On the origins of classical Green functors

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Reference:
D., "Green 2-functors", *Trans. AMS* (to appear)
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Some structure in the representation theory of finite groups

$\mathcal{M}(G) := mod(kG)$: the category of k-linear representations of a finite group G.

By varying G, we get the following basic structures:

- The **restriction**, **induction** and **conjugation** functors ($H \leq G$, $g \in G$):

$$\begin{array}{ccc} \mathcal{M}(G) & & \\ Ind \uparrow \;\; \downarrow Res & & \\ \mathcal{M}(H) & \xrightarrow[\sim]{Conj_g} & \mathcal{M}({}^gH) \end{array}$$

- The **adjunctions** $Ind \dashv Res \dashv Ind$ ($\Leftarrow$ index finite & categories additive!)
- **Conjugation natural isos** between composites, e.g.

$$conj_g \colon Conj_g \circ Res^G_H \cong Res^G_{{}^gH}$$

- Other relations, most notably the **Mackey formula** (for $K, L \leq G$):

$$Res^G_L \circ Ind^G_K \cong \bigoplus_{[g] \in L \backslash G / K} Ind^L_{L \cap {}^gK} \circ Conj_g \circ Res^K_{L^g \cap K} \,.$$

An axiomatization: Mackey 2-functors

gpd : the 2-category of finite groupoids, functors, natural isomorphisms
ADD : the 2-category of additive categories, additive functors, natural transf.

Definition [Balmer–D. 2020]

A **Mackey 2-functor** is a 2-functor

$$\mathcal{M}\colon gpd^{op} \longrightarrow ADD$$

satisfying the following four axioms.

1. **Additivity axiom**

$$\mathcal{M}(G_1 \sqcup G_2) \overset{\sim}{\to} \mathcal{M}(G_1) \oplus \mathcal{M}(G_2) \quad \text{and} \quad \mathcal{M}(\emptyset) \overset{\sim}{\to} 0.$$

$\rightsquigarrow$ groupoids decompose into groups $G \simeq \sqcup_i G_i$, hence the *structure* of the Mackey 2-functor $\mathcal{M}$ is determined by the data associated to finite groups, group homomorphisms and their conjugation relations (see previous slide!).

A very nice induction

2. **Induction axiom:** For every *faithful* $i\colon H \to G$, the 'restriction' functor $\mathcal{M}(i) = i^*$ has a left adjoint i_ℓ and a right adjoint i_r:

$$\begin{array}{ccc} G & \mapsto & \mathcal{M}(G) \\ {\scriptstyle i}\uparrow & & {\scriptstyle i_\ell}\uparrow \ \downarrow{\scriptstyle i^*} \ \uparrow{\scriptstyle i_r} \\ H & \mapsto & \mathcal{M}(H) \end{array}$$

Note: the adjoints are not really part of the structure.

3. **Ambidexterity axiom:** For every faithful i, there is an isomorphism $i_\ell \cong i_r$.

The above are easy to check in examples, but we get more:

Rectification theorem

Under the four axioms, there exist for all i unique isomorphisms $\theta_i\colon i_\ell \cong i_r$ fully compatible with given left and right adjunctions.

Base-Change axiom = canonical Mackey formulas

4. **Base-Change axiom:** each iso-comma square γ in *gpd* with two faithful sides defines, via $\mathcal{M}$ and the left/right adjunctions, two mates γ_ℓ and $(\gamma^{-1})_r$:

$$\gamma = \begin{array}{c} \overset{p}{\swarrow} \quad \overset{q}{\searrow} \\ \overset{\sim}{\Rightarrow} \\ \underset{i}{\searrow} \quad \underset{f}{\swarrow} \end{array} \quad \rightsquigarrow \quad \gamma_\ell := \begin{array}{c} \overset{p^*}{\nearrow} \quad \overset{q_\ell}{\searrow} \\ \Downarrow \\ \underset{i_\ell}{\searrow} \quad \underset{f^*}{\nearrow} \end{array} \quad \text{and} \quad (\gamma^{-1})_r := \begin{array}{c} \overset{p^*}{\nearrow} \quad \overset{q_r}{\searrow} \\ \Uparrow \\ \underset{i_r}{\searrow} \quad \underset{f^*}{\nearrow} \end{array}$$

We require both to be invertible: $f^* i_\ell \cong q_\ell p^*$ and $f^* i_r \cong q_r p^*$.

Convenient fact: via the rectification θ's, the two mates are mutual inverses!

Motivating example: for i, f two subgroup inclusions $K, L \leq G$

$$\text{iso-comma} \quad \begin{array}{c} (i/f) \\ K \overset{p}{\swarrow} \quad \overset{q}{\searrow} L \\ \overset{\sim}{\Rightarrow} \\ \underset{i}{\searrow} \quad \underset{f}{\swarrow} \\ G \end{array} \quad \rightsquigarrow \quad (i/f) \simeq \coprod_{[g] \in L \backslash G / K} L \cap {}^{g}K$$

get a Mackey formula

Variations on the axioms

Variations are possible:

- Note: The previous definition is actually more analogous to inflation functors, because it has 'restrictions' f^* along non-faithful morphisms $f\colon H \to G$.
- For the analogue of global Mackey functor: replace *gpd* by gpd_f (only allow faithful functors).
- For the G_0-local version (only allow $G \leq G_0$ for a fixed group G_0): replace *gpd* with the comma 2-category gpd_f / G_0.
  Note: this is equivalent to the ordinary category of finite left G_0-sets:

$$G_0\text{-}set \xrightarrow{\sim} gpd_f / G_0$$

$$X \longmapsto \left(G_0 \ltimes X \,,\; G_0 \ltimes X \overset{forget}{\to} G_0 \right)$$

- We can also vary the target 2-category:
  abelian categories, triangulated categories, additive derivators, . . .

Examples of Mackey 2-functors

Each of the following families of additive (in fact, abelian or triangulated) categories $\mathcal{M}(G)$ defines a Mackey 2-functor:

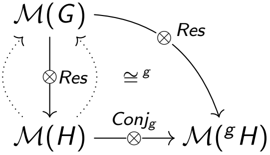
- **From (linear) representation theory**:
  $\mathcal{M}(G) = mod(kG)$ or $Mod(kG)$ f.dim. or all representations over k
  $\mathcal{M}(G) = D^b(mod\, kG)$ or $D(kG)$ (un)bounded derived category
  $\mathcal{M}(G) = \underline{mod}(kG)$ stable module category ($\mathcal{M}$ only defined on gpd_f!)
- **From (formal) representation theory**:
  $\mathcal{M}(G) = Mack_k(G)$ or $CoMack_k(G)$ ordinary (cohom.) Mackey functors!
- **From topology**:
  $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$ the homotopy category of genuine G-spectra.
- **From noncommutative geometry**:
  $\mathcal{M}(G) = KK^G$ the equivariant Kasparov category of G-C*-algebras

Note: all these are *tensor* categories $\rightsquigarrow$ examples of "Green 2-functors"?

Multiplicative structure in the examples

Let $\mathcal{M}(G) = mod(kG)$, say:

- Each category $\mathcal{M}(G)$ is **(symmetric) monoidal** via $- \otimes_k -$, with unit k.
- Restriction and conjugation functors are **strong tensor functors**:



- The conjugation natural isos are **monoidal natural transformations**.
- There is a **projection formula** (two of them, related by the symmetry):

$$Ind_H^G(\ Res_H^G(X) \otimes_H Y\ ) \ \cong\ X \otimes_G Ind_H^G(Y)$$

All of this data seems to be 'coherent'...

An axiomatization: Green 2-functors

Definition

A **(symmetric) Green 2-functor** is a Mackey 2-functor $\mathcal{M}$ equipped with a lifting

$$\begin{array}{ccc} & & PsMon(ADD) \\ & \nearrow & \downarrow{\scriptstyle forget} \\ gpd^{op} & \xrightarrow{\mathcal{M}} & ADD \end{array}$$

to (symm.) pseudo-monoids in ADD, satisfying the **projection formulas** below.

Given a lift of $\mathcal{M}$ and a faithful $i\colon H \rightarrowtail G$, we have in particular a strong monoidal structure on i^*, hence by taking mates for the adjunctions $i_\ell \dashv i^*$ and $i^* \dashv i_r$ we get natural maps *Lproj* and *Rproj* making the two squares commute:

$$\begin{array}{ccc} X \otimes i_r(Y) & \xrightarrow{Rproj} & i_r(X \otimes i^* Y) \\ \simeq\uparrow\theta & & \theta\uparrow\simeq \\ X \otimes i_\ell(Y) & \xleftarrow{Lproj} & i_\ell(X \otimes i^* Y) \end{array} \qquad \begin{array}{ccc} i_r(Y) \otimes X & \xrightarrow{Rproj} & i_r(i^* Y \otimes X) \\ \theta\uparrow\simeq & & \theta\uparrow\simeq \\ i_\ell(Y) \otimes X & \xleftarrow{Lproj} & i_\ell(i^* Y \otimes X) \end{array}$$

An axiomatization: Green 2-functors

Definition

$\mathcal{M}$ satisfies the **projection formulas** if in aech square the maps *Rproj*, *Lproj* are invertible, hence mutually inverse modulo θ.

Examples:
all previous Mackey 2-functors are Green 2-functors for the usual tensor structures.

Theorem (Origins of the projection formulas)

The projection formulas hold for all *i* if and only if the **external tensor products**

$$\mathcal{M}(G_1) \times \mathcal{M}(G_2) \xrightarrow{\quad \overline{\otimes} \quad} \mathcal{M}(G_1 \times G_2)$$

associated with the given internal ones $\otimes$ "preserve inductions" in both variables.

Warning:
This reformulation holds on *gpd* or *G-set*, but not gpd_f (which is not Cartesian!)

An application: decategorifications

A Mackey 2-functor $\mathcal{M}$ can be 'decategorified' in at least two different ways:

K-decategorification - [Dress 1973] [Balmer–D. 2020]

If $\mathcal{M}$ is essentially small, the composite $K_0^{add} \circ \mathcal{M}$ is an ordinary Mackey functor.
Variants: If $\mathcal{M}$ takes the appropriate values, we can use K_0^{triang}, K_0^{ex}, $K_*^{Quillen}$, ...

Clearly: if $\mathcal{M}$ is a Green 2-functor, its K-decategorifications are *Green* functors!

Hom-decategorification - [Balmer–D. 2022]

If the Mackey 2-functor $\mathcal{M}$ is equipped with two **coherent families of objects**

$$\Big\{ X_G, Y_G \in \mathcal{M}(G) \text{ for all } G, \quad i^* X_G \cong X_H \,,\; i^* Y_G \cong Y_H \text{ for all } i \colon H \to G \Big\}$$

then there is an ordinary Mackey functor M such that

$$M(G) = Hom_{\mathcal{M}(G)}(X_G, Y_G).$$

Variants: with graded Homs ...

Ordinary Green functors via Hom-decategorifications

Theorem ("Endo-style")

Choosing $X_G \equiv Y_G$ in a Hom-decategorification yields a Green functor M with

$$M(G) = End_{\mathcal{M}(G)}(X_G).$$

Example ("Unit-style")

If $\mathcal{M}$ is a Green 2-functor, we can always take $X_G = Y_G = \mathbf{1}_G$ the tensor unit, $\forall G$.

More generally:

Theorem ("Convolution-style")

Given a Green 2-functor $\mathcal{M}$ equipped with

- a coherent family of *comonoids* $X_G \in \mathcal{M}(G)$
- a coherent family of *monoids* $Y_G \in \mathcal{M}(G)$

then the associated Mackey functor $G \mapsto M(G) = Hom_{\mathcal{M}(G)}(X_G, Y_G)$, equipped with the convolution products, is an ordinary Green functor.

The origins of classical Green functors

Fact: all classical examples of Green functors from algebra and topology arise in one of these two of three ways from some naturally occurring Green 2-functor.

For instance:

- **Group cohomology:** $M(G) = H^*(G; k) = Hom^*_{D(kG)}(k, k)$
  $\rightsquigarrow$ unit-style (graded) Hom-decat. from the Green 2-functor $D(kG)$.
- **The Burnside ring:** $M(G) := K_0(G\text{-}set) \cong K_0(Span(G\text{-}set))$
  $\rightsquigarrow$ K-decat. from $Mack_{\mathbb{Z}}^{f.g.free}(G) = \mathbb{Z}Span(G\text{-}set)$ ($\leftsquigarrow$ made additive)
  $\rightsquigarrow$ also: unit-style Hom-decat. from $Ho(Sp^G)$!
- **The complex character ring:** $M(G) = Rep_{\mathbb{C}}(G) = K_0(\mathbb{C}G\text{-}mod)$
  $\rightsquigarrow$ K-decat from $\mathbb{C}G\text{-}mod$, obviously
  $\rightsquigarrow$ also: unit-style Hom-decat. from the Kasparov category KK^G
- **Fixed points:** $H \mapsto A^H = Hom_{kH}(k, A)$, for A a G-algebra (G fixed, $H \leq G$)
  $\rightsquigarrow$ convolution-style Hom-decat. from $kG\text{-}mod$, with $X_H = \mathbf{1}_H$ and $Y_H = A$.

These methods yield an industrial production of Green functors, as well as *modules over them.*

Thank you for your attention!

The motivic approach

An ordinary Green functor is a monoid in the tensor category of Mackey functors. This should categorify in the Cartesian case (for $\mathcal{M}$ defined on *gpd* or *G-set*):

Theorem [Balmer-D. 2020]

There is an additive 2-category *Mot* of **Mackey 2-motives**, through which every Mackey 2-functor $\mathcal{M}$ factors uniquely as an additive 2-functor:

$$
\begin{array}{ccc}
G \in & \qquad gpd^{op} & \xrightarrow{\forall\ \mathcal{M}\ \ Mackey} ADD \\
\big\downarrow & \text{univ}\big\downarrow & \nearrow_{\exists!\ \widehat{\mathcal{M}}\ \ \text{additive}} \\
\text{(the motive of) } G \in & \qquad Mot &
\end{array}
$$

Conjecture

- Cartesian products induce an additive symmetric monoidal structure on *Mot*.
- By Day convolution, this extends to a symmetric monoidal structure on the 2-category $2Mack \simeq 2Fun^{ind}_{add}(Mot, ADD)$ of Mackey to functors.
- A Green 2-functor is the same as a pseudo-monoid in $2Mack$.