The spectrum of equivariant Kasparov theory for cyclic groups of prime order

Ivo Dell'Ambrogio
Université de Lille

Quantum Groups Seminar
Copenhagen, 30 November 2020

Reference:
Ivo Dell'Ambrogio and Ralf Meyer, arXiv:2009.05424 (Sept 2020)

1. The equivariant Kasparov category

G : a 2nd countable locally compact group
$\rightsquigarrow$ KK^G : the ***G*-equivariant Kasparov category** (Kasparov 1988)

- Objects: separable complex G-C*-algebras
- Hom sets: $Hom(A,B) = KK_0^G(A,B)$
  Kasparov cycles $/\sim$, or generalized equiv. $*$-homomorphisms, or ...
- composition & a symmetric monoidal structure: 'Kasparov product'

(Meyer-Nest 2006)

KK^G is a **tensor triangulated category**, that is:

- additive category: can sum morphisms and objects (usual direct sums)
- suspension functor $\Sigma\colon A \mapsto C_0(\mathbb{R}) \otimes A$ (invertible by Bott: $\Sigma^2 \cong \mathrm{Id}$)
- triangles: $\Sigma C \to A \to B \to C$ (mapping cones / cpc-split extensions)
- bi-exact tensor product: $A \otimes_{\min} B$ with diagonal G-action

This algebraic structure captures for example:

- **Homological algebra:** get LES from triangles $\Sigma C \stackrel{\partial}{\to} A \to B \to C$:

$$\ldots \to Hom(D, \Sigma C) \stackrel{\partial_*}{\to} Hom(D, A) \to Hom(D, B) \to Hom(D, C) \stackrel{\partial_*}{\to} \ldots$$

$$\ldots \leftarrow Hom(\Sigma C, D) \stackrel{\partial^*}{\leftarrow} Hom(A, D) \leftarrow Hom(B, D) \leftarrow Hom(C, D) \stackrel{\partial^*}{\leftarrow} \ldots$$

- **Bootstrap-like constructions:** $\mathcal{S}$ any set of objects:
  Thick$(\mathcal{S})$:= closure of $\mathcal{S}$ under $\Sigma^{\pm}$, sums, mapping cones, retracts, and isomorphic objects.
  Loc$(\mathcal{S})$:= as above + closed under *infinite* direct sums.

  Both constructions yield (full) triangulated subcategories.

  $\mathrm{Thick}_{\otimes}(\mathcal{S}), \mathrm{Loc}_{\otimes}(\mathcal{S})$: variants closed under tensoring with any objects
  $\rightsquigarrow$ these are (thick, localizing) *tensor ideals*.

3. The Balmer spectrum

Can also 'do geometry':

(Balmer 2005)

Every (essentially small) tensor triangulated category $\mathcal{T}$ admits a 'universal support theory', namely:

- A topological space $\mathrm{Spc}(\mathcal{T})$, the **spectrum** of $\mathcal{T}$.
- For each $A \in \mathcal{T}$, a closed subset $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{T})$, its **support**.
- This data yields a rough geometric classification of objects:
  $\mathrm{Thick}_{\otimes}(A) = \mathrm{Thick}_{\otimes}(B) \quad \Leftrightarrow \quad \mathrm{supp}(A) = \mathrm{supp}(B)$

Examples:

- (Thomason 1997) V an quasi-compact and quasi-separated scheme,
  $\mathcal{T} = D^{perf}(V) \quad \rightsquigarrow \quad \mathrm{Spc}(\mathcal{T}) \cong V$.
  In particular for $V = \mathrm{Spec}(R) \quad \rightsquigarrow \quad \mathrm{Spc}(D^b(\text{proj-}R)) \cong \mathrm{Spec}(R)$.
- (Benson-Carlson-Rickard 1997) G a finite group, $\mathrm{char}(k) \mid |G|$,
  $\mathcal{T} = \mathrm{stmod}(kG) \quad \rightsquigarrow \quad \mathrm{Spc}(\mathcal{T}) \cong \mathrm{Proj}(H^*(G; k))$.

4. So, what about $\mathcal{T} = KK^G$?

A very nice characterisation of the Baum-Connes assembly map:

(Meyer-Nest 2006)

The inclusion functor of the following subcategory

$$\mathcal{CI} := \mathrm{Loc}_{(\otimes)}\Big(\bigcup_{H \leq G \text{ compact}} \mathrm{Ind}_H^G(KK^H) \Big) \quad \subset \quad KK^G$$

has a right adjoint $A \mapsto \tilde{A} \in \mathcal{CI}$. Applying $K_*(G \ltimes -)$ to the counit of adjunction $\varepsilon_A \colon \tilde{A} \to A$ we get the Baum-Connes assembly map with coefficients in $A \in KK^G$.

Tantalizingly:

(D. 2008)

If the natural map $(\mathrm{Res}_H^G)_H^* \colon \bigcup_{H \text{ cpt}} \mathrm{Spc}(KK^H) \to \mathrm{Spc}(KK^G)$ is surjective, we have $\mathcal{CI} = KK^G$, hence $\tilde{A} \overset{\simeq}{\to} A$, hence BC holds for G and all A.

5. Towards to spectrum of Kasparov theory

Unfortunately, the computation of $\mathsf{Spc}(KK^G)$ seems well out of reach! Only general result known:

(Balmer 2010)

For any (essentially small) tensor triangulated $\mathcal{T}$, there is a natural continuous map

$$\rho_{\mathcal{T}} \colon \mathsf{Spc}(\mathcal{T}) \longrightarrow \mathsf{Spec}(\mathsf{End}_{\mathcal{T}}(\mathbf{1}))$$

to the Zariski spectrum of the endomorphism ring of the tensor unit object $\mathbf{1}$. It is surjective as soon as $\mathsf{End}_{\mathcal{T}}(\mathbf{1})_*$ is noetherian.

Corollary

For G a compact Lie group, we have a surjective map

$$\mathsf{Spc}(KK^G) \longrightarrow \mathsf{Spec}(\mathrm{R}_{\mathbb{C}}(G))$$

onto the Zariski spectrum of its complex character ring.

6. Bootstrap categories are nicer

Main technical difficulties:

- KK^G has no good **generation properties**.
- KK^G has (countable) infinite direct sums, but $\mathrm{Spc}(-)$ is best for (sub-)categories of **compact** and **dualizable** objects A: those which
  - satisfy $Hom(A, \bigoplus_i B_i) \cong \bigoplus_i Hom(A, B_i)$
  - and have a tensor-dual $A^\vee$: $Hom(A \otimes B, C) \cong Hom(B, A^\vee \otimes C)$.

Definition: G-cell algebras

$Cell^G := \mathrm{Loc}(\{C(G/H) \colon H \leq G \text{ a closed subgroup}\}) \quad \subset KK^G$

- $Cell^1$ is the usual Rosenberg-Schochet bootstrap category.
- For G compact, $Cell^G$ is again a tensor triangulated category, and:
  - it is 'countably compactly-rigidly generated'.
  - its compact and dualizable objects agree $\rightsquigarrow$ they form a nice ttc $Cell^G_c$.

7. The spectrum of compact G-cell algebras

(D. 2010)

For G finite, the map $\rho\colon \mathrm{Spc}(Cell_c^G) \longrightarrow \mathrm{Spec}(\mathrm{R}_{\mathbb{C}}(G))$ is split surjective.

(D.-Meyer 2020)

For G cyclic of prime order, the map $\rho\colon \mathrm{Spc}(Cell_c^G) \xrightarrow{\sim} \mathrm{Spec}(\mathrm{R}_{\mathbb{C}}(G))$ is injective, hence a homeomorphism.

From now on, ideas for the proof. Set $G \cong \mathbb{Z}/p\mathbb{Z}$ for a prime p.
Recall:

$$\mathrm{R}_{\mathbb{C}}(G) \cong \mathbb{Z}[\hat{G}] \cong \mathbb{Z}[x]/(x^p - 1)$$

and $x^p - 1$ has two irreducible factors:

$$x - 1 \quad \text{and} \quad \Phi_p = 1 + x + \ldots + x^{p-1}.$$

8. Computation for $G \cong \mathbb{Z}/p\mathbb{Z}$

Modding them out in turn:

$$\mathbb{Z} \xleftarrow{\text{mod } x-1} \underbrace{\mathbb{Z}[x]/(x^p - 1)}_{\mathrm{R}_{\mathbb{C}}(G)} \xrightarrow{\text{mod } \Phi_p} \mathbb{Z}[x]/(\Phi_p) := \mathbb{Z}[\vartheta]$$

Two irreducible components, their intersection is the unique closed point over p. By inverting p on the RHS, get a disjoint union decomposition:

$$\operatorname{Spec} \mathbb{Z} \rightarrowtail \operatorname{Spec} \mathrm{R}_{\mathbb{C}}(G) \leftarrowtail \operatorname{Spec} \mathbb{Z}[\vartheta, p^{-1}]$$

Now, lift 'the same' decomposition to $Cell^G$, as follows:

$$Cell^1 \xleftarrow{\mathrm{Res}^G_1} Cell^G \xrightarrow{\text{localisation}} Cell^G/\mathrm{Loc}\{C(G)\} =: \mathcal{Q}^G$$

9. Computation for $G \cong \mathbb{Z}/p\mathbb{Z}$

Restrict these two tensor-exact functors to compact objects and apply $\mathrm{Spc}(-)$ to get the top row:

$$\begin{array}{ccccc}
\mathrm{Spc}\, \mathit{Cell}^1_c & \rightarrowtail & \mathrm{Spc}\, \mathit{Cell}^G_c & \leftarrowtail & \mathrm{Spc}\, \mathcal{Q}^G_c \\
\rho \downarrow \cong & & \rho \downarrow & & \cong \downarrow \rho \\
\mathrm{Spec}\, \mathbb{Z} & \rightarrowtail & \mathrm{Spec}\, \mathrm{R}_{\mathbb{C}}(G) & \leftarrowtail & \mathrm{Spec}\, \mathbb{Z}[\vartheta, p^{-1}]
\end{array}$$

- The top row is also a disjoint union decomposition (Balmer 2005+15).
- The left ρ is known to be bijective (D. 2010).
- $\mathrm{End}(\mathbf{1})_* \cong \mathbb{Z}[\vartheta, p^{-1}, \beta^{\pm 1}]$ in $\mathcal{Q}^G$, computed thanks to Köhler's UCT.
- In particular, the right square commutes!
- The right ρ is bijective by an abstract criterion (D.-Stanley 2016), since $\mathcal{Q}^G_c = \mathrm{Thick}\{\mathbf{1}\}$ by construction and $\mathrm{End}(\mathbf{1})_*$ is regular as seen.

Hence the middle ρ is bijective as well. QED