The Green correspondence in algebraic geometry

Ivo Dell'Ambrogio
Université de Lille

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References:

- Paul Balmer and Ivo Dell'Ambrogio. *Green equivalences in equivariant mathematics*. Preprint 2020, 25 pages (arXiv:2001.10646).
- Paul Balmer and Ivo Dell'Ambrogio. *Mackey 2-functors and Mackey 2-motives*. Preprint 2019, 209 pages (arXiv:1808.04902). To appear in EMS Monographs in Mathematics.

1. The Green correspondence

A classical result in modular representation theory of finite groups:

- k a field of characteristic $p > 0$
- G a finite group
- For M an indecomposable kG-module $\rightsquigarrow$ the **vertex** of M: the minimal subgroup $Q \le G$ such that $M \le \mathrm{Ind}_Q^G(N)$ for some kQ-module N (this Q is a p-group, unique up to conjugation)

The Green correspondence (J. A. Green 1958, 1964)

For all $Q \le H \le G$ with $N_G(Q) \subseteq H$, there is a bijection

$$\begin{tikzcd} \boxed{\begin{matrix}\text{indec. } kH\text{-modules} \\ \text{with vertex } Q\end{matrix}} \arrow[r, shift left, "\text{“Ind”}"] \arrow[r, phantom, "\sim"] & \boxed{\begin{matrix}\text{indec. } kG\text{-modules} \\ \text{with vertex } Q\end{matrix}} \arrow[l, shift left, "\text{“Res”}"] \end{tikzcd}$$

where N and M correspond iff $N \le \mathrm{Res}_H^G(M)$ iff $M \le \mathrm{Ind}_H^G(N)$.

2. The classical Green equivalence

Notations:

- $\mathcal{M}(G) := kG\text{-mod}$, the category of finitely generated kG-modules
- $\mathcal{M}(G;S) := \{M \mid M \leq \mathrm{Ind}(N) \text{ for some } N \in \mathcal{M}(S)\} \overset{\text{full}}{\subset} \mathcal{M}(G)$ the full subcategory of ***S*-objects**, for $S \leq G$ a subgroup
- $\mathcal{M}(G;\mathbb{S})$ defined similarly for a set $\mathbb{S}$ of subgroups of G

The Green equivalence (J. A. Green 1974)

For $Q \leq H \leq G$ as before, there is an equivalence of "additive quotients"

$$\frac{\mathcal{M}(H;Q)}{\mathcal{M}(H;\mathbb{X})} \underset{\text{“Res”}}{\overset{\mathrm{Ind}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \frac{\mathcal{M}(G;Q)}{\mathcal{M}(G;\mathbb{X})}$$

where $\mathbb{X} = \mathbb{X}(G,H,Q) := \{Q \cap gQg^{-1} \mid g \in G \smallsetminus H\}$.

3. The classical Green equivalence: explanations

Recall **additive quotient categories**:

- Given $\mathcal{A}$ an additive category, $\mathcal{B} \subset \mathcal{A}$ full additive subcategory
- $\mathcal{A}/\mathcal{B}$ or $\frac{\mathcal{A}}{\mathcal{B}}$: category with same objects as $\mathcal{A}$ and Hom groups:

$$\mathcal{A}/\mathcal{B}(X,Y) = \frac{\mathcal{A}(X,Y)}{\left\{ \varphi\colon X \to Y \;\middle|\; \exists \begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, dotted] & & Y \\ & Z \arrow[ur, dotted] & \end{tikzcd} \text{ with } Z \in \mathcal{B} \right\}}$$

- Example: $kG\text{-}\underline{\mathrm{mod}} := kG\text{-mod}/kG\text{-proj}$, the stable module category
  Not abelian, but a triangulated category!
- For similar reasons, the 'relative stable categories' in the Green equivalence are also triangulated

4. Green equiv + Krull-Schmidt $\Rightarrow$ Green corr

- kG-mod is a **Krull-Schmidt category**, in particular:

> $\forall$ object M $\exists$ decomposition $M \simeq M_1 \oplus \cdots \oplus M_n$ with
> $M_1, \ldots, M_n$ indecomposable and unique (up to iso and a perm.)

- Property preserved by taking subcategories and quotient categories
- The Green equivalence preserves vertices
  $\Rightarrow$ can derive the Green correspondence from the Green equivalence:

$$
\begin{tikzcd}
 & \mathcal{M}(H) \arrow[rr, "\mathrm{Ind}"] & & \mathcal{M}(G) & \\
N \arrow[d, "\text{vert}=Q"] & \mathcal{M}(H;Q) \arrow[u, hook] \arrow[rr, dotted] \arrow[d, two heads] & & \mathcal{M}(G;Q) \arrow[u, hook] \arrow[d, two heads] & \mathrm{Ind}(N) \geq M \arrow[d, "\exists! M \ \text{vert}=Q"] \\
N & \mathcal{M}(H;Q)/\mathcal{M}(H;\mathbb{X}) \arrow[rr, dotted, "\sim"] & & \mathcal{M}(G;Q)/\mathcal{M}(G;\mathbb{X}) & M
\end{tikzcd}
$$

5. Generalizations

Note:

- The *statement* of the Green eq. makes sense as soon as we have: add cats $\mathcal{M}(S)$ ($S \leq G$) & add functors $\mathrm{Ind}, \mathrm{Res} \rightsquigarrow \mathcal{M}(G; Q)$ etc.
- Green eq. $\Rightarrow$ Green corr. always, as soon as the $\mathcal{M}(S)$ are all KS
- Can prove the Green eq. for other categories $\mathcal{M}(S) = ??$

Previous results:

1. Benson-Wheeler 2001: $\mathcal{M}(G) = kG$-Mod, ∞-dim representations
2. Carlson-Wang-Zhang 2020: $\mathcal{M}(G) =$ derived and homotopy categories of complexes (variously bounded) of kG-modules

   Note: $\mathcal{M}(G) = D^b(kG\text{-mod})$ is Krull-Schmidt

   $\Rightarrow$ a Green correspondence for indecomposable complexes!

6. The 'right' context

Our contribution: use the setting of *Mackey 2-functors*!

- Idea: the *proofs* really only need additive categories $\mathcal{M}(S)$ ($S \leq G$), adjoint induction and restriction functors, the Mackey formula!
- More precisely:

Definition (Balmer-Dell'Ambrogio 2019)

A Mackey 2-functor for G is a 2-functor $\mathcal{M}\colon (gpd^f/G)^{op} \to ADD$ satisfying the axioms:

1. **Additivity:** $\mathcal{M}(G_1 \sqcup G_2) \simeq \mathcal{M}(G_1) \times \mathcal{M}(G_2)$
2. **Induction:** for $K \overset{i}{\hookrightarrow} L \leq G$ faithful (e.g. a subgroup inclusion), the functor $\mathrm{Res}^L_K := \mathcal{M}(i)\colon \mathcal{M}(L) \to \mathcal{M}(K)$ has a right-and-left adjoint Ind^L_K
3. **The Mackey formula:** Both the left and the right adjunctions satisfy Base-Change for iso-comma squares in gpd^f/G ($\simeq$ pullbacks in G-sets).

7. The 'right' result

The general Green equivalence (Balmer-D. 2020)

$\mathcal{M}$ any Mackey 2-functor for G, and $Q \leq H \leq G$ any subgroups.
Then the induction functor induces an equivalence of categories

$$\left(\frac{\mathcal{M}(H;Q)}{\mathcal{M}(H;\mathbb{X})}\right)^{\natural} \xrightarrow[\sim]{\quad\mathrm{Ind}\quad} \left(\frac{\mathcal{M}(G;Q)}{\mathcal{M}(G;\mathbb{X})}\right)^{\natural}$$

where $\mathbb{X} = \{Q \cap {}^{g}Q \mid g \in G \smallsetminus H\}$ and $(-)^{\natural}$ is idempotent-completion.

Remarks:

- No conditions on coefficients, or on $N_G(Q)$!
- No idempotent-completion needed in any of the examples, for different reasons: (e.g. if the $\mathcal{M}(S)$ are KS or nicely triangulated)
- In the Krull-Schmidt case, we always get the Green correspondence

8. Mackey 2-functors are everywhere!

There is a Mackey 2-functor $\mathcal{M}$ for G for each of the following families of abelian or triangulated categories $\mathcal{M}(S)$ (for all $S \leq G$):

1. In (linear) representation theory:
   $\mathcal{M}(S) = kS\text{-mod}$, $kS\text{-Mod}$, $D^b(kS\text{-mod})$, $D(kS\text{-Mod})$, $kS\text{-}\underline{\text{mod}}$ ...
2. In stable homotopy theory:
   $\mathcal{M}(S) = Ho(\mathcal{S}p^S)$, the homotopy category of genuine S-spectra
3. In noncommutative topology / operator algebras:
   $\mathcal{M}(S) = KK^S$ or E^S, equivariant Kasparov theory or E-theory
4. In geometry:
   Fix X a locally ringed space (e.g. scheme) with a G-action
   $\mathcal{M}(S) = Sh(X/\!/S)$ or $D(Sh(X/\!/S))$, S-equivariant sheaves

9. Application to equivariant sheaves

In algebraic geometry:

- X a scheme
- Assume G acts on X (hence each $S \leq G$ too)
- There is a Mackey 2-functor $\mathcal{M}$ such that $\mathcal{M}(S) = \mathrm{Coh}(X /\!\!/ S)$, the category of ***S*-equivariant sheaves of coherent $\mathcal{O}_X$-modules**: $M = (M, \{\gamma_g \colon M \overset{\sim}{\to} g^*(M)\}_{g \in S})$ with $\begin{cases} M \in \mathrm{Coh}(X) \\ \text{cocycle condition on } \gamma_g\text{'s} \end{cases}$
- For $X = \mathrm{Spec}(k)$ we have $\mathrm{Coh}(X /\!\!/ S) = kS\text{-mod}$
- We obtain the Green equivalence

$$\frac{\mathrm{Coh}(X /\!\!/ H; Q)}{\mathrm{Coh}(X /\!\!/ H; \mathbb{X})} \xrightarrow[\sim]{\mathrm{Ind}} \frac{\mathrm{Coh}(X /\!\!/ G; Q)}{\mathrm{Coh}(X /\!\!/ G; \mathbb{X})}$$

and similarly with $\mathrm{Qcoh}(X /\!\!/ S)$, $D(\mathrm{Qcoh}(X /\!\!/ S))$, ...

10. Application to equivariant sheaves

For X nice enough the categories $\mathrm{Coh}(X /\!\!/ S)$, and thus $\mathrm{D^b}(\mathrm{Coh}(X /\!\!/ S))$, are Krull-Schmidt $\rightsquigarrow$ get the correspondence:

Theorem (The 'global' Green correspondence)

G a finite group acting on a regular proper (e.g. smooth projective) variety X over k, a field of characteristic $p > 0$ dividing the order of G.

Then for every p-subgroup $Q \leq G$ and every $H \leq G$ containing $N_G(Q)$:

$$\boxed{\begin{array}{c}\textit{indec. } H\textit{-equiv. coherent} \\ \mathcal{O}_X\textit{-modules with vertex } Q\end{array}} \ \underset{\text{“Res”}}{\overset{\text{“Ind”}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \ \boxed{\begin{array}{c}\textit{indec. } G\textit{-equiv. coherent} \\ \mathcal{O}_X\textit{-modules with vertex } Q\end{array}}$$

- The same holds with complexes in $D^b(\mathrm{Coh}(X /\!\!/ S))$ instead of sheaves
- With $X = \mathrm{Spec}(k)$ we recover all previous results ☺