An introduction to Mackey 2-functors

Ivo Dell'Ambrogio
19 April 2021

Université de Lille
Faculté des Sciences et Technologies
Département de Mathématiques
Laboratoire Paul Painlevé

ItaCa Fest 2021

References:

1. Paul Balmer and Ivo Dell'Ambrogio. Mackey 2-functors and Mackey 2-motives. *EMS Monographs in Mathematics. European Mathematical Society*. Zürich (2020), viii+227. (arXiv:1808.04902)
2. Paul Balmer and Ivo Dell'Ambrogio. Green equivalences in equivariant mathematics. *Math. Ann.* (2021) (arXiv:2001.10646)
3. Paul Balmer and Ivo Dell'Ambrogio. Cohomological Mackey 2-functors. Preprint 2021 (arXiv:2103.03974)
4. Jun Maillard. A categorification of the Cartan-Eilenberg formula. Preprint 2021 (arXiv:2102.07554)

What is representation theory?

At the very least: *finite groups acting on vector spaces!*

For G a finite group, k a field
$\rightsquigarrow$ study the category of (left, fin. gen.) kG-modules and kG-linear maps:

$$mod(kG)$$

Classical dichotomy:

- **The semi-simple case**, when $\operatorname{char} k \nmid |G|$, e.g. $\operatorname{char} k = 0$:
  Maschke's theorem: Every finite dim'l representation is uniquely a direct sum of simple ones.
  $\rightsquigarrow$ enough to study the representation ring $R_k(G) = K_0(mod\, kG)$.

- **The modular case**, when $\operatorname{char} k \mid |G|$:
  Now the abelian category $mod(kG)$ has non-trivial extensions.
  $\rightsquigarrow$ need categorical and homological tools:
  derived category $D^b(mod\, kG)$, stable module category $\underline{mod}(kG)$.

An extra layer of 2-categorical structure

Let $\mathcal{M}(G)$ be either $mod(kG)$, $D^b(mod\, kG)$, or $\underline{mod}(kG)$. Let us vary G!

The following are basic tools, used all the time:

- The **restriction**, **induction** and **conjugation** functors ($H \leq G$):

$$\begin{array}{ccc} \mathcal{M}(G) & & \\ {\scriptstyle Ind}\uparrow\ \downarrow{\scriptstyle Res} & & \\ \mathcal{M}(H) & \xrightarrow[\sim]{Conj_g} & \mathcal{M}({}^gH) \end{array}$$

- The **adjunctions** $Ind \dashv Res \dashv Ind$
  (Note: have an ambijunction only because index finite & categories additive!)
- **Conjugation natural isos** between composites functors, e.g.

$$conj_g \colon\ Conj_g \circ Res^G_H \cong Res^G_{{}^gH}$$

- The **Mackey formula** (for $K, L \leq G$):

$$Res^G_L \circ Ind^G_K \cong \bigoplus_{[g] \in L \backslash G / K} Ind^L_{L \cap {}^gK} \circ Conj_g \circ Res^K_{L^g \cap K} \,.$$

Axiomatization: Mackey 2-functors

gpd_f : the 2-category of finite groupoids, faithful functors, natural isos
ADD : the 2-category of additive categories, additive functors, nat. transf.

Definition [Balmer-D. 2020]

A (global) **Mackey 2-functor** is a 2-functor

$$\mathcal{M}\colon gpd_f^{op} \longrightarrow ADD$$

satisfying the following axioms:

1. Additivity: $\mathcal{M}(G_1 \sqcup G_2) \overset{\sim}{\to} \mathcal{M}(G_1) \oplus \mathcal{M}(G_2)$ and $\mathcal{M}(\emptyset) \overset{\sim}{\to} 0$.
2. Every 'restriction' $i^* := \mathcal{M}(i)$ admits a left adjoint $i_!$ and a right adjoint i_*.
3. Left and right adjunctions satisfy Base-Change for iso-comma squares.
4. There is a natural isomorphism $i_! \cong i_*$ for all $i\colon H \to G$.

For the 'local' variant for a fixed G: replace gpd_f with $gpd_f/G \simeq G\text{-}set$.

Comments on the axioms

Inspired by:

- Ordinary, abelian-group-valued Mackey functors [Green, Dress 60s]
- Additive derivators [Grothendieck 80s]
- Similar, but complementary, to the Mackey $(\infty, 1)$-functors of [Barwick '17]

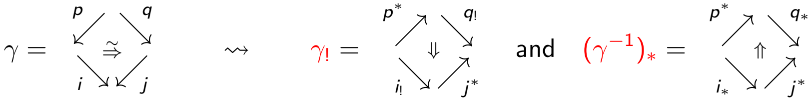
Explanations:

- **Additivity axiom 1:** groupoids decompose into groups $G \simeq \bigsqcup_i G_i$
  $\rightsquigarrow$ the data of the 2-functor $\mathcal{M}$ is determined by what it does to the 2-full 2-subcategory of finite groups: the *Res*, *Cong* / *Isos* and *cong* !
- **Induction 2:** as is the case with derivators, (co)induction functors $i_!$ and i_* are not really part of the data.
- **Ambidexterity 4:** any isomorphisms $i_! \cong i_*$ will do, so the axiom is easy to check in examples!

  Fact (rectification theorem): if axiom 4 holds, there exist unique canonical isomorphisms $\theta_i \colon i_! \cong i_*$ fully compatible with given left and right adjunctions.

Base-Change = canonical Mackey formulas

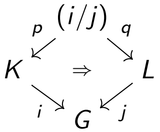
Base-Change axiom 3: every iso-comma square γ in gpd_f defines, via the 2-functor $\mathcal{M}$ and the left/right adjunctions, two mates $\gamma_!$ and $(\gamma^{-1})_*$:



The axiom requires both to be invertible: $j^* i_! \cong q_! p^*$ and $j^* i_* \cong q_* p^*$.
Convenient fact: via the rectification isos θ, they are mutual inverses!

Motivating example: for two subgroups $K, L \leq G$

iso-comma



⇝

$$(i/j) \simeq \coprod_{[g] \in L \backslash G / K} L \cap {}^g K$$

get the Mackey formula!

Note: the iso-comma groupoid (i/j) and the Base-Change isos are canonical, but the decomposition into groups depends on choices!

Exemples: Mackey 2-functors are everywhere!

There is a Mackey 2-functor $\mathcal{M}$ for each of the following families of abelian or triangulated categories $\mathcal{M}(G)$:

- **In (linear) representation theory**:
  $\mathcal{M}(G) = mod\, kG$, $Mod\, kG$, $D^b(mod\, kG)$, $D(Mod\, kG)$, $\underline{mod}(kG)$...
- **In topology**:
  $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$, the homotopy category of genuine G-spectra.
- **In noncommutative geometry**:
  $\mathcal{M}(G) = KK^G$ or E^G, equivariant Kasparov theory or Higson-Connes E-theory of C*-algebras.
- **In geometry** (only defined 'locally' for a fixed group G):
  Fix X a locally ringed space (e.g. scheme) with a G-action.
  For $H \leq G$, $\mathcal{M}(H) = Sh(X /\!\!/ H)$ the category of H-equivariant $\mathcal{O}_X$-modules.
  Variants: take the derived category $D(Sh(X /\!\!/ H))$, or constructible sheaves, or coherent sheaves if X is a noetherian scheme, etc.

First results: monadicity

So... What can we prove with these axioms? Let's see some applications.

Among the many good properties of the canonical isos θ_i, we can show the ambijunction at every $i\colon H \to G$ has the **special Frobenius property**:

$$\Big(\ \mathrm{Id}_{\mathcal{M}(H)} \xRightarrow{\ unit\ } i^* i_! \overset{\theta_i}{\cong} i^* i_* \xRightarrow{\ counit\ } \mathrm{Id}_{\mathcal{M}(H)} \ \Big) \quad = \quad \mathrm{id}$$

By a 'separable' version of Beck's monadicity criterion, this implies:

Co/monadicity of restrictions [Balmer-D. 2020]

Suppose $\mathcal{M}$ takes values in idempotent complete categories.
For every $i\colon H \to G$ the adjunction $i^* \dashv i_*$ is monadic and $i_! \dashv i^*$ comonadic:

$$\mathcal{M}(G)^{\,i_! i^*} \overset{\sim}{\leftarrow} \mathcal{M}(H) \overset{\sim}{\rightarrow} \mathcal{M}(G)^{\,i_* i^*}$$

Thus, we can always recover the 'smaller' $\mathcal{M}(H)$ from the 'bigger' $\mathcal{M}(G)$.

Motivic approach

A Mackey 2-functor is ***k*-linear** if its values are k-linear categories and functors (over some commutative ring k).

Theorem (Mackey 2-motives)

There is a 2-category Mot_k of ***k*-linear Mackey 2-motives**, through which every k-linear $\mathcal{M}$ factors uniquely:

$$\begin{array}{ccc} \mathit{gpd}_f^{op} & \xrightarrow{\ \forall\, \mathcal{M}\ \ \mathit{Mackey}\ } & \mathit{ADD}_{k\text{-lin}} \\ {\scriptstyle \mathrm{univ}}\downarrow & \nearrow {\scriptstyle \exists!\ \widehat{\mathcal{M}}\ \ k\text{-linear}} & \\ \mathit{Mot}_k & & \end{array}$$

Corollary

The 2-cell endomorphism ring $\mathit{End}_{\mathit{Mot}_k}(\mathrm{Id}_G)$ of (the motive of) a group G acts on the category $\mathcal{M}(G)$, for every k-linear Mackey 2-functor $\mathcal{M}$.

Mot_k has concrete models (via spans, bimodules, or string diagrams): we can compute in it!

Universal block decompositions

For instance:

Motivic endomorphism ring

$End_{Mot_k}(\mathrm{Id}_G)$ is isomorphic to the **crossed Burnside *k*-algebra** [Yoshida '97]

$$B_k^c(G) = k \otimes_{\mathbb{Z}} K_0\big(G\text{-}sets/G^{conj},\ \text{a certain braided} \otimes\big)$$

or concretely: a finite free k-module generated by G-conjugacy classes of pairs (H, a) with $H \leq G$ and $a \in C_G(H)$, with multiplication given by:

$$(K, b) \cdot (H, a) = \sum_{[g] \in K \backslash G/H} (K \cap {}^gH, bgag^{-1}) \,.$$

Example: well-known that the (usual) **Burnside ring** $B(G) = K_0(G\text{-set})$ acts on $\mathcal{M}(G) = Ho(\mathcal{S}p^G)$, since $B(G) \cong End(S^0)$ and the sphere S^0 is the tensor unit.

But now $B_{\mathbb{Z}}^c(G)$ also acts, a bigger ring $\rightsquigarrow$ get a finer direct sum decomposition:

$$Ho(\mathcal{S}p^G) \simeq \bigoplus_{e \in PrimIdem(B_{\mathbb{Z}}^c(G))} e \cdot Ho(\mathcal{S}p^G).$$

The Cartan-Eilenberg formula

A classical result links cohomology and fusion in groups:

The stable elements formula [Cartan-Eilenberg '56]

Let $0 < p = \mathrm{char}\, k \mid |G|$. Group cohomology is computed by the limit

$$H^*(G;k) \cong \lim_{P \in \mathcal{F}_p(G)} H^*(P;k)$$

taken over the ***p*-fusion category** $\mathcal{F}_p(G) = \begin{cases} \text{objects: } p\text{-subgroups of } G \\ \text{morphisms: incl. and conj. maps.} \end{cases}$

But group cohomology is only a small piece of the derived category:

$$H^*(G;k) \cong \mathrm{Hom}_{D^b(mod\, kG)}(k, \Sigma^* k)\,.$$

Question: Can we similarly reconstruct $D^b(mod\, kG)$ from $D^b(mod\, kP)$ for p-subgroups P, together with restrictions & conjugations?

Yes!

We say $\mathcal{M}$ is **cohomological** if $(\mathrm{Id}_{\mathcal{M}(G)} \Rightarrow i_* i^* \overset{\theta^{-1}}{\cong} i_! i^* \Rightarrow \mathrm{Id}_{\mathcal{M}(G)}) = [G : H]$ for every subgroup inclusion $i \colon H \to G$.

Categorified Cartan-Eilenberg formula [Maillard '21]

Suppose $\mathcal{M}$ is idempotent-complete and k-linear, with char $k = p > 0$.
If $\mathcal{M}$ is cohomological, there is an equivalence

$$\mathcal{M}(G) \simeq \operatorname*{bilim}_{P \in \mathcal{T}_p(G)} \mathcal{M}(P)$$

with the bilimit taken in $ADD_{k\text{-lin}}$ over the ***p*-transporter 2-category** $\mathcal{T}_p(G)$.

- $\mathcal{T}_p(G)$ refines the fusion category $\mathcal{F}_p(G)$, which it admits as a quotient.
- $\mathcal{T}_p(G) \simeq \mathcal{O}_p(G)$, the usual orbit category of the orbits G/P.
- These $\mathcal{M}$'s are cohomological: $\mathcal{M} = mod(k-)$, $D^b(k-)$, and $\underline{mod}(k-)$.
- But also equivariant sheaves on a G-variety X over k: $Sh(X /\!\!/ G)$ etc. Previous examples: just the case $X = \mathrm{Spec}(k)$ with trivial G-action! ☺

Thank you for your attention!