Green 2-functors

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D., "Green 2-functors", *Trans. AMS* (to appear)
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Some structure in the representation theory of finite groups

Let $\mathcal{M}(G)$ be either $mod(kG)$, $D^b(mod\, kG)$, or $\underline{mod}(kG)$. Let us vary G!

The following are basic tools, used all the time:

- The **restriction**, **induction** and **conjugation** functors ($H \leq G$):

$$\begin{array}{ccc} \mathcal{M}(G) & & \\ {\scriptstyle Ind} \uparrow \downarrow {\scriptstyle Res} & & \\ \mathcal{M}(H) & \xrightarrow[\sim]{Conj_g} & \mathcal{M}({}^g H) \end{array}$$

- The **adjunctions** $Ind \dashv Res \dashv Ind$
  (Note: have an ambijunction only because index finite & categories additive!)
- **Conjugation natural isos** between composites functors, e.g.

$$conj_g \colon Conj_g \circ Res^G_H \cong Res^G_{{}^g H}$$

- Other relations, most notably the **Mackey formula** (for $K, L \leq G$):

$$Res^G_L \circ Ind^G_K \cong \bigoplus_{[g] \in L \backslash G / K} Ind^L_{L \cap {}^g K} \circ Conj_g \circ Res^K_{L^g \cap K} \ .$$

An axiomatization: Mackey 2-functors

gpd : the 2-category of finite groupoids, functors, natural isomorphisms
ADD : the 2-category of additive categories, additive functors, natural transf.

Definition [Balmer–D. 2020]

A **Mackey 2-functor** is a 2-functor

$$\mathcal{M} \colon gpd^{op} \longrightarrow ADD$$

satisfying the following axioms:

1. *Additivity:* $\mathcal{M}(G_1 \sqcup G_2) \xrightarrow{\sim} \mathcal{M}(G_1) \oplus \mathcal{M}(G_2)$ and $\mathcal{M}(\emptyset) \xrightarrow{\sim} 0$.
2. *Induction:* Every 'restriction' $i^* := \mathcal{M}(i)$ along a *faithful* functor i between groupoids admits both a left adjoint $i_!$ and a right adjoint i_*.
3. *Base-Change / Mackey formula:* the left and right adjunctions satisfy base-change with respect to pseudo-pullbacks (iso-comma squares).
4. *Ambidexterity:* There is a natural isomorphism $i_! \cong i_*$ for all faithful i.

Mackey 2-functors: comments

Explanations:

- **Additivity axiom:** groupoids decompose into groups $G \simeq \coprod_i G_i$
  $\rightsquigarrow$ the data of the 2-functor $\mathcal{M}$ is determined by what it does on groups.
- **Induction:** as for derivators, (co)induction $i_!$ and i_* are not part of the data.
- **Ambidexterity:** any isomorphisms $i_! \cong i_*$ will do, so the axiom is easy to check in examples!

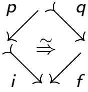
  Fact ("rectification theorem"): the axioms imply there exist unique isomorphisms $\theta_i \colon i_! \cong i_*$ fully compatible with given left and right adjunctions.

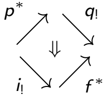
Variations are possible:

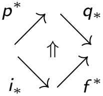
- NB: The previous definition is actually more analogous to inflation functors, because it has 'restrictions' f^* along non-faithful morphisms f.
- For the 'correct' analogue of global Mackey functor: replace *gpd* by *gpd*$_f$ (only allow faithful functors).
- For the G_0-local version (only $G \leq G_0$): replace *gpd* with $gpd_f / G_0 \simeq G_0$-set.
- We can vary the target 2-category: triangulated cats, add. derivators,...

Base-Change = canonical Mackey formulas

Base-Change axiom: an iso-comma square γ in *gpd* with two faithful sides defines, via $\mathcal{M}$ and the left/right adjunctions, two mates $\gamma_!$ and $(\gamma^{-1})_*$:

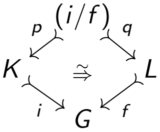
$\gamma =$ 

$\rightsquigarrow$ $\gamma_! =$ 

and $(\gamma^{-1})_* =$ 

The axiom requires both to be invertible: $f^* i_! \cong q_! p^*$ and $f^* i_* \cong q_* p^*$.
Convenient fact: via the rectification isos θ, they are mutual inverses!

Motivating example: for i, f two subgroup inclusions $K, L \leq G$

iso-comma 

$\rightsquigarrow$ $$(i/f) \simeq \coprod_{[g] \in L \backslash G / K} L \cap {}^{g}K$$

get a Mackey formula!

Note: the iso-comma groupoid (i/f) and the Base-Change isos are canonical, but the decomposition into groups depends on choices!

Exemples of Mackey 2-functors

There is a Mackey 2-functor $\mathcal{M}$ for each of the following families of abelian or triangulated categories $\mathcal{M}(G)$:

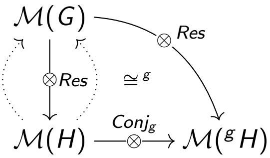
- **In (linear) representation theory**:
  $\mathcal{M}(G) = mod(kG)$, $Mod(kG)$, $D(kG)$, $\underline{mod}(kG)$ ($\leftsquigarrow$ only on gpd_f), ...
- **In (formal) representation theory**:
  $\mathcal{M}(G) = Mack_k(G)$ or $CoMack_k(G)$, categories of ordinary Mackey functors!
- **In topology**:
  $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$, the homotopy category of genuine G-spectra.
- **In geometry** (only defined 'locally' for a fixed group G_0):
  Fix X a locally ringed space (e.g. scheme) with a G_0-action.
  For $G \leq G_0$, $\mathcal{M}(G) = Sh(X /\!\!/ G)$ the category of G-equivariant $\mathcal{O}_X$-modules.
  Variants: take the derived category $D(Sh(X /\!\!/ G))$, or constructible sheaves, or coherent sheaves if X is a noetherian scheme, etc.

Note: all these are *tensor* categories $\rightsquigarrow$ examples of "Green 2-functors"?

What do we find in the examples?

Let $\mathcal{M}(G)$ be any of the previous examples, say $mod(kG)$.

- Each category $\mathcal{M}(G)$ is **(symmetric) monoidal**.
- Restriction and conjugation functors are **strong tensor functors**:



- The conjugation natural isos are **monoidal natural transformations**.
- There is a **projection formula** (two of them, related by the symmetry):

$$Ind_H^G(\ Res_H^G(X) \otimes_H Y\ ) \ \cong\ X \otimes_G Ind_H^G(Y)$$

All of this seems to be 'coherent'...

An axiomatization: Green 2-functors

Definition

A **(symmetric) pre-Green 2-functor** is a Mackey 2-functor $\mathcal{M}$ equipped with a lifting

$$\begin{array}{ccc} & & PsMon(ADD) \\ & \nearrow & \downarrow{\scriptstyle forget} \\ gpd^{op} & \xrightarrow{\mathcal{M}} & ADD \end{array}$$

to (symm.) pseudo-monoids in ADD.

A pre-Green 2-functor $\mathcal{M}$ and a faithful $i \colon H \rightarrowtail G$ define commutative squares

$$\begin{array}{ccc} X \otimes i_*(Y) & \xrightarrow{Rproj} & i_*(X \otimes i^* Y) \\ \simeq \uparrow \theta & & \theta \uparrow \simeq \\ X \otimes i_!(Y) & \xleftarrow{Lproj} & i_!(X \otimes i^* Y) \end{array} \qquad \begin{array}{ccc} i_*(Y) \otimes X & \xrightarrow{Rproj} & i_*(i^* Y \otimes X) \\ \theta \uparrow \simeq & & \theta \uparrow \simeq \\ i_!(Y) \otimes X & \xleftarrow{Lproj} & i_!(i^* Y \otimes X) \end{array}$$

where *Lproj* and *Rproj* are the evident mate transformations of the strong monoidal structure of i^* under the two *rectified* adjunctions $i_! \dashv i^* \dashv i_*$.

An axiomatization: Green 2-functors

Definition

The **projection formulas** hold for i if in both squares the maps $Rproj, Lproj$ are invertible, hence mutually inverse modulo θ.
A (symm.) pre-Green 2-functor is a **(symm.) Green 2-functor** if this holds $\forall i$.

Examples: all previous Mackey 2-functors, for the usual tensor structures.

Theorem (Origins of the projection formulas)

The projection formulas hold for all i if and only if the **external tensor products**

$$\mathcal{M}(G_1) \times \mathcal{M}(G_2) \xrightarrow{\overline{\otimes}} \mathcal{M}(G_1 \times G_2)$$

associated with the given internal ones $\otimes$ "preserves inductions" in both variables.

Warning! This holds on *gpd* or *G-set*, but not gpd_f (which is not Cartesian!)

Application: "finite-index Res_H^G are finite étale extensions"

Let $\mathcal{M}$ be a Green 2-functor and $i\colon H \rightarrowtail G$ faithful.
$\Rightarrow$ The (co)induction functors $i_!, i_*\colon \mathcal{M}(H) \to \mathcal{M}(G)$ are colax / lax monoidal.
$\Rightarrow$ The object $A(i) := i_*(\mathbf{1}) \cong i_!(\mathbf{1})$ is both a monoid and comonoid in $\mathcal{M}(G)$.

Theorem (Induced Frobenius algebras)

$A(i)$ is a special Frobenius monoid, commutative when $\mathcal{M}$ is symmetric.

Corollary (Separable tensor monadicity - [Balmer-Sanders-D. 2016])

If $\mathcal{M}$ is idempotent complete (each category is), there are canonical equivalences

$$
\begin{array}{ccccc}
 & & \mathcal{M}(G) & & \\
 & \swarrow^{\textit{Cofree}} & \downarrow{\scriptstyle i^*} & \searrow^{\textit{Free}} & \\
A(i)\text{-}\mathit{Comod} & \xleftarrow{\;\sim\;} & \mathcal{M}(H) & \xrightarrow{\;\sim\;} & A(i)\text{-}\mathit{Mod}
\end{array}
$$

of *symmetric monoidal* categories if $\mathcal{M}$ is symmetric, and of tensor *triangulated* categories if moreover $\mathcal{M}$ is triangulated.

Application: decategorifications

A Mackey 2-functor $\mathcal{M}$ can be 'decategorified' in at least two different ways:

K-decategorification - [Dress 1973] [Balmer–D. 2020]

If $\mathcal{M}$ is essentially small, the composite $K_0^{add} \circ \mathcal{M}$ is an ordinary Mackey functor.
Variants: If $\mathcal{M}$ takes the appropriate values, we can use K_0^{triang}, K_0^{ex}, $K_*^{Quillen}$, ...

Clearly: if $\mathcal{M}$ is a Green 2-functor, its K-decategorifications are *Green* functors!

Hom-decategorification - [Balmer–D. 2022]

If the Mackey 2-functor $\mathcal{M}$ is equipped with two **coherent families of objects**

$$\Big\{X_G, Y_G \in \mathcal{M}(G) \text{ for all } G, \quad i^*X_G \cong X_H \,,\, i^*Y_G \cong Y_H \text{ for all } i\colon H \rightarrowtail G\Big\}$$

then there is an ordinary Mackey functor M such that

$$M(G) = Hom_{\mathcal{M}(G)}(X_G, Y_G).$$

Variants: with graded Homs ...

Ordinary Green functors via Hom-decategorifications

Theorem (“Endo-style”)

Choosing $X_G \equiv Y_G$ in a Hom-decategorification yields a Green functor M with

$$M(G) = End_{\mathcal{M}(G)}(X_G).$$

Example (“Unit-style”)

If $\mathcal{M}$ is a Green 2-functor, we can always take $X_G = Y_G = \mathbf{1}_G$ the tensor unit, $\forall G$.

More generally:

Theorem (“Convolution-style”)

Given a Green 2-functor $\mathcal{M}$ equipped with

- a coherent family of *comonoids* $X_G \in \mathcal{M}(G)$
- a coherent family of *monoids* $Y_G \in \mathcal{M}(G)$

then the associated Mackey functor $G \mapsto M(G) = Hom_{\mathcal{M}(G)}(X_G, Y_G)$ is an ordinary Green functor via the convolution products.

The origins of classical Green functors

Fact: all classical examples of Green functors from algebra and topology arise in one of these two of three ways from some naturally occurring Green 2-functor.

For instance:

- **Group cohomology:** $M(G) = H^*(G; k) = Hom^*_{D(kG)}(k, k)$
  $\rightsquigarrow$ Unit-style (graded) Hom-decat. from the Green 2-functor $D(kG)$.
- **The Burnside ring:** $M(G) := K_0(G\text{-}set) \cong K_0(Span(G\text{-}set))$
  $\rightsquigarrow$ K-decat. from $Mack_{\mathbb{Z}}^{f.g.free}(G) = \mathbb{Z}Span(G\text{-}set)$
  $\rightsquigarrow$ also: unit-style Hom-decat. from $SH(G)$!
- **The complex character ring:** $M(G) = Rep_{\mathbb{C}}(G) = K_0(\mathbb{C}G\text{-}mod)$
  $\rightsquigarrow$ K-decat from $\mathbb{C}G\text{-}mod$, obviously
  $\rightsquigarrow$ also: unit-style Hom-decat from the Kasparov category $KK(G)$
- **Fixed points:** $H \mapsto A^H = Hom_{kH}(k, A)$, for A a G-algebra (G fixed, $H \leq G$)
  $\rightsquigarrow$ Convolution-style Hom-decat. from $kG\text{-}mod$, with $X_H = \mathbf{1}_H$ and $Y_H = A$.

These methods yield an industrial production of Green functors, as well as *modules over them.*

Thank you for your attention!