We obtain the Green equivalence

$$\frac{\text{Coh}(X//H; \mathbb{Q})}{\text{Coh}(X//H; \mathbb{X})} \xrightarrow[\sim]{\text{Ind}} \frac{\text{Coh}(X//G; \mathbb{Q})}{\text{Coh}(X//G; \mathbb{X})}$$

and similarly with $\text{Qcoh}(X//S)$, $D(\text{Qcoh}(X//S))$, ...

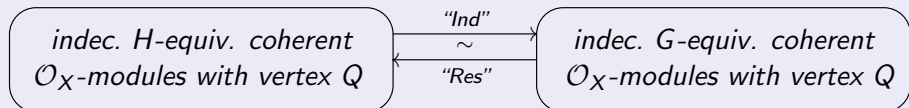
10. Application to equivariant sheaves

For X nice enough the categories $\text{Coh}(X//S)$, and thus $D^b(\text{Coh}(X//S))$, are Krull-Schmidt \rightsquigarrow get the correspondence:

Theorem (The 'global' Green correspondence)

G a finite group acting on a regular proper (e.g. smooth projective) variety X over k , a field of characteristic $p > 0$ dividing the order of G .

Then for every p -subgroup $Q \leq G$ and every $H \leq G$ containing $N_G(Q)$:



- The same holds with complexes in $D^b(\text{Coh}(X//S))$ instead of sheaves
- With $X = \text{Spec}(k)$ we recover all previous results ☺