Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 212 (2008) 1039–1045
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

The Witt groups of the spheres away from two

Ivo Dell'Ambrogio*, Jean Fasel

ETH Zentrum, 8092 Zürich, Switzerland

Received 11 May 2007; received in revised form 13 June 2007; accepted 23 July 2007
Available online 14 September 2007

Communicated by F. Morel

Abstract

We calculate the Witt groups of the spheres up to 2-primary torsion.

1. Introduction

Let A be a Gorenstein ring of finite Krull dimension with $\frac{1}{2} \in A$. Denote the coordinate ring of the n-sphere over A by

$$S_A^n := A[X_1, \dots, X_{n+1}]/(X_1^2 + \cdots + X_{n+1}^2 - 1),$$

and let

$$\mathbb{P}S_A^n := \operatorname{Proj} A[X_0, \dots, X_{n+1}]/(X_0^2 + \cdots + X_n^2 - X_{n+1}^2)$$

be the corresponding projective scheme.

We are interested in determining the (total) coherent Witt groups of these varieties, which are still unknown, in terms of the Witt groups of the base A. As a first step we obtain the following two theorems, with the hope one day of determining also the 2-primary torsion. We abbreviate:

$$\overline{\mathrm{W}}^i(X) := \tilde{\mathrm{W}}^i(X) \otimes \mathbb{Z}[1/2], \qquad \overline{\mathrm{W}}^{\mathrm{tot}}(X) := \bigoplus_{0 \le i \le 3} \overline{\mathrm{W}}^i(X).$$

Theorem 1.1. *For all i, we have isomorphisms*

$$\overline{\mathrm{W}}^i(S_A^n) \simeq \overline{\mathrm{W}}^i(\mathbb{P}S_A^n) \simeq \overline{\mathrm{W}}^i(A) \oplus \overline{\mathrm{W}}^{i-n}(A).$$

* Corresponding address: Department of Mathematics, ETH Zentrum, 8092 Zürich, Switzerland.
E-mail addresses: ambrogio@math.ethz.ch (I. Dell'Ambrogio), jean.fasel@math.ethz.ch (J. Fasel).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.07.015

If the base A is regular, then S^n_A is also regular and its coherent and derived (locally free) Witt groups are isomorphic. The tensor product induces a natural structure of $\mathbb{Z}/4$-graded rings on $\mathrm{W}^{\mathrm{tot}}(A)$ and $\mathrm{W}^{\mathrm{tot}}(S^n)$, denoted by $\star$ (see [9]). If $q : \operatorname{Spec} S^n \to \operatorname{Spec} A$ denotes the canonical projection, then q^* makes $\mathrm{W}^{\mathrm{tot}}(S^n)$ a $\mathbb{Z}/4$-graded $\mathrm{W}^{\mathrm{tot}}(A)$-algebra. All this remains true of course for $\overline{\mathrm{W}}^{\mathrm{tot}}$. Our geometric proof of Theorem 1.1 lets us easily find this multiplicative structure:

Theorem 1.2. *Moreover let A be regular. Then*

$$\overline{\mathrm{W}}^{\mathrm{tot}}(S^n_A) \simeq \overline{\mathrm{W}}^{\mathrm{tot}}(A)[\alpha]/(\alpha^2),$$

with α sitting in degree n.

Remarks. (a) By Proposition 3.1 below, we see that our calculation produces non-trivial groups only when the base ring A has characteristic 0 (that is, when $\mathbb{Z} \subset A$).

(b) Theorem 1.1 follows also from the general results of Brumfiel in [3,4]. In [3,4] Brumfiel sketches the development of a theory $\mathrm{KO}^{-n}(\operatorname{Sper} A)$ inspired by the usual (real) topological K-theory of spaces, and which depends only on the real spectrum $\operatorname{Sper} A$ (equipped with its sheaf of abstract semi-algebraic functions). One of the main results of Brumfiel is that there are natural isomorphisms

$$\mathrm{W}^K_n(A) \otimes \mathbb{Z}[1/2] \simeq \mathrm{KO}^{-n}(\operatorname{Sper} A) \otimes \mathbb{Z}[1/2] \quad (n \geq 0),$$

where W^K_n are Karoubi's Witt groups. By identifying $\overline{\mathrm{W}}^n(A) \simeq \mathrm{W}^K_n(A) \otimes \mathbb{Z}[\frac{1}{2}]$ on the left-hand side [10, Lemma A.3], and by reducing Brumfiel's theory to the usual K-theory of some space on the right-hand side, one can use topological results to calculate Balmer–Witt groups tensored with $\mathbb{Z}[\frac{1}{2}]$. We can avoid all this, our approach here being much simpler and completely geometric in nature. Our main tools will be the 12-periodic localization long exact sequence, homotopy invariance and dévissage (see [1] for an overview and references).

2. Transfer and dévissage

We recall here some technical facts, for the convenience of the reader. Given a finite dimensional Gorenstein scheme X with $\frac{1}{2} \in \mathcal{O}_X(X)$ and a line bundle $\mathcal{L}$ on it, the coherent Witt groups $\tilde{\mathrm{W}}^i(X, \mathcal{L})$ depend on X, $\mathcal{L}$ and also on a choice of an injective resolution of $\mathcal{L}$ [5]. Thus when we write $\tilde{\mathrm{W}}^i(X) := \tilde{\mathrm{W}}^i(X, \mathcal{O}_X)$ we tacitly choose such a resolution. Given a finite morphism $f : Y \to X$ of pure relative dimension d, Gille constructed transfer (push-forward) morphisms

$$f_* : \tilde{\mathrm{W}}^i(Y, \overline{f}^* \mathrm{Ext}^d_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)) \to \tilde{\mathrm{W}}^{i+d}(X)$$

(see *e.g.* [8, section 2]). Transfers are functorial: for two composable morphisms f and g, one can choose the injective resolutions so that $(gf)_* = g_* f_*$.

Let now Y be a connected Gorenstein closed subscheme of X of codimension d, and let $f : Y \hookrightarrow X$ be the inclusion. Then the transfers induce "dévissage" isomorphisms

$$f_* : \tilde{\mathrm{W}}^i(Y, \overline{f}^* \mathrm{Ext}^d_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)) \to \tilde{\mathrm{W}}^{i+d}_Y(X),$$

for all i ([8, Thm.3.2]), where the groups on the right are the Witt groups with support appearing in the localization sequence.

It happens sometimes that the sheaf $\overline{f}^* \mathrm{Ext}^d_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)$ is isomorphic to $\mathcal{O}_Y$, *e.g.* when $Y = \operatorname{Spec} A$ is affine and $f : \operatorname{Spec} A/I \to \operatorname{Spec} A$ is a closed immersion whose defining ideal I can be given by a regular sequence $(a_1, \ldots, a_d)$ in A. In this case, every choice of a sequence gives an isomorphism $\mathcal{O}_Y \simeq \overline{f}^* \mathrm{Ext}^d_{\mathcal{O}_Y}(j_*\mathcal{O}_X, \mathcal{O}_Y)$ by sending $1 \in \mathcal{O}_Y$ to the Koszul complex of $A/(a_1 \ldots, a_d)$, and therefore an isomorphism of Witt groups

$$\phi : \tilde{\mathrm{W}}^i(Y) \to \tilde{\mathrm{W}}^i(Y, \overline{f}^* \mathrm{Ext}^d_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \mathcal{O}_X)).$$

In what follows we will make this choice tacitly and will still write $f_* : \tilde{\mathrm{W}}^i(Y) \simeq \tilde{\mathrm{W}}^{i+d}_Y(X)$ for the composite isomorphism $f_* \circ \phi$.

Observe also that for flat morphisms $g : X \to Y$ the usual contravariant functoriality of derived Witt groups, written as $g^* : \mathrm{W}^i(Y) \to \mathrm{W}^i(X)$, works just as well for coherent Witt groups.

3. Proof of Theorem 1.1

The case $n = 0$ is trivial. We have isomorphisms

$$\mathrm{Spec}\ S^0 \simeq \mathrm{Proj}\ A[X_0, X_1]/(X_0^2 - X_1^2) \simeq \mathrm{Spec}\ (A \times A)$$

and thus $\tilde{\mathrm{W}}^i(S^0) \simeq \tilde{\mathrm{W}}^i(\mathbb{P}S^0) \simeq \tilde{\mathrm{W}}^i(A) \oplus \tilde{\mathrm{W}}^i(A)$.

For the rest of the proof we can assume $n \geq 1$. The following proposition is the main component of our calculation.

Proposition 3.1. *Let X be a finite dimensional Gorenstein scheme with $\frac{1}{2} \in \mathcal{O}_X(X)$, such that none of its residue fields admits a total ordering (i.e., it has no 'real points'). Let $Z \subset X$ be a closed subset, and let $\mathcal{L}$ be a line bundle over X. Then all the coherent Witt groups $\tilde{\mathrm{W}}^i_Z(X, \mathcal{L})$ of X with support in Z and values in $\mathcal{L}$ are 2-primary torsion groups.*

Proof. For any field K, the classical Pfister local–global principle (see *e.g.* [11, VIII Thm. 3.2]) implies that if K is nonreal (*i.e.*, iff K does not admit a total ordering, iff -1 is a sum of squares in K) then $\mathrm{W}(K)$ is a 2-primary torsion group. The groups in the initial page of the Gersten–Witt spectral sequence for $(X, \mathcal{L})$ with support in Z (see Balmer and Walter [2, Thm. 7.2], Gille [5, Thm. 3.14]) consist of sums of Witt groups $\mathrm{W}(k(x))$ of the residue fields of X, thus with the above hypothesis they are 2-primary torsion. But the spectral sequence converges to the groups $\tilde{\mathrm{W}}^i_Z(X, \mathcal{L})$ because X is finite dimensional. □

Corollary 3.2. *Let X be a finite dimensional separated Gorenstein scheme with $\frac{1}{2} \in \mathcal{O}_X(X)$, and let $X_{\mathbb{Q}} := X \times_{\mathrm{Spec}\ \mathbb{Z}}$ Spec $\mathbb{Q}$. Then the canonical projection $X_{\mathbb{Q}} \to X$ induces isomorphisms $\overline{\mathrm{W}}^i(X) \simeq \overline{\mathrm{W}}^i(X_{\mathbb{Q}})$ for all i.*

Proof. Let first $X = \mathrm{Spec}\ R$ be affine. Notice that a ring R satisfies the non-reality hypothesis of Proposition 3.1 if -1 is a sum of squares in R. Thus we can assume that the characteristic of R is 0, the other case being trivial. Consider for $m \geq 2$ the localization long exact sequence

$$\cdots \longrightarrow \tilde{\mathrm{W}}^i_{(m)}(A) \longrightarrow \tilde{\mathrm{W}}^i(A) \longrightarrow \tilde{\mathrm{W}}^i(A[1/m]) \longrightarrow \cdots .$$

Dévissage and Proposition 3.1 imply that $\overline{\mathrm{W}}^i_{(m)}(A) \simeq 0$, so we obtain $\overline{\mathrm{W}}^i(A) \simeq \overline{\mathrm{W}}^i(A[1/m])$. Now $A \otimes \mathbb{Q} \simeq \mathrm{colim}_m\ A[1/m]$ and Witt groups commute with filtering colimits [6, Thm. 1.6], therefore $\overline{\mathrm{W}}^i(A) \simeq \overline{\mathrm{W}}^i(A \otimes \mathbb{Q})$. For X separated, the global case can be obtained with Mayer-Vietoris. □

Notation. In the following, we will write

$$C^n = C^n_A := \frac{A[X_1, \ldots, X_n]}{(1 + X_1^2 + \cdots + X_n^2)}.$$

An immediate corollary of Proposition 3.1 is that $\overline{\mathrm{W}}^i(C^n) \simeq 0$ for all i.

Now, the intersection of the projective sphere $\mathbb{P}S^n = V(\Sigma_k X_k^2 - X_{n+1}^2) \subset \mathbb{P}^{n+1}$ with the affine open $D(X_{n+1}) = \{X_{n+1} \neq 0\} \subset \mathbb{P}^{n+1}$ is isomorphic to the affine sphere Spec S^n. We have

Lemma 3.3. *The open immersion* Spec $S^n \simeq D(X_{n+1}) \hookrightarrow \mathbb{P}S^n$ *induces isomorphisms* $\overline{\mathrm{W}}^i(\mathbb{P}S^n) \simeq \overline{\mathrm{W}}^i(S^n)$.

Proof. We have a localization sequence

$$\cdots \longrightarrow \tilde{\mathrm{W}}^i_{(X_{n+1})}(\mathbb{P}S^n) \longrightarrow \tilde{\mathrm{W}}^i(\mathbb{P}S^n) \longrightarrow \tilde{\mathrm{W}}^i(S^n) \longrightarrow \cdots . \quad (1)$$

Denote by $\mathbb{P}C^n$ the closed projective subscheme $V(X_{n+1}) \subset \mathbb{P}S^n$, *i.e.* $\mathbb{P}C^n = V(X_0^2 + \cdots + X_n^2) \subset \mathbb{P}^n$. By dévissage we have isomorphisms

$$\tilde{\mathrm{W}}^i_{(X_{n+1})}(\mathbb{P}S^n) \simeq \tilde{\mathrm{W}}^{i-1}(\mathbb{P}C^n, \mathcal{L}_n), \quad (2)$$

where $\mathcal{L}_n$ is some line bundle over $\mathbb{P}C^n$ ($n \geq 1$). On the affine opens $D(X_i) = \{X_i \neq 0\}$ ($i = 0, \ldots, n$), the scheme $\mathbb{P}C^n$ is isomorphic to C^n; thus $\mathbb{P}C^n$ clearly satisfies the hypothesis of Proposition 3.1 and therefore $\overline{\mathrm{W}}^i(\mathbb{P}C^n, \mathcal{L}_n) = 0$. Then the exactness of (1) implies the claim. □

Lemma 3.4. *Write $f := 1 + X_1^2 + \cdots + X_n^2 \in A[X_1, \ldots, X_n]$, and denote by $\mathbb{A}^n_f$ the corresponding open subscheme in the affine plane. Then we have isomorphisms $S^n_{1-X_{n+1}} \simeq \mathbb{A}^n_f$.*

Proof. The isomorphism is given by the stereographic projection. Indeed, writing $U := S^n_{1-X_{n+1}}$ and $V := A[Y_1, \ldots, Y_n]_{1+Y_1^2+\cdots+Y_n^2}$ for the localized rings, we define a morphism $V \to U$ by

$$Y_\ell \mapsto \frac{X_\ell}{1 - X_{n+1}} \quad (\ell = 1, \ldots, n).$$

Its inverse is given by

$$X_{n+1} \mapsto \frac{\Sigma_k Y_k^2 - 1}{\Sigma_k Y_k^2 + 1}, \qquad X_\ell \mapsto \frac{2Y_\ell}{\Sigma_k Y_k^2 + 1} \quad (\ell = 1, \ldots, n).$$

(Recall that 2 is invertible in A.) □

Lemma 3.5. *The projection $p : \mathbb{A}^n_f \to \operatorname{Spec} A$ induces an isomorphism $p^* : \overline{\mathrm{W}}^i(A) \simeq \overline{\mathrm{W}}^i(\mathbb{A}^n_f)$.*

Proof. We use the localization long exact sequence:

$$\cdots \longrightarrow \overline{\mathrm{W}}^i_{(f)}(\mathbb{A}^n) \longrightarrow \overline{\mathrm{W}}^i(\mathbb{A}^n) \longrightarrow \overline{\mathrm{W}}^i(\mathbb{A}^n_f) \longrightarrow \cdots.$$

Since $f = 1 + X_1^2 + \cdots + X_n^2$ is a regular element, by dévissage we have

$$\overline{\mathrm{W}}^i_{(f)}(\mathbb{A}^n) \simeq \overline{\mathrm{W}}^i(C^n).$$

Because of Proposition 3.1, the latter is zero. Now we use homotopy invariance of coherent Witt groups to conclude. □

Lemma 3.6. *There are split short exact sequences*

$$0 \longrightarrow \overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n) \longrightarrow \overline{\mathrm{W}}^i(S^n) \longrightarrow \overline{\mathrm{W}}^i(A) \longrightarrow 0.$$

Proof. Consider the long exact sequence associated with the localization $S^n \to S^n_{1-X_{n+1}}$:

$$\cdots \to \overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n) \to \overline{\mathrm{W}}^i(S^n) \to \overline{\mathrm{W}}^i(S^n_{1-X_{n+1}}) \to \overline{\mathrm{W}}^{i+1}_{(1-X_{n+1})}(S^n) \to \cdots,$$

and consider the projection $q : \operatorname{Spec} S^n \to \operatorname{Spec} A$. We then have a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} S^n & \xleftarrow{\;i\;} & \operatorname{Spec} S^n_{1-X_{n+1}} \\ & \searrow^{q} & \downarrow p \\ & & \operatorname{Spec} A \end{array}$$

where i is the inclusion. Using Lemma 3.5 and the fact that $\operatorname{Spec} S^n_{1-X_{n+1}} \simeq \mathbb{A}^n_f$ (Lemma 3.4), we obtain a commutative diagram

$$\begin{array}{ccc} \overline{\mathrm{W}}^i(S^n) & \xrightarrow{\;i^*\;} & \overline{\mathrm{W}}^i(\mathbb{A}^n_f) \\ & \nwarrow^{q^*} & \uparrow p^* \simeq \\ & & \overline{\mathrm{W}}^i(A) \end{array}$$

The above long exact sequence yields finally the split sequences

$$0 \longrightarrow \overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n) \longrightarrow \overline{\mathrm{W}}^i(S^n) \longrightarrow \overline{\mathrm{W}}^i(A) \longrightarrow 0. \quad \square$$

Corollary 3.7. *For any i, we have* $\overline{\mathrm{W}}^i(S^n) \simeq \overline{\mathrm{W}}^i(A) \oplus \overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n)$.

Next we compute $\overline{\mathrm{W}}^i_{(1-X_{n+1})}(S^n)$. In order to do this, we introduce another notation:

Notation.

$$B^n = B^n_A := \frac{A[X_1, \dots, X_n]}{(X_1^2 + \dots + X_n^2)}.$$

Thus we have isomorphisms $S^n/(1 - X_{n+1}) \simeq B^n$. For $n \geq 1$, the element $1 - X_{n+1} \in S_n$ is regular and we can use dévissage to obtain:

$$\tilde{\mathrm{W}}^i_{(1-X_{n+1})}(S^n) \simeq \tilde{\mathrm{W}}^{i-1}(B^n). \tag{3}$$

Notice *en passant* that the rings B^n are singular. This makes the use of coherent (rather that derived) Witt groups necessary for our calculation, even when A is regular.

Lemma 3.8. *For $n \geq 1$ and any i, we have* $\overline{\mathrm{W}}^i(B^n) \simeq \overline{\mathrm{W}}^{i-n+1}(A)$.

Proof. We will exploit the recursive property $B^n/(X_n) \simeq B^{n-1}$. Consider the long exact sequence

$$\cdots \longrightarrow \tilde{\mathrm{W}}^i_{(X_n)}(B^n) \longrightarrow \tilde{\mathrm{W}}^i(B^n) \longrightarrow \tilde{\mathrm{W}}^n(B^n_{X_n}) \longrightarrow \cdots$$

associated with the localization $B^n \to B^n_{X_n}$. Notice that $B^n_{X_n} \simeq C^{n-1}[X_n, X_n^{-1}]$. It is a result of Gille that $\tilde{\mathrm{W}}^i(R[T, T^{-1}]) \simeq \tilde{\mathrm{W}}^i(R) \oplus \tilde{\mathrm{W}}^i(R)$ for R a finite dimensional Gorenstein ring (see [5, Thm. 5.6]). Thus we have

$$\overline{\mathrm{W}}^i(B^n_{X_n}) \simeq \overline{\mathrm{W}}^i(C^{n-1}) \oplus \overline{\mathrm{W}}^i(C^{n-1}) = 0,$$

where the vanishing is due to Proposition 3.1. For $n \geq 2$, the element $X_n \in B^n$ is regular, so dévissage yields

$$\tilde{\mathrm{W}}^i_{(X_n)}(B^n) \simeq \tilde{\mathrm{W}}^{i-1}(B^n/X_n) = \tilde{\mathrm{W}}^{i-1}(B^{n-1}).$$

Altogether, we obtain the formula $\overline{\mathrm{W}}^i(B^n) \simeq \overline{\mathrm{W}}^{i-1}(B^{n-1})$ $(n \geq 2)$. We finish the proof by remarking that $\tilde{\mathrm{W}}^j(B^1) \simeq \tilde{\mathrm{W}}^j(A)$ for all j. (For example, one can use the generalization of affine dévissage to zero dimensional ideals, see [6, Thm. 3.5]: $\tilde{\mathrm{W}}^j(A[X]/X^2) = \tilde{\mathrm{W}}^j_{(X)}(A[X]/X^2) \simeq \tilde{\mathrm{W}}^j((A[X]/X^2)/X) = \tilde{\mathrm{W}}^j(A)$.) $\square$

Finally, we have $\overline{\mathrm{W}}^i(S^n) \simeq \overline{\mathrm{W}}^i(A) \oplus \overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n)$ by Corollary 3.7 and $\overline{\mathrm{W}}^i_{1-X_{n+1}}(S^n) \simeq \overline{\mathrm{W}}^{i-n}(A)$ by Eq. (3) and the last lemma. Together with Lemma 3.3, this ends the proof of Theorem 1.1.

4. Proof of Theorem 1.2

From now on, the base A and thus also S^n_A will be assumed to be regular rings. We will still denote by $q : \operatorname{Spec} S^n \to \operatorname{Spec} A$ the structure morphism.

For any ring R, we will denote by $e_R \in \mathrm{W}^0(R)$ the multiplicative unit of $\mathrm{W}^{\mathrm{tot}}(R)$; this is just the diagonal form $\langle 1 \rangle = [\mathrm{id} : R \to R]$ (we make the usual identification $R = R^\vee$). In this proof we will also abbreviate

$$P := S^n/(X_1, \dots, X_n)$$
$$\mathfrak{n} := (X_{n+1} - 1) \qquad \mathfrak{s} := (X_{n+1} + 1),$$

so that $P/\mathfrak{n} \simeq A$ and $P/\mathfrak{s} \simeq A$ are the North Pole and the South Pole of the sphere. Write i_P, i_N and i_S for the corresponding closed immersions. We will further write

$$\alpha_N := i_{N*}(e_{P/\mathfrak{n}}) \in \mathrm{W}^n(S^n), \qquad \alpha_S := i_{S*}(e_{P/\mathfrak{s}}) \in \mathrm{W}^n(S^n),$$

and we will keep the same notation for the images in $\overline{\mathrm{W}}^n(S^n)$ of these forms.

The next lemma is just a corollary of the proof of Theorem 1.1.

Lemma 4.1.

$$\overline{\mathrm{W}}^{\mathrm{tot}}(S^n) = \overline{\mathrm{W}}^{\mathrm{tot}}(A) \cdot e_{S^n} \oplus \overline{\mathrm{W}}^{\mathrm{tot}}(A) \cdot \alpha_N.$$

Proof. From the proof of Lemma 3.6 we see that $q^* : \overline{\mathrm{W}}^{\mathrm{tot}}(A) \to \overline{\mathrm{W}}^{\mathrm{tot}}(S^n)$ is injective. Since $q^*(e_A) = e_{S^n}$, we recognize the first direct summand. By the proof of Lemma 3.8 and functoriality of the transfer (see Section 2 above), we see that the other summand is the image of $i_{N*} : \overline{\mathrm{W}}^{\mathrm{tot}}(P/\mathfrak{n}) \to \overline{\mathrm{W}}^{\mathrm{tot}}(S^n)$. This image is $\overline{\mathrm{W}}^{\mathrm{tot}}(A) \cdot \alpha_N$ by the following lemma. □

Lemma 4.2. *Let* $i : \operatorname{Spec} A \hookrightarrow \operatorname{Spec} S^n$ *be the closed immersion corresponding either to the North or the South pole, and let* $i_* : \mathrm{W}^j(A) \to \mathrm{W}^{j+n}(S^n)$ *be the induced transfer morphism. Then for every form* $\beta \in \mathrm{W}^j(A)$*:*

$$i_*(\beta) = q^*(\beta) \star i_*(e_A) \in \mathrm{W}^{j+n}(S^n).$$

Proof. This follows from the projection formula for coherent Witt groups (Gille [7, Thm. 5.2]), applied to the finite morphism i:

$$i_*(\beta) = i_*(\underbrace{i^* q^*}_{\mathrm{id}}(\beta) \star e_A) = q^*(\beta) \star i_*(e_A). \quad \square$$

Proposition 4.3. *The relation* $\alpha_N = -\alpha_S$ *holds in* $\mathrm{W}^n(S^n)$.

As an immediate consequence of this, the form $\alpha_N^2 = -\alpha_N \star \alpha_S$ is supported on the intersection of the North and the South Poles, which is empty: $\operatorname{Spec} P/\mathfrak{n} \cap \operatorname{Spec} P/\mathfrak{s} = \emptyset$; so it is trivially equal to zero.

To prove Proposition 4.3, we first recall straight from [7, Section 9] some facts about forms on Koszul complexes. For any ring R and any regular sequence $(x_1, \ldots, x_n)$ in R, we will denote by $K_\bullet(x_1, \ldots, x_n)$ the Koszul complex for this sequence, and we set it in (homological) degrees from n to 0. (Below we will specialize to the ring S^n and the regular sequence $(X_1, \ldots, X_n)$ in it.) For $1 \le i \le n$ and any unit $r \in R^\times$, the complex $K_\bullet(x_i)$ can be equipped by the following symmetric 1-form:

$$\begin{array}{ccccccccccccc} K_\bullet(x_i): & & \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot x_i} & R & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \ell_r :=} & & & & \downarrow & & {\scriptstyle \cdot r}\downarrow & & \downarrow{\scriptstyle \cdot(-r)} & & \downarrow & & \\ K_\bullet(x_i): & & \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot(-x_i)} & R & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

For any choice of n units in R, the product $(K_\bullet(x_1), \ell_{r_1}) \star \cdots \star (K_\bullet(x_n), \ell_{r_n})$ is a symmetric n-space on the complex $K_\bullet(x_1, \ldots, x_n) \simeq K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$. We denote its form by $\ell_{r_1 \cdots r_n}$.

By displaying an explicit Lagrangian, it is easy to see that $[K_\bullet(x_i), \ell_{r_i}] = 0 \in \mathrm{W}^n(R)$, and therefore $[K_\bullet(x_1, \ldots, x_n), \ell_{r_1 \cdots r_n}] = 0 \in \mathrm{W}^n(R)$ for all symmetric spaces as above.

Lemma 4.4. *If* $\phi : K_\bullet(x_1, \ldots, x_n) \to \operatorname{Hom}_R(K_\bullet(x_1, \ldots, x_n), R)[n]$ *is a quasi-isomorphism of complexes, then there exists a unit* $r \in R^\times$ *such that* $\ell_{r \cdot 1 \ldots 1}$ *is chain homotopic to* ϕ.

Proof. This is a slight generalization of [7, Lemma 9.1]. The same proof goes through. □

We have an isomorphism $\mathrm{W}^0(P) \simeq \mathrm{W}^0(P/\mathfrak{n}) \oplus \mathrm{W}^0(P/\mathfrak{s})$, induced by $P \simeq P/\mathfrak{n} \times P/\mathfrak{s}$, which identifies e_P with (e_N, e_S). Under this isomorphism, the transfer $i_{P*} : \mathrm{W}^0(P) \to \mathrm{W}^n(S^n)$ identifies with (i_{N*}, i_{S*}), and in particular

$$\alpha_N + \alpha_S = i_{N*}(e_N) + i_{S*}(e_S) = i_{P*}(e_P).$$

But the last term is zero in $\mathrm{W}^n(S^n)$. In fact, $i_{P*}(e_P)$ can be represented by $(F_\bullet, \psi)$, where $F_\bullet$ is a projective resolution of the S^n-module $P = S^n/(X_1, \ldots, X_n)$, and where ψ is a symmetric quasi-isomorphism between $F_\bullet$ and its n-shifted dual, lying above the morphism $\mathrm{id} : A \to A$. Since $(X_1, \ldots, X_n)$ is a regular sequence in S^n, we can take $F_\bullet$ to be the Koszul complex for this sequence. By the above lemma (or by direct inspection) we have $i_{A*}(e_A) = [K_\bullet(X_1, \ldots, X_n), \ell_1] = 0 \in \mathrm{W}^n(S^n)$.

This ends the proof of our second theorem.

Acknowledgements

The authors sincerely thank Paul Balmer, Stefan Gille and Burt Totaro for their careful reading and useful comments.

References

[1] Paul Balmer, Witt groups, in: Handbook of K-Theory, vol. 2, Springer, Berlin, 2005, pp. 539–576.
[2] P. Balmer, C. Walter, A Gersten–Witt spectral sequence for regular schemes, Ann. Sci. École Norm. Sup. (4) 35 (1) (2002) 127–152.
[3] G.W. Brumfiel, Witt rings and K-theory, Rocky Mountain J. Math. 14 (4) (1984) 733–765.
[4] G.W. Brumfiel, The real spectrum of an ideal and KO-theory exact sequences, K-Theory 1 (3) (1987) 211–235.
[5] Stefan Gille, On Witt groups with support, Math. Ann. 322 (1) (2002) 103–137.
[6] Stefan Gille, Homotopy invariance of coherent Witt groups, Math. Z. 244 (2003) 211–233.
[7] Stefan Gille, A transfer morphism for Witt groups, J. Reine Angew. Math. 564 (2003) 215–233.
[8] Stefan Gille, The general dévissage theorem for Witt groups of schemes, Arch. Math. 88 (2007) 333–343.
[9] S. Gille, A. Nenashev, Pairings in triangular Witt theory, J. Algebra 261 (2) (2003) 292–309.
[10] J. Hornbostel, M. Schlichting, Localization in Hermitian K-theory of rings, J. London Math. Soc. (2) 70 (1) (2004) 77–124.
[11] Tsit-Yuen Lam, Introduction to quadratic forms over fields, in: Graduate Studies in Math., vol. 67, American Math. Soc., Providence, 2005.