A survey of tensor triangular geometry and applications

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We gave a mini-survey of Paul Balmer's geometric theory of tensor triangulated categories, or *tensor triangular geometry*, and applications. In the following, $\mathcal{K} = (\mathcal{K}, \otimes, 1)$ will denote a tensor triangulated category, i.e., a triangulated category $\mathcal{K}$ equipped with a tensor product (a symmetric monoidal structure) $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ with unit object 1, such that $a \otimes -$ and $- \otimes a$ are exact functors $\mathcal{K} \to \mathcal{K}$ for every object $a \in \mathcal{K}$. The main tool of tensor triangular geometry is the spectrum of a tensor triangulated category:

Definition 1 ([Ba05])**.** Let $\mathcal{K}$ be an essentially small $\otimes$-triangulated category. A *prime ideal* $\mathcal{P}$ of $\mathcal{K}$ is a proper (i.e., $\mathcal{P} \neq \mathcal{K}$) full triangulated subcategory $\mathcal{P} \subset \mathcal{K}$ which is: thick (i.e., $a \oplus b \in \mathcal{P} \Rightarrow a, b \in \mathcal{P}$), $\otimes$-ideal ($a \in \mathcal{P}, x \in \mathcal{K} \Rightarrow a \otimes x \in \mathcal{K}$) and prime ($a \otimes b \in \mathcal{P} \Rightarrow a \in \mathcal{P}$ or $b \in \mathcal{P}$). The *spectrum of* $\mathcal{K}$ is the set of its prime ideals:

$$\mathrm{Spc}(\mathcal{K}) := \{\mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is a prime ideal of } \mathcal{K}\}.$$

We give $\mathrm{Spc}(\mathcal{K})$ the topology determined by the following basis of closed subsets:

$$\mathrm{supp}(a) := \{\mathcal{P} \mid a \notin \mathcal{P}\} = \{\mathcal{P} \mid a \not\simeq 0 \text{ in } \mathcal{K}/\mathcal{P}\} \subseteq \mathrm{Spc}(\mathcal{K}) \quad (\text{for } a \in \mathcal{K}).$$

Remarks 2. (a) The space $\mathrm{Spc}(\mathcal{K})$ is always non-empty (if $\mathcal{K} \not\simeq 0$) and spectral, in the sense of Hochster [Ho69]: it is quasi-compact, it has an open basis of quasi-compact opens, and every irreducible closed subset has a unique generic point.

(b) $\mathrm{Spc}(\mathcal{K})$ is naturally equipped with a sheaf of rings $\mathcal{O}_{\mathcal{K}}$. The ringed space

$$\mathrm{Spec}(\mathcal{K}) := (\mathrm{Spc}(\mathcal{K}), \mathcal{O}_{\mathcal{K}})$$

is always a locally ringed space ([Ba09b]) and sometimes a scheme (cf. Ex. 5.a-c).

(c) Every monoidal exact functor $F : \mathcal{K} \to \mathcal{L}$ induces a continuous map $\mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ by $\mathcal{P} \mapsto F^{-1}\mathcal{P}$. This defines a functor Spec from the category of $\otimes$-triangulated categories to that of locally ringed (spectral) spaces.

Universal property and classification. The *support* assignment $\mathrm{supp} : \mathrm{Ob}(\mathcal{K}) \to \mathrm{Closed}(\mathrm{Spc}(\mathcal{K}))$, $a \mapsto \mathrm{supp}(a)$, is compatible with the $\otimes$-triangulated structure, and is the finest such:

Proposition 3 (Universal property of $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$)**.** *We have the following:*

(1) $\mathrm{supp}(0) = \emptyset$ *and* $\mathrm{supp}(1) = \mathrm{Spc}(\mathcal{K})$
(2) $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$
(3) $\mathrm{supp}(T(a)) = \mathrm{supp}(a)$, *where* $T : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$ *is the translation of* $\mathcal{K}$
(4) $\mathrm{supp}(b) \subseteq \mathrm{supp}(a) \cup \mathrm{supp}(c)$ *for every exact triangle* $a \to b \to c \to T(a)$
(5) $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$.

Moreover, if (X, σ) *is a pair where* X *is a topological space and* σ *is an assignment from objects of* $\mathcal{K}$ *to closed subsets of* X *satisfying (1)-(5) above (we say that* (X, σ) *is a* support datum*), then there exists a unique continuous map* $f : X \to \mathrm{Spc}(\mathcal{K})$ *such that* $\sigma(a) = f^{-1}(\mathrm{supp}(a))$ *for all objects* $a \in \mathcal{K}$.

Theorem 4 (Classification [Ba05] [BKS07]). *There is a bijection*

$$\begin{array}{rcl} \{\textit{radical thick } \otimes\textit{-ideals of } \mathcal{K}\} & \simeq & \{\textit{Thomason subsets of } \operatorname{Spc}(\mathcal{K})\} \\ \mathcal{J} & \mapsto & \operatorname{supp}(\mathcal{J}) := \cup_{a\in\mathcal{J}}\operatorname{supp}(a) \\ \{a \in \mathcal{K} \mid \operatorname{supp}(a) \subseteq Y\} =: \mathcal{K}_Y & \leftarrow\!\shortmid & Y \end{array}$$

(a $\otimes$-ideal $\mathcal{J}$ is radical *if $a^{\otimes n} \in \mathcal{J}$ for some $n \geq 1$ implies $a \in \mathcal{J}$, and a subset Y of the spectrum is* Thomason *if it is a union of closed subsets, each with quasi-compact open complement). Moreover, if (X, σ) is a support datum inducing the above bijection, then the canonical map $f : X \to \operatorname{Spc}(\mathcal{K})$ is a homeomorphism.*

By exploiting existing classifications of $\otimes$-ideals, the Classification theorem can be used to provide concrete descriptions of the spectrum $\operatorname{Spc}(\mathcal{K})$ in examples ranging over the most disparate branches of mathematics.

Examples 5. (a) **(Algebraic geometry).** Let X be a quasi-compact and quasi-separated scheme, and let $\mathcal{K} := D^{\mathrm{perf}}(X)$ be its derived category of perfect complexes with $\otimes = \otimes_X^L$ and $1 = \mathcal{O}_X$. From Thomason's classification of thick tensor ideals [Th97] we deduce a natural isomorphism $\operatorname{Spec}(D^{\mathrm{perf}}(X)) \simeq X$ of schemes. Thus tensor triangular geometry generalizes algebraic geometry ([Ba02] [Ba05]).

(b) **(Commutative algebra)** As a special case of (a), if R is any commutative ring and $\mathcal{K} := K^b(R - \mathrm{proj})$ its bounded derived category of finitely generated projective modules, then $\operatorname{Spec}(K^b(R - \mathrm{proj})) \simeq \operatorname{Spec}(R)$ is the Zariski spectrum.

(c) **(Modular representation theory).** Let G be a finite group (or a finite group scheme), and let k be a field with $\operatorname{char}(k) > 0$. From the classification in [BCR97] (resp., in [FP07]) of the thick $\otimes$-ideals in the stable category $\mathcal{K} := kG - \mathrm{stab}$ of finite dimensional modules, with $\otimes = \otimes_k$ and $1 = k$, one deduces an isomorphism $\operatorname{Spec}(kG - \mathrm{stab}) \simeq \operatorname{Proj}(H^*(G, k))$ of projective varieties. Similarly, $\operatorname{Spec}(D^b(kG - \mathrm{mod})) \simeq \operatorname{Spec}^{\mathrm{h}}(H^*(G, k))$, the spectrum of homogeneous primes.

(d) **(Stable homotopy).** Let $\mathcal{K} := SH^{\mathrm{fin}}$ be the homotopy category of finite spectra (of topology), i.e., the stable homotopy category of finite based CW-complexes. The famous Thick Subcategory theorem of Hopkins and Smith [HS98] translates neatly into a description of $\operatorname{Spc}(SH^{\mathrm{fin}})$ in terms of the chromatic towers at all prime numbers ([Ba09b]). Note that the ringed space $\operatorname{Spec}(SH^{\mathrm{fin}})$ is *not* a scheme.

Remark 6. Other concrete classifications known so far are: The category of perfect complexes over a Deligne-Mumford stack [Kr08]; The category $\mathcal{K} = Boot_c$ of compact objects in the Bootstrap category of separable C*-algebras (the latter simply yields $\operatorname{Spec}(Boot_c) \simeq \operatorname{Spec}(\mathbb{Z})$ [De09]).

Hypothesis 7. *From now on, we assume that our tensor triangulated category $\mathcal{K}$ is* rigid, *i.e., that there is an equivalence $D : \mathcal{K}^{\mathrm{op}} \xrightarrow{\sim} \mathcal{K}$ with $\operatorname{Hom}(a \otimes b, c) \simeq \operatorname{Hom}(a, D(b) \otimes c)$. Moreover, we assume that $\mathcal{K}$ is* idempotent complete*: if $e = e^2 : a \to a$ is an idempotent morphism in $\mathcal{K}$, then $a \simeq \operatorname{Ker}(e) \oplus \operatorname{Im}(e)$. Both are light hypotheses; e.g., they are satisfied by all categories in Example 5.*

Decomposition of objects. The support $\text{supp}(a)$ can be used to decompose the object a in $\mathcal{K}$, or to test its indecomposability:

Theorem 8 ([Ba07]). *Let $\mathcal{K}$ be a $\otimes$-triangulated category (see Hypothesis 7). Let $a \in \mathcal{K}$ be an object such that $\text{supp}(a) = Y_1 \cup Y_2$, where Y_1 and Y_2 are disjoint Thomason subsets of $\text{Spc}(\mathcal{K})$ (as in Thm. 4). Then there is a decomposition $a \simeq a_1 \oplus a_2$ in $\mathcal{K}$ with $\text{supp}(a_i) = Y_i$ (for $i = 1, 2$).*

In modular representation theory (Example 5.c), for instance, the latter result generalizes to finite group schemes a celebrated theorem of Carlson [Ca84], saying that the projective support variety of a finitely generated indecomposable module is connected. The corresponding statement, of course, is now available in all examples.

Topological filtrations and local-to-global spectral sequences. Given a reasonable notion of "dimension" for the closed subsets of $\text{Spc}(\mathcal{K})$ (such as the usual Krull dimension, or minus the Krull codimension in $\text{Spc}(\mathcal{K})$), one can produce filtrations of the category $\mathcal{K}$ of the form

$$0 \subseteq \mathcal{K}_{(-\infty)} \subseteq \cdots \subseteq \mathcal{K}_{(n-1)} \subseteq \mathcal{K}_{(n)} \subseteq \mathcal{K}_{(n+1)} \subseteq \cdots \subseteq \mathcal{K}_{(+\infty)} = \mathcal{K}$$

where $\mathcal{K}_{(n)} \subseteq \mathcal{K}$ is the subcategory of those objects whose support has dimension at most n ($n \in \mathbb{Z} \cup \{\pm\infty\}$). Every term in the filtration is a thick triangulated subcategory of the next one up, so the subquotients $\mathcal{K}_{(n)}/\mathcal{K}_{(n-1)}$ are again triangulated. Each has a decomposition into a sum of local terms. More precisely:

Theorem 9 ([Ba07]). *Assume that the space $\text{Spc}(\mathcal{K})$ is noetherian (i.e., every open subset is quasi-compact). Then the quotient functors $q_{\mathcal{P}} : \mathcal{K} \to \mathcal{K}/\mathcal{P}$ induce a fully faithful triangulated functor*

$$\mathcal{K}_{(n)}/\mathcal{K}_{(n-1)} \longrightarrow \coprod_{\mathcal{P} \in \text{Spc}(\mathcal{K}) \; s.t. \, \dim(\overline{\{\mathcal{P}\}}) = n} (\mathcal{K}/\mathcal{P})_{(0)}$$

which moreover is cofinal (that is, essentially surjective up to direct summands).

In algebraic geometry, the above decomposition is well known for regular schemes and hides behind various local-to-global spectral sequences. Indeed, Theorem 9 becomes an essential ingredient in the following generalization to singular schemes of Quillen's [Qu73] classical construction of a local-to-global spectral sequence for the algebraic K-theory of regular schemes:

Theorem 10 ([Ba09a]). *Let X be any (topologically) noetherian scheme of finite Krull dimension. Then there exists a cohomological spectral sequence*

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(\mathcal{O}_{X,x} \text{ on } \{x\}) \quad \overset{n=p+q}{\Longrightarrow} \quad K_{-n}(X)$$

converging to the algebraic K-theory of X; the E_1-page contains Thomason's non-connective K-theory of the local ring $\mathcal{O}_{X,x}$ with support on the closed point x.

Gluing of morphisms and objects. To each quasi-compact open set $U \subseteq \mathrm{Spc}(\mathcal{K})$ we associate the (again, rigid and idempotent complete) $\otimes$-triangulated category $\mathcal{K}(U) := \widetilde{\mathcal{K}/\mathcal{K}_Y}$ obtained by idempotent completing (see [BS01]) the quotient of $\mathcal{K}$ by all objects supported on the complement $Y := \mathrm{Spc}(\mathcal{K}) \smallsetminus U$. Given a covering $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$, it is natural to ask if and how it is possible to glue information in $\mathcal{K}(U_i)$ $(i = 1, 2)$, compatible over $\mathcal{K}(U_1 \cap U_2)$, in order to provide information in $\mathcal{K}$. The "gluing technique" of Balmer-Favi [BF07] provides some general answers:

Theorem 11 (Mayer-Vietoris for morphisms)**.** *There is a long exact sequence*

$$\cdots \mathrm{Hom}_{12}(a, T^{-1}b) \xrightarrow{\partial} \mathrm{Hom}(a, b) \to \mathrm{Hom}_1(a, b) \oplus \mathrm{Hom}_2(a, b) \to \mathrm{Hom}_{12}(a, b) \xrightarrow{\partial} \cdots$$

of Hom groups for every two objects $a, b \in \mathcal{K}$ (here we use the short-hand notation $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{K}}$, $\mathrm{Hom}_i = \mathrm{Hom}_{\mathcal{K}(U_i)}$ *and* $\mathrm{Hom}_{12} = \mathrm{Hom}_{\mathcal{K}(U_1 \cap U_2)}$, *and we keep writing a and b for the canonical images of a and b in the appropriate categories).*

Theorem 12 (Gluing of two objects)**.** *Given two objects $a_i \in \mathcal{K}(U_i)$ $(i = 1, 2)$ and an isomorphism $\sigma : a_1 \xrightarrow{\sim} a_2$ over $U_1 \cap U_2$, i.e., in $\mathcal{K}(U_1 \cap U_2)$, there exists an (up to isomorphism, unique) object $a \in \mathcal{K}$ mapping to a_i in $\mathcal{K}(U_i)$ $(i = 1, 2)$.*

The Picard group. For any $\otimes$-triangulated category $\mathcal{K}$, define its *Picard group* $\mathrm{Pic}(\mathcal{K})$ to be the abelian group of $\otimes$-invertible objects (i.e., those $a \in \mathcal{K}$ such that there exists $b \in \mathcal{K}$ and an isomorphism $a \otimes b \simeq 1$), with $\otimes$ as group operation.

Examples 13. (a) For a scheme X, we have $\mathrm{Pic}(D^{\mathrm{perf}}(X)) \simeq \mathrm{Pic}(X) \oplus \mathbb{Z}^\ell$, where ℓ is the number of connected components of X.

(b) For a finite group G and a field k, we recognise $\mathrm{Pic}(kG - \mathrm{stab})$ as the group of endotrivial kG-modules, usually denoted $T(G)$.

Theorem 12 supplies the connecting map δ used in the next result.

Theorem 14 (Mayer-Vietoris for Picard [BF07])**.** *Let $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$ as above. There is a long exact sequence (extending to the left as in Theorem 11)*

$$\begin{aligned} &\cdots \to \mathrm{Hom}_{\mathcal{K}(U_1 \cap U_2)}(1, T^{-1}1) \xrightarrow{1+\partial} \\ &\mathbb{G}_{\mathrm{m}}(\mathcal{K}) \to \mathbb{G}_{\mathrm{m}}(\mathcal{K}(U_1)) \oplus \mathbb{G}_{\mathrm{m}}(\mathcal{K}(U_2)) \to \mathbb{G}_{\mathrm{m}}(\mathcal{K}(U_1 \cap U_2)) \xrightarrow{\delta} \\ &\mathrm{Pic}(\mathcal{K}) \to \mathrm{Pic}(\mathcal{K}(U_1)) \oplus \mathrm{Pic}(\mathcal{K}(U_2)) \to \mathrm{Pic}(\mathcal{K}(U_1 \cap U_2)). \end{aligned}$$

Here $\mathbb{G}_{\mathrm{m}}(\mathcal{L}) := \mathrm{End}_{\mathcal{L}}(1)^\times$ denotes the automorphism group of the tensor unit 1 in a $\otimes$-triangulated category $\mathcal{L}$.

Applications of gluing to modular representation theory. The authors of [BBC08] compare the above gluing techniques with similar-minded uses of Rickard's idempotent modules ([Ri97]) in modular representation theory. Among other things, they provide a new proof for Alperin's computation ([Al01] [Ca06]) of the rank of the group $T(G)$ in terms of the number of conjugacy classes of maximal elementary abelian subgroups of G. They also show that the above gluing technique provides a subgroup of finite index inside $T(G)$. Further enquiry along these lines brings to light the following deep connection between algebraic geometry and modular representation theory:

Theorem 15 ([Ba08]). *Let G be a finite group and k a field of positive characteristic. Then the gluing construction induces an isomorphism*

$$\mathrm{Pic}\big(\mathrm{Proj}(H^*(G,k))\big) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} T(G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

which rationally identifies the Picard group of line bundles on the projective variety of G with the group of endotrivial kG-modules.

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